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ON SOME PROPERTIES OF HAMMERSLEY'S ESTIMATOR OF AN INTEGER MEAN[1]

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Let $X_1, \cdots, X_n$ be i.i.d. $N(i, 1)$, $i = 0, \pm 1, \pm 2, \cdots$. Hammersley [2] proposed $[\bar{X}_n]$, the nearest integer to the sample mean, as an estimator of i. It is proved that d is minimax and admissible relative to zero-one loss. However, it is shown that relative to squared error loss, the estimator is neither admissible nor minimax.

1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be independently and identically distributed $N(i, 1)$ random variables, and set $S_n = X_1 + \cdots + X_n$. The mean i is an unknown integer $0, \pm 1, \pm 2, \cdots$. Using the method of maximum likelihood, Hammersley [2] proposed $d = [S_n/n]$ (nearest integer to the sample mean) as an estimator of i. He showed that d is unbiased for i and computed its variance.

Lindley suggested that the proposed estimator is minimax relative to zero-one loss, and Stein conjectured its minimaxity relative to squared error loss (see the discussion in [2]). In Section 2 it is proved that d is in fact minimax and admissible relative to zero-one loss. In Section 3 it is shown, however, that relative to squared error loss the estimator is neither admissible nor minimax.

2. Minimax property and admissibility relative to zero one loss. We consider the loss function

$$L(a, i) = 0\,, \quad \text{if} \quad a = i\,,$$
$$= 1\,, \quad \text{if} \quad a \neq i\,.$$

To show that the estimator $d = [S_n/n]$ is minimax and admissible, we use the Bayesian argument. The joint probability density function of $(X_1, \cdots, X_n)$ given i is

$$f(X_1, X_2, \cdots, X_n \mid i) = (2\pi)^{-n/2} \exp[-\tfrac{1}{2} \textstyle\sum_{j=1}^{n} (X_j - i)^2]\,. \tag{1}$$

Assume that the prior ζ_σ is given by

$$P(i = r) = K_\sigma \exp(-r^2/2\sigma^2)\,, \qquad r = 0, \pm 1, \pm 2, \cdots \tag{2}$$

where $K_\sigma^{-1} = \sum_{r=-\infty}^{\infty} \exp(-r^2/2\sigma^2)$. Then the posterior probability function is

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given by

$$(3)\qquad P(i\mid X_1,\cdots,X_n)=\frac{\exp\left\{-\frac{(n+(1/\sigma^2))}{2}\left(i-\frac{S_n}{n+(1/\sigma^2)}\right)^2\right\}}{\sum_{i=-\infty}^{\infty}\exp\left\{-\frac{(n+(1/\sigma^2))}{2}\left(i-\frac{S_n}{n+(1/\sigma^2)}\right)^2\right\}},$$
$$i=0,\pm1,\pm2,\cdots.$$

It is known (cf. [1]) that the Bayes estimator relative to zero-one loss is the mode of the posterior distribution. Thus it is easily seen that the Bayes estimator of i is

$$(4)\qquad d_\sigma=[S_n/(n+(1/\sigma^2))]\,.$$

Recall that $d=[S_n/n]$, so that $\lim_{\sigma\to\infty}d_\sigma=d$. Thus the estimator d is a limit of Bayes estimators. Let $R(\delta,i)$ denote the risk of an estimator δ, i.e., $R(\delta,i)=E_iL(\delta,i)=P_i(\delta\neq i)$. Then

$$(5)\qquad \begin{aligned}R(d,i)&=P_i\left(\left|\frac{S_n}{n}-i\right|\geqq\tfrac12\right)=P_0\left(n^{\frac12}\left|\frac{S_n}{n}\right|\geqq\frac{n^{\frac12}}{2}\right)\\&=2\left(1-\Phi\left(\frac{n^{\frac12}}{2}\right)\right)\end{aligned}$$

where $\Phi(x)=(2\pi)^{-\frac12}\int_{-\infty}^{x}e^{-u^2/2}\,du$. We obtain the minimax property by showing that

$$(6)\qquad \lim_{\sigma\to\infty}B(d_\sigma,\zeta_\sigma)=2\left(1-\Phi\left(\frac{n^{\frac12}}{2}\right)\right),$$

where $B(\cdot,\zeta_\sigma)$ refers to the Bayes risk relative to the discrete normal prior given by (2). Now

$$\begin{aligned}R(d_\sigma,i)&=P_i(d_\sigma\neq i)=P_i\left\{\left|\frac{S_n}{n+(1/\sigma^2)}-i\right|\geqq\tfrac12\right\}\\&=P_0\left\{Z\geqq\frac{n^{\frac12}}{2}\left(1+\frac{1}{n\sigma^2}\right)+\frac{i}{\sigma^2n^{\frac12}}\ \text{ or }\ Z\leqq-\frac{n^{\frac12}}{2}\left(1+\frac{1}{n\sigma^2}\right)+\frac{i}{\sigma^2n^{\frac12}}\right\},\end{aligned}$$

where Z denotes a $N(0,1)$ variable. Setting $c_n=c_n(\sigma)=\frac12n^{\frac12}(1+1/n\sigma^2)$, $u_n(i)=c_n+(i/\sigma^2n^{\frac12})$, $v_n(i)=-c_n+(i/\sigma^2n^{\frac12})$, and $w_n(i)=c_n-(i/\sigma^2n^{\frac12})$, it follows that

$$(7)\qquad B(d_\sigma,\zeta_\sigma)=K_\sigma\sum_{i=-\infty}^{\infty}\left[\int_{u_n(i)}^{\infty}\varphi(y)\,dy+\int_{-\infty}^{v_n(i)}\varphi(y)\,dy\right]e^{-i^2/2\sigma^2}$$

where $\varphi(y)=e^{-y^2/2}/(2\pi)^{\frac12}$. We can rewrite $B(d_\sigma,\zeta_\sigma)$ as

$$(8)\qquad \begin{aligned}B(d_\sigma,\zeta_\sigma)&=K_\sigma\sum_{i=0}^{\infty}\left[\int_{u_n(i)}^{\infty}\varphi(y)\,dy+\int_{-\infty}^{v_n(i)}\varphi(y)\,dy\right]e^{-i^2/2\sigma^2}\\&\quad+K_\sigma\sum_{i=-\infty}^{-1}\left[\int_{u_n(i)}^{\infty}\varphi(y)\,dy+\int_{-\infty}^{v_n(i)}\varphi(y)\,dy\right]e^{-i^2/2\sigma^2}\\&\equiv G_+(\sigma)+G_-(\sigma)\,,\quad\text{say.}\end{aligned}$$

Now

$$\begin{aligned}G_+(\sigma)&=K_\sigma\sum_{i=0}^{\infty}\left[\left\{\int_{c_n}^{\infty}-\int_{c_n}^{u_n(i)}+\int_{-\infty}^{-c_n}+\int_{-c_n}^{v_n(i)}\right\}\varphi(y)\,dy\right]e^{-i^2/2\sigma^2}\\&=2(1-\Phi(c_n))K_\sigma\sum_{i=0}^{\infty}e^{-i^2/2\sigma^2}\\&\quad+K_\sigma\sum_{i=0}^{\infty}\left[\int_{-c_n}^{v_n(i)}\varphi(y)\,dy-\int_{c_n}^{u_n(i)}\varphi(y)\,dy\right]e^{-i^2/2\sigma^2}\,.\end{aligned}$$

Also

$$G_-(\sigma) = K_\sigma \sum_{i=-\infty}^{-1} [\{\int_{u_n(i)}^{c_n} + \int_{c_n}^{\infty} + \int_{-\infty}^{-c_n} - \int_{v_n(i)}^{-c_n}\}\varphi(y)\, dy]e^{-i^2/2\sigma^2}$$
$$= 2(1 - \Phi(c_n))K_\sigma \sum_{i=-\infty}^{-1} e^{-i^2/2\sigma^2}$$
$$+ K_\sigma \sum_{i=-\infty}^{-1} [\int_{u_n(i)}^{c_n} \varphi(y)\, dy - \int_{v_n(i)}^{-c_n} \varphi(y)\, dy]e^{-i^2/2\sigma^2}$$
$$= 2(1 - \Phi(c_n))K_\sigma \sum_{i=-\infty}^{-1} e^{-i^2/2\sigma^2}$$
$$+ K_\sigma \sum_{i=1}^{\infty} [\int_{-c_n}^{v_n(i)} \varphi(y)\, dy - \int_{c_n}^{u_n(i)} \varphi(y)\, dy]e^{-i^2/2\sigma^2}\, .$$

Hence from (8) we obtain

$$(9) \qquad B(d_\sigma, \zeta_\sigma) = 2(1 - \Phi(c_n)) + 2K_\sigma \sum_{i=0}^{\infty} [\int_{-c_n}^{v_n(i)} \varphi(y)\, dy - \int_{c_n}^{u_n(i)} \varphi(y)\, dy]e^{-i^2/2\sigma^2}$$
$$= 2(1 - \Phi(c_n)) + 2K_\sigma F_n(\sigma)\, .$$

In what follows we shall use the notations c_n, $u_n(i)$, $v_n(i)$ and $w_n(i)$, which were introduced before the equation number (7). All that we prove in this section is based on the following lemma.

LEMMA. *If* $F_n(\sigma) = \sum_{i=0}^{\infty} [\int_{-c_n}^{v_n(i)} \varphi(y)\, dy - \int_{c_n}^{u_n(i)} \varphi(y)\, dy]e^{-i^2/2\sigma^2}$, *then*

$$(10) \qquad \lim_{\sigma\to\infty} F_n(\sigma) = 0\, ,$$

and

$$(11) \qquad \lim_{\sigma\to\infty} K_\sigma F_n(\sigma) = 0\, .$$

PROOF. We rewrite $F_n(\sigma)$ as

$$F_n(\sigma) = \sum_{i=0}^{\infty} [\int_{w_n(i)}^{c_n} \varphi(y)\, dy - \int_{c_n}^{u_n(i)} \varphi(y)\, dy]e^{-i^2/2\sigma^2}$$
$$= \sum_{i=0}^{\infty} (\int_{w_n(i)}^{c_n} (\varphi(y) - \varphi(y + (i/\sigma^2 n^{\frac{1}{2}})))\, dy)e^{-i^2/2\sigma^2}\, .$$

Now

$$\varphi(y) - \varphi(y + (i^2/\sigma n^{\frac{1}{2}})) = \frac{e^{-y^2/2}}{(2\pi)^{\frac{1}{2}}} (1 - \exp(-\tfrac{1}{2}[(i^2/n\sigma^4) + (2iy/\sigma^2 n^{\frac{1}{2}})]))$$
$$\leqq \frac{e^{-y^2/2}}{(2\pi)^{\frac{1}{2}}} \left(\frac{i^2}{2n\sigma^4} + \frac{iy}{\sigma^2 n^{\frac{1}{2}}}\right),$$

and it follows that

$$F_n(\sigma) \leqq \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} \left(\int_{w_n(i)}^{c_n} e^{-y^2/2} \left(\frac{i^2}{2n\sigma^4} + \frac{iy}{\sigma^2 n^{\frac{1}{2}}}\right) dy\right) e^{-i^2/2\sigma^2}$$
$$\leqq \frac{1}{2n\sigma^6(2n\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} i^3 e^{-i^2/2\sigma^2} + \frac{1}{\sigma^2(2n\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} i(e^{-\frac{1}{2}w_n^2(i)} - e^{-\frac{1}{2}c_n^2})e^{-i^2/2\sigma^2}$$
$$\leqq \frac{1}{2n\sigma^6(2n\pi)^{\frac{1}{2}}} \int_0^\infty x^3 e^{-x^2/2\sigma^2}\, dx$$
$$+ \frac{1}{\sigma^2(2n\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} i e^{-\frac{1}{2}w_n^2(i)} \left(\frac{ic_n}{\sigma^2 n^{\frac{1}{2}}} - \frac{i^2}{2n\sigma^4}\right) e^{-i^2/2\sigma^2}$$
$$\leqq \frac{1}{2n\sigma^6(2n\pi)^{\frac{1}{2}}} \int_0^\infty x^3 e^{-x^2/2\sigma^2}\, dx + \frac{c_n}{\sigma^4(2n^2\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} i^2 e^{-\frac{1}{2}w_n^2(i)} e^{-i^2/2\sigma^2}$$
$$\leqq \frac{1}{2n\sigma^2(2n\pi)^{\frac{1}{2}}} \int_0^\infty w^3 e^{-w^2/2}\, dw + \frac{c_n}{\sigma(2n^2\pi)^{\frac{1}{2}}} \int_0^\infty w^2 e^{-w^2/2}\, dw\, .$$

The right-hand side approaches 0 as $\sigma \to \infty$, and hence

$$\limsup_{\sigma\to\infty} F_n(\sigma) \leqq 0 . \tag{12}$$

We may directly verify

$$F_n(\sigma) \geqq \sum_{i=0}^{\infty} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{w_n(i)}^{c_n} (e^{-\frac{1}{2}\max^2\{c_n, |w_n(i)|\}} - e^{-\frac{1}{2}c_n^2})\, dy e^{-i^2/2\sigma^2} ,$$

and thus

$$\begin{aligned} F_n(\sigma) &\geqq \frac{1}{\sigma^2(2n\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} i(e^{-\frac{1}{2}\max^2\{c_n, |w_n(i)|\}} - e^{-\frac{1}{2}c_n^2})e^{-i^2/2\sigma^2} \\ &\geqq -\psi_n(\sigma) , \end{aligned} \tag{13}$$

where

$$\psi_n(\sigma) = \frac{1}{\sigma^2(2n\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} i(e^{-\frac{1}{2}c_n^2} - e^{-\frac{1}{2}\max^2\{c_n, |w_n(i)|\}})e^{-i^2/2\sigma^2} .$$

Note that $\psi_n(\sigma) \geqq 0$. However,

$$\begin{aligned} \psi_n(\sigma) &= \frac{1}{\sigma^2(2n\pi)^{\frac{1}{2}}} \sum_{i\geqq 0: c_n \leqq |w_n(i)|} i(e^{-\frac{1}{2}c_n^2} - e^{-\frac{1}{2}w_n^2(i)})e^{-i^2/2\sigma^2} \\ &\leqq \frac{1}{\sigma^2(2n\pi)^{\frac{1}{2}}} \sum_{i\geqq 0: c_n \leqq |w_n(i)|} ie^{-\frac{1}{2}c_n^2}\left(\frac{i^2}{2n\sigma^4} - \frac{ic_n}{\sigma^2 n^{\frac{1}{2}}}\right) e^{-i^2/2\sigma^2} \\ &\leqq \frac{e^{-\frac{1}{2}c_n^2}}{2n\sigma^6(2n\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} i^3 e^{-i^2/2\sigma^2} \leqq \frac{1}{2n\sigma^6(2n\pi)^{\frac{1}{2}}} \int_0^\infty x^3 e^{-x^2/2\sigma^2}\, dx , \end{aligned}$$

which approaches 0 as $\sigma \to \infty$.
Thus $\lim_{\sigma\to\infty} \psi_n(\sigma) = 0$. It follows from (13) that

$$\liminf_{\sigma\to\infty} F_n(\sigma) \geqq 0 . \tag{14}$$

Hence (10) follows from (12) and (14). Moreover, it is easy to see that $[1 + \sigma(2\pi)^{\frac{1}{2}}]^{-1} \leqq K_\sigma \leqq 1$, and essentially the preceding argument also proves (11). Alternatively, it is easy to show that $\lim_{\sigma\to\infty} K_\sigma = 0$, and hence (11) follows from the fact that $F_n(\sigma)$ remains bounded. This completes the proof of the lemma.

It follows from (9) and (11) that

$$\lim_{\sigma\to\infty} B(d_\sigma, \zeta_\sigma) = 2\left(1 - \Phi\left(\frac{n^{\frac{1}{2}}}{2}\right)\right) = R(d, i) .$$

Hence d is minimax. As usual, we now prove admissibility by contradiction. Assume on the contrary that d is not admissible. Then there exists a δ such that

$$R(\delta, i) \leqq R(d, i) \qquad \text{for all} \quad i , \tag{15}$$

and

$$R(\delta, i_0) < R(d, i_0) \qquad \text{for some} \quad i_0 .$$

So there exists an $\varepsilon > 0$ such that

$$R(\delta, i_0) - R(d, i_0) \leqq -\varepsilon \qquad \text{for some} \quad i_0 . \tag{16}$$

Since d_σ is Bayes, we have

(17) $$\liminf_{\sigma\to\infty} K_\sigma^{-1}[B(\delta, \zeta_\sigma) - B(d_\sigma, \zeta_\sigma)] \geqq 0 .$$

Now

(18) $$\begin{aligned} K_\sigma^{-1}&[B(\delta, \zeta_\sigma) - B(d_\sigma, \zeta_\sigma)] \\ &= \textstyle\sum_{i=-\infty}^{\infty} (R(\delta, i) - R(d_\sigma, i))e^{-i^2/2\sigma^2} \\ &= \textstyle\sum_{i=-\infty}^{\infty} [R(\delta, i) - R(d, i) + R(d, i) - R(d_\sigma, i)]e^{-i^2/2\sigma^2} \\ &\leqq -\varepsilon e^{-i_0^2/2\sigma^2} + K_\sigma^{-1}[B(d, \zeta_\sigma) - B(d_\sigma, \zeta_\sigma)] . \end{aligned}$$

From (9) we have

$$\begin{aligned} B(d_\sigma, \zeta_\sigma) &= 2(1 - \Phi(c_n)) + 2K_\sigma F_n(\sigma) \\ &= B(d, \zeta_\sigma) + 2\left(\Phi\left(\frac{n^{\frac{1}{2}}}{2}\right) - \Phi(c_n)\right) + 2K_\sigma F_n(\sigma) . \end{aligned}$$

Thus

(19) $$K_\sigma^{-1}[B(d, \zeta_\sigma) - B(d_\sigma, \zeta_\sigma)] = 2K_\sigma^{-1}\left[\Phi(c_n) - \Phi\left(\frac{n^{\frac{1}{2}}}{2}\right)\right] - 2F_n(\sigma) .$$

It follows from (18) and (19) that

(20) $$\begin{aligned} K_\sigma^{-1}&[B(\delta, \zeta_\sigma) - B(d_\sigma, \zeta_\sigma)] \\ &\leqq -\varepsilon e^{-i_0^2/2\sigma^2} + 2K_\sigma^{-1}\left[\Phi(c_n) - \Phi\left(\frac{n^{\frac{1}{2}}}{2}\right)\right] - 2F_n(\sigma) . \end{aligned}$$

It is easy to see that

$$\begin{aligned} K_\sigma^{-1}\left[\Phi(c_n) - \Phi\left(\frac{n^{\frac{1}{2}}}{2}\right)\right] &= K_\sigma^{-1} \int_{\frac{1}{2}n^{\frac{1}{2}}}^{\frac{1}{2}n^{\frac{1}{2}}+1/2n^{\frac{1}{2}}\sigma^2} \varphi(y)\,dy \\ &\leqq \frac{K_\sigma^{-1}}{2\sigma^2(2n\pi)^{\frac{1}{2}}} \leqq \frac{1 + \sigma 2^{\frac{1}{2}}}{2\sigma^2(2n\pi)^{\frac{1}{2}}} \to 0 \quad \text{as} \quad \sigma \to \infty . \end{aligned}$$

Hence from (10) and (20) we obtain

(21) $$\liminf_{\sigma\to\infty} K_\sigma^{-1}[B(\delta, \zeta_\sigma) - B(d_\sigma, \zeta_\sigma)] \leqq -\varepsilon < 0 .$$

Since (21) contradicts (17), d is admissible.

3. Inadmissibility and the non-minimax property relative to squared error loss. Once and for all we set $d_1 = [S_n/n]$, and $d_2 = S_n/n$. We will show that d_1 is inadmissible and non-minimax relative to squared error loss. We note that $E_i d_1 = E_i d_2 = i$, and $\sigma_{d_2}^2 = 1/n$. Moreover, from Hammersley [2] we have

(22) $$\sigma_{d_1}^2 = \psi\left(\frac{n^{\frac{1}{2}}}{2}\right) - 2\psi(n^{\frac{1}{2}}) ,$$

where

$$\psi(y) = (2/\pi)^{\frac{1}{2}} \textstyle\sum_{j=1}^{\infty} j \int_{jy}^{\infty} e^{-x^2/2}\,dx = 2 \sum_{j=1}^{\infty} j(1 - \Phi(jy)) .$$

It is interesting to note that in the regular case $N(\mu, 1)$, $-\infty < \mu < \infty$, $\bar{X}_n = S_n/n$ is a complete sufficient statistic, UMVU estimator, and also admissible for

μ relative to the loss function $|a - \mu|^k$, $k = 1, 2$. Though $\bar{X}_n$ continues to be sufficient, it fails to be complete when $\Omega = \{0, \pm 1, \pm 2, \cdots\}$. This is trivially seen on taking $g(\bar{X}_n) = d_1 - d_2$.

Before showing inadmissibility and non-minimaxity we observe the following relevant fact. We then have

$$\begin{aligned}\operatorname{Cov}(d_1, d_2) &= E_i(\bar{X}_n - i)([\bar{X}_n] - i)\\ &= \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} (y - i)([y] - i)e^{-n(y-i)^2/2}\, dy\,.\end{aligned}$$

It is easy to show that $[y] - i = [y - i]$, therefore

$$\operatorname{Cov}(d_1, d_2) = \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x[x]e^{-nx^2/2}\, dx\,. \tag{23}$$

Thus the $\operatorname{Cov}(d_1, d_2)$ (hence the correlation ρ) is independent of i. Now we will exhibit a uniformly better estimator than d_1 relative to squared error loss. To this end, we set

$$d_\alpha = \alpha d_1 + (1 - \alpha)d_2\,, \qquad \alpha \quad \text{real.} \tag{24}$$

Then d_α is also unbiased for i since d_1 and d_2 are so. We have

$$V(\alpha) = \sigma_{d_\alpha}^2 = \alpha^2\sigma_{d_1}^2 + (1 - \alpha)^2\sigma_{d_2}^2 + 2\alpha(1 - \alpha)\rho\sigma_{d_1}\sigma_{d_2}\,.$$

It is easily seen that the optimum α_0 which minimizes $V(\alpha)$ is given by

$$\alpha_0 = (\sigma_{d_2}^2 - \rho\sigma_{d_1}\sigma_{d_2})/(\sigma_{d_1}^2 + \sigma_{d_2}^2 - 2\rho\sigma_{d_1}\sigma_{d_2})\,. \tag{25}$$

We now show that $\alpha_0 \neq 1$. It suffices to show that $\operatorname{Cov}(d_1, d_2) \neq \sigma_{d_1}^2$. From [2] we know that

$$\sigma_{d_1}^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{j=1}^{\infty} j^2 \int_{n^{\frac{1}{2}}(j-\frac{1}{2})}^{n^{\frac{1}{2}}(j+\frac{1}{2})} e^{-u^2/2}\, du\,. \tag{26}$$

From (23) we have

$$\begin{aligned}\operatorname{Cov}(d_1, d_2) &= \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x[x]e^{-nx^2/2}\, dx\\ &= \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \lim_{k\to\infty} \sum_{m=-k}^{k} m \int_{(m-\frac{1}{2})}^{(m+\frac{1}{2})} xe^{-nx^2/2}\, dx \qquad (27)\\ &= \frac{1}{(2n\pi)^{\frac{1}{2}}} \lim_{k\to\infty} \sum_{m=-k}^{k} m \int_{n^{\frac{1}{2}}(m-\frac{1}{2})}^{n^{\frac{1}{2}}(m+\frac{1}{2})} xe^{-x^2/2}\, dx\\ &= \left(\frac{2}{n\pi}\right)^{\frac{1}{2}} \sum_{j=1}^{\infty} j \int_{n^{\frac{1}{2}}(j-\frac{1}{2})}^{n^{\frac{1}{2}}(j+\frac{1}{2})} xe^{-x^2/2}\, dx\,.\end{aligned}$$

Now,

$$\begin{aligned} j \int_{n^{\frac{1}{2}}(j-\frac{1}{2})}^{n^{\frac{1}{2}}(j+\frac{1}{2})} (y/n^{\frac{1}{2}})e^{-y^2/2}\, dy &< j^2 \int_{n^{\frac{1}{2}}(j-\frac{1}{2})}^{n^{\frac{1}{2}}(j+\frac{1}{2})} e^{-y^2/2}\, dy\\ &\Leftrightarrow \int_{n^{\frac{1}{2}}(j-\frac{1}{2})}^{n^{\frac{1}{2}}(j+\frac{1}{2})} (y - n^{\frac{1}{2}}j)e^{-y^2/2}\, dy < 0\\ &\Leftrightarrow n \int_{-\frac{1}{2}}^{\frac{1}{2}} ue^{-n(u+j)^2/2}\, du < 0\\ &\Leftrightarrow \int_0^{\frac{1}{2}} ue^{-n(u+j)^2/2} - \int_0^{\frac{1}{2}} ue^{-n(u-j)^2/2}\, du < 0 \quad \forall j \geqq 1\,,\end{aligned}$$

which is true. This together with (26) and (27) implies

$$\operatorname{Cov}(d_1, d_2) \neq \sigma_{d_1}^2 \Leftrightarrow \alpha_0 \neq 1 .$$

Therefore $d_{\alpha_0} = \alpha_0 d_1 + (1 - \alpha_0)d_2$, where α_0 is given by (25), is uniformly better than any unbiased estimator given by (24) and hence, in particular, uniformly better than d_1 (i.e. Hammersley's rounded mean). Since d_{α_0} has constant risk and is uniformly better than d_1, this shows that d_1 is neither admissible nor minimax relative to squared error loss.

That the sequential methods have potential for such problems has been shown by Robbins [5]. Though the basic object of [3] is to decide among a countable set of probability distributions, the methods are applicable to the problem of estimating restricted parameters (see also [4]).

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