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THE NON-SINGULARITY OF GENERALIZED SAMPLE COVARIANCE MATRICES[1]

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Let $X = (X_1, \cdots, X_n)$ where the $X_i : p \times 1$ are independent random vectors, and let $A : n \times n$ be positive semi-definite symmetric. This paper establishes necessary and sufficient conditions that the random matrix XAX' be positive definite w.p. 1. The results are applied to cases where A has a particular form or $X_1, \cdots, X_n$ are i.i.d. In particular, it is shown that in the i.i.d. case, the sample covariance matrix $\Sigma(X_i - \bar{X})(X_i - \bar{X})'$ is positive definite w.p. 1 iff $P[X_1 \in F] = 0$ for every proper flat $F \subset R^p$.

1. Introduction. In a recent paper, Dykstra (1970) demonstrated the non-singularity w.p. 1 of the sample covariance matrix $S = \sum_1^n (Y_i - \bar{Y})(Y_i - \bar{Y})'$ for $Y_1, \cdots, Y_n$ i.i.d. $N_p(\mu, \Sigma)$ where Σ is positive definite (henceforth written $\Sigma > 0$). A similar result also appears in lecture notes of C. Stein (1969). These demonstrations depend heavily on normality. Let $Y : p \times n$ have columns $Y_1, \cdots, Y_n$ of dimension p and write $S = YA_0Y'$ where $A_0 = I - (1/n)ee'$, $e' = (1, 1, \cdots, 1) \in R^n$. In this paper, we obtain conditions under which a random matrix of the form XAX' is positive definite w.p. 1 where $A : n \times n$ is positive semi-definite and $X = (X_1, \cdots, X_n) : p \times n$ is a random matrix whose columns are independent but not necessarily normal or identically distributed.

2. Main results. Let $n - r$ denote the rank of $A = \{a_{ij}\}$ (assumed to be positive semi-definite) and assume that $n - r \geqq p$ and $a_{ii} > 0$, $i = 1, \cdots, n$. We now reduce the problem of studying the non-singularity of XAX' to the case $r = 0$. Let $\mathscr{V} \subseteqq R^n$ be the range of A and let $\Gamma : r \times n$ be any matrix of rank r with row space $\mathscr{V}^\perp \equiv$ the orthogonal complement of $\mathscr{V}$. Now, XAX' is singular iff $\exists a \neq 0$ in R^p such that $a'XAX'a = 0$ iff $\exists a \neq 0$ in R^p such that $a'X \in \mathscr{V}^\perp$ iff $\exists a \neq 0$ in R^p and $b \in R^r$ such that $a'X = b'\Gamma$ iff the matrix $\tilde{X} = \binom{X}{\Gamma} : (p + r) \times n$ is singular. Thus, studying the non-singularity of XAX' is equivalent to studying the non-singularity of $\tilde{X}$ (or equivalently of $\tilde{X}\tilde{X}'$) where $\tilde{X} : \tilde{p} \times n$ again has independent columns and $n \geqq \tilde{p} = p + r$.

REMARK. The assumption that $a_{ii} > 0$ is without essential loss of generality. For if $a_{ii} = 0$, then X_i does not occur in XAX', so conditions on X_i are irrelevant.

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We now study the case of $r = 0$. Let $X_1, \cdots, X_n$ in R^p be independent random vectors, $n \geqq p$, and $X = (X_1, \cdots, X_n)$.

THEOREM 2.1. *The following are equivalent*:

(i) $P\{X \text{ is singular}\} > 0$,

(ii) *for some* s, $1 \leqq s \leqq p$, $\exists\, n - p + s$ *columns of* X, $\{X_{i_\alpha} \mid \alpha = 1, \cdots, n-p+s\}$ *and* $\exists$ *an* $(s-1)$-*dimensional linear manifold* $M^{(s-1)} \subset R^p$ *such that* $P\{X_{i_\alpha} \in M^{(s-1)};\ \alpha = 1, \cdots, n-p+s\} > 0$.

PROOF. Clearly (ii) implies (i). The proof that (i) implies (ii) is deferred to Section 4.

Replacing X by $\tilde{X}$ in Theorem 2.1 gives the necessary and sufficient condition that $XAX' > 0$ w.p. 1. We illustrate the application of Theorem 2.1 for the case $r = 1$, $n \geqq p + 1$ and $\Gamma = e' = (1, 1, \cdots, 1) \in R^n$. Recall that a q-dimensional *flat*, $F^{(q)}$, in R^p is the translate of a q-dimensional manifold.

THEOREM 2.2. *Let* $A: n \times n$ *have rank* $n - 1$ *and null space spanned by* e. *The following are equivalent*:

(i) $P\{XAX' \text{ is singular}\} > 0$,

(ii) *for some* t, $1 \leqq t \leqq p$, $\exists\, n - p + t$ *columns of* X, $\{X_{i_\alpha} \mid \alpha = 1, \cdots, n-p+t\}$ *and* $\exists$ *a* $(t-1)$-*dimensional flat*, $F^{(t-1)}$, *such that* $P\{X_{i_\alpha} \in F^{(t-1)};\ \alpha = 1, \cdots, n - p + t\} > 0$.

PROOF. Apply Theorem 2.1 to $\tilde{X} = \binom{X}{e'}$ with p replaced by $p+1$ and $t = s-1$.

In the case $A_0 = I - (1/n)ee'$, Theorem 2.2 gives necessary and sufficient conditions that the sample covariance matrix be positive definite w.p. 1.

A very simple sufficient condition which guarantees (for any $r \leqq n - p$) the positive definiteness of XAX' w.p. 1 is given below.

THEOREM 2.3. *Assume that under the distribution of each* $X_i \in R^p$, *every flat of dimension* $p - 1$ *has probability* 0. *Then* $XAX' > 0$ w.p. 1.

PROOF. Consider $\tilde{X} = \binom{X}{\Gamma}: \tilde{p} \times n$, $\tilde{p} = p + r$. Let $1 \leqq s \leqq \tilde{p}$ and choose $n - \tilde{p} + s$ columns of $\tilde{X}$, say $\tilde{X}_1, \cdots, \tilde{X}_{n-\tilde{p}+s}$, relabeling if necessary. If $M^{(s-1)} \subset R^p$ is an $(s-1)$-dimensional manifold, then $M^{(s-1)} = \{z \mid \Delta z = 0\}$ where $\Delta: (\tilde{p} - s + 1) \times \tilde{p}$ has rank $\tilde{p} - s + 1$. To apply Theorem 2.1, we must show that $P\{\Delta\tilde{X}_i = 0,\ i = 1, \cdots, n - \tilde{p} + s\} = 0$. Partition $\Delta = (\Delta_1, \Delta_2)$, $\Delta_1: (\tilde{p} - s + 1) \times p$, $\Delta_2: (\tilde{p} - s + 1) \times r$ and write

$$(\tilde{X}_1, \cdots, \tilde{X}_{n-\tilde{p}+s}) = \begin{pmatrix} \dot{X} \\ \dot{\Gamma} \end{pmatrix}.$$

Thus, we must show that

$$P\left\{(\Delta_1 \Delta_2)\begin{pmatrix} \dot{X} \\ \dot{\Gamma} \end{pmatrix} = \Delta_1 \dot{X} + \Delta_2 \dot{\Gamma} = 0\right\} = 0\,. \tag{2.1}$$

If $\Delta_1 \neq 0$, then (2.1) $= 0$ from the assumption that all proper flats have probability

0. If $\Delta_1 = 0$, then rank $(\Delta_2) = \tilde{p} - s + 1$ so $\Delta_2 \dot{\Gamma}$ cannot be zero: otherwise, dim (range $\dot{\Gamma}$) $\leqq r - (\tilde{p} - s + 1)$ so that $r =$ dim (range Γ) $\leqq$ dim (range $\dot{\Gamma}$) + dim (range $\ddot{\Gamma}$) $\leqq r - (\tilde{p} - s + 1) + (\tilde{p} - s) = r - 1$ where $\Gamma = (\dot{\Gamma} \ \ \ddot{\Gamma})$, a contradiction. Hence, in either case, (2.1) is 0.

The sufficient condition of Theorem 2.3 is satisfied, for example, if the distribution of X_i has a density with respect to Lebesgue measure or is orthogonally invariant.

For pedagogical purposes we give a direct proof of Theorem 2.3 which does not appeal to Theorem 2.1. Partition

$$\tilde{X} = \begin{pmatrix} \dot{X} & \ddot{X} \\ \dot{\Gamma} & \ddot{\Gamma} \end{pmatrix}$$

with $\dot{X}: p \times (n-r)$, $\ddot{X}: p \times r$, $\dot{\Gamma}: r \times (n-r)$, and $\ddot{\Gamma}: r \times r$. Permuting columns if necessary, we may assume $\ddot{\Gamma}$ is non-singular. Now $\tilde{X}$ is singular iff $W \equiv \dot{X} - \ddot{X}\ddot{\Gamma}^{-1}\dot{\Gamma}$ is singular (to see this, premultiply $\tilde{X}$ by the non-singular $\begin{pmatrix} I_p & \ddot{X}\ddot{\Gamma}^{-1} \\ 0 & I_r \end{pmatrix}$), so to prove $P\{\tilde{X} \text{ singular}\} = 0$ it suffices to show that $P\{W \text{ singular} \mid \ddot{X}\} = 0$. For each $i = 1, \cdots, n-r$ set $S_i = \text{Span}\{W_j : j \neq i\}$, where $W = (W_1, \cdots, W_{n-r})$. Then

$$\begin{aligned} &P\{W \text{ singular} \mid \ddot{X}\} \\ (2.2) \qquad &\leqq \textstyle\sum_{i=1}^{n-r} P\{W_i \in S_i \text{ and } \dim(S_i) \leqq p-1 \mid \ddot{X}\} \\ &= \textstyle\sum_{i=1}^{n-r} E[P\{W_i \in S_i \text{ and } \dim(S_i) \leqq p-1 \mid W_j, j \neq i, \ddot{X}\} \mid \ddot{X}]\,. \end{aligned}$$

Since $\dot{X}$ and $\ddot{X}$ are independent, however, for fixed $\ddot{X}$ the random vectors $(W_1, \cdots, W_{n-r})$ are independent, and $P\{W_i \in F \mid \ddot{X}\} = 0$ for every proper flat F in R^p. Therefore the conditional probability in the last expression in (2.2) is zero.

If we drop the assumption that $X_1, \cdots, X_n$ are independent but assume instead that the distribution of X is absolutely continuous with respect to np-dimensional Lebesgue measure, the above argument remains valid if the statements involving independence are replaced by the observation that the conditional distribution of W_i given $\ddot{X}$ and W_j, $j \neq i$, is absolutely continuous with respect to p-dimensional Lebesgue measure. This provides an alternate proof of the theorem of Okamoto (1973).

3. Special cases. In this section we obtain necessary and sufficient conditions that $XAX' > 0$ w.p. 1 in the special cases (a) $p = 1$ or (b) $X_1, \cdots, X_n$ are i.i.d.

THEOREM 3.1. *Let $U = (U_1, \cdots, U_n)$ where the U_i are independent real-valued random variables. Let $A: n \times n$ be positive semi-definite of rank $n - r \geqq 1$ with range space $\mathscr{V} \subseteq R^n$ and $a_{ii} > 0$, $i = 1, \cdots, n$. The following are equivalent*:

(i) $P\{UAU' > 0\} = 1$,
(ii) $P\{U \in \mathscr{V}^\perp\} = 0$,
(iii) *U has no atoms in $\mathscr{V}^\perp$.*

PROOF. Clearly, (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). We show that not (i) implies not (iii). First, we remark that the assumption $a_{ii} > 0$, $i = 1, \cdots, n$ is equivalent to the statement that $\mathscr{V}^\perp$ contains no coordinate axis $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)' \in R^n$, which is equivalent to the statement that every subset of $(n-1)$ columns of Γ span R^r.

Not (i) implies, by Theorem 2.1, that there is an s, $1 \leqq s \leqq \tilde{p} = r + 1$, there are $n - \tilde{p} + s$ columns $\{\tilde{X}_{i_\alpha} \mid \alpha = 1, \cdots, n - \tilde{p} + s\}$ of $\tilde{X} = \binom{U}{\Gamma}$ and there is an $(s-1)$-dimensional manifold $M^{(s-1)} \subset R^{\tilde{p}}$ such that

$$P\{\tilde{X}_{i_\alpha} \in M^{(s-1)}, \alpha = 1, \cdots, n - \tilde{p} + s\} > 0\,. \tag{3.1}$$

If $n - \tilde{p} + s$ columns of $\tilde{X}$ lie in $M^{(s-1)}$ with positive probability, then the corresponding $n - \tilde{p} + s$ columns of Γ lie in a manifold contained in R^r of dimension $\leqq s - 1$. If $s \leqq r$, then some $n - 1$ columns of Γ lie in a manifold of dimension $\leqq r - 1$ which cannot happen. Thus, $s = r + 1$ so $M^{(s-1)} = \{z \mid z \in R^{r+1}, \delta' z = 0\}$ where $\delta \neq 0$, $\delta \in R^{r+1}$. Hence (3.1) becomes

$$P\{\delta\tilde{X} = \delta_1 U + \ddot{\delta}\Gamma = 0\} > 0 \tag{3.2}$$

where $\delta = (\delta_1, \ddot{\delta})$, $\delta_1 \in R^1$. Since Γ has full rank, (3.2) implies that $\delta_1 \neq 0$, so

$$P\{U = -(1/\delta_1)\ddot{\delta}\Gamma \in \mathscr{V}^\perp\} > 0\,.$$

This completes the proof.

We now turn to the case of i.i.d. random vectors in R^p. The following is an immediate consequence of Theorem 2.1.

THEOREM 3.2. *Let $X = (X_1, \cdots, X_n)$ where the X_i are* i.i.d. *random vectors in R^p, $n \geqq p$. The following are equivalent*:

(i) $P\{X$ *is non-singular*$\} = 1$,

(ii) $P\{X_1 \in M\} = 0$ *for all proper manifolds* $M \subset R^p$.

For $w \in R^p$, $w \neq 0$, and $c \in R^1$, let $F(w, c) = \{x \in R^p \mid w'x = c\}$ be the $(p-1)$-dimensional flat determined by w and c.

THEOREM 3.3. *Let $X = (X_1, \cdots, X_n)$ where the $X_i \in R^p$ are* i.i.d. *and let $A: n \times n$ be positive semi-definite with range space $\mathscr{V} \subseteqq R^n$ and rank $n - r \geqq p$. The following are equivalent*:

(i) $P\{XAX' > 0\} = 1$,

(ii) $\prod_{i=1}^n P\{X_1 \in F(w, v_i)\} = 0$ *for all* $w \in R^p$, $w \neq 0$ *and* $v = (v_1, \cdots, v_n) \in \mathscr{V}^\perp$.

PROOF. If (ii) does not hold, then $\exists w \neq 0$ in R^p and $v = b'\Gamma \in \mathscr{V}^\perp$ such that $P\{w'X = b'\Gamma\} > 0$. Hence $P\{XAX'$ is singular$\} = P\{\binom{X}{\Gamma}$ is singular$\} > 0$ and (i) does not hold.

Next, assume (ii) holds. Setting $v = 0$, it follows that $P\{X_1 \in M\} = 0$ for all proper manifolds $M \subset R^p$ and therefore $P\{X_1 \in F\} = 0$ for all flats of dimension $\leqq p - 2$. If (i) does not hold, then by Theorem 2.1, $\exists s$, $1 \leqq s \leqq \tilde{p} = p + r$, $\exists n - \tilde{p} + s$ columns of $\tilde{X} = \binom{X}{\Gamma}$, say $\tilde{X}_1, \cdots, \tilde{X}_{n-\tilde{p}+s}$ (relabeling if necessary) and $\exists$ a manifold $M^{(s-1)} \subset R^p$ such that

$$P\{\tilde{X}_i \in M^{(s-1)}, i = 1, \cdots, n - \tilde{p} + s\} > 0\,. \tag{3.3}$$

Note that s must be $\geqq 2$ since $P\{X_1 = 0\} = 0$. Now, $M^{(s-1)} = \{z \mid \Delta z = 0\}$ where $\Delta: (\tilde{p} + s - 1) \times \tilde{p}$ has rank $\tilde{p} + s - 1$, so (3.3) becomes

$$P\left\{\Delta \begin{pmatrix} \dot{X} \\ \dot{\Gamma} \end{pmatrix} = 0\right\} = P\{\Delta_1 X_i + \Delta_2 \gamma_i = 0, i = 1, \cdots, n - \tilde{p} + s\} > 0 \tag{3.4}$$

where $\Delta = (\Delta_1 \Delta_2)$,

$$\begin{pmatrix} \dot{X} \\ \dot{\Gamma} \end{pmatrix} = (\tilde{X}_1, \cdots, \tilde{X}_{n-\tilde{p}+s}),$$

and $\Gamma = (\gamma_1, \cdots, \gamma_{n-\tilde{p}+s})$. As in the proof of Theorem 2.3, $\Delta_1 \neq 0$. However, if rank $\Delta_1 \geqq 2$ then $\{x \mid x \in R^p, \Delta_1 x = -\Delta_2 \gamma_i\}$ is a flat of dimension $\leqq p - 2$, contradicting (3.4). Thus rank $\Delta_1 = 1$ so $\Delta_1 = aw'$, $a: (\tilde{p} - s + 1) \times 1$ and $w: p \times 1$.

Since

$$\Delta \begin{pmatrix} \dot{X} \\ \dot{\Gamma} \end{pmatrix} = 0$$

implies that

$$\Delta \begin{pmatrix} X \\ \Gamma \end{pmatrix}$$

is singular, (3.4) yields

$$\begin{aligned} 0 &< P\left\{\Delta \begin{pmatrix} X \\ \Gamma \end{pmatrix} \text{ singular}\right\} \\ &= P\{aw'X + \Delta_2 \Gamma \text{ singular}\} \\ &= P\left\{(a \quad I) \begin{pmatrix} w'X \\ \Delta_2 \Gamma \end{pmatrix} \text{ singular}\right\}, \end{aligned} \tag{3.5}$$

where $I: (\tilde{p} - s + 1) \times (\tilde{p} - s + 1)$, and

$$\begin{pmatrix} w'X \\ \Delta_2 \Gamma \end{pmatrix} : (\tilde{p} - s + 2) \times n .$$

Using the fact that $s \geqq 2$ and $(a \quad I)$ has rank $\tilde{p} - s + 1$, we conclude that

$$P\left\{\begin{pmatrix} w'X \\ \Delta_2 \Gamma \end{pmatrix} \text{ singular}\right\} > 0 . \tag{3.6}$$

Since $U \equiv w'X: 1 \times n$ has independent components, Theorem 3.1 implies that there is a $y: (\tilde{p} - s + 1) \times 1$ such that

$$P\{U = y'\Delta_2 \Gamma\} > 0 . \tag{3.7}$$

Setting $v = y'\Delta_2 \Gamma \in \mathscr{Y}^{\perp}$, (3.7) becomes

$$\prod_{i=1}^{n} P\{X_1 \in F(w, v_i)\} > 0 ,$$

contradicting (ii). Hence (i) must hold.

As an immediate consequence of Theorem 3.3, we obtain

THEOREM 3.4. *Let* $X = (X_1, \cdots, X_n)$ *where the* $X_i \in R^p$ *are* i.i.d., *and let*

$A: n \times n$, $n \geqq p + 1$, have rank $n - 1$ with null space spanned by $e \in R^n$. The following are equivalent:

(i) $P\{XAX' \text{ non-singular}\} = 1$,

(ii) $P\{X_1 \in F\} = 0$ *for all proper flats* $F \subset R^p$.

In particular, condition (ii) is necessary and sufficient for the non-singularity w.p. 1 of the sample covariance matrix $\sum (X_i - \bar{X})(X_i - \bar{X})'$ in the i.i.d. case.

Theorem 3.3 has an interesting geometric interpretation. Let $\mathscr{S}(X) \subseteqq R^n$ be the random subspace spanned by the row vectors $X_{(1)}, \cdots, X_{(p)}$ of X. Theorem 3.3 states that

$$P\{\dim [\mathscr{S}(X) \cap \mathscr{Y}^\perp] \geqq 1\} > 0$$

iff there exists a fixed vector $v \in \mathscr{Y}^\perp$ and a fixed nonzero linear combination $\sum_1^p w_i X_{(i)}$ of $X_{(1)}, \cdots, X_{(p)}$ such that

$$P\{\textstyle\sum_1^p w_i X_{(i)} = v\} > 0 \, .$$

4. Proof of Theorem 2.1. Throughout this section V^p will denote a p-dimensional real vector space and $M^{(d)}$ (with or without subscripts) a manifold of dimension d in V^p.

LEMMA 4.1. *Let $Z \in V^p$ be a random vector with $P\{Z = 0\} = 0$. Then for each $d = 1, \cdots, p - 1$ there are at most countably many manifolds, say $\{M_i^{(d)} \mid i = 1, \cdots\}$, such that*

(a) $$P\{Z \in M_i^{(d)}\} > 0$$

and

(b) $$P\{Z \in M^{(d-1)}\} = 0 \qquad \text{for every} \quad M^{(d-1)} \subset M_i^{(d)} \, .$$

PROOF. If $M_1^{(d)}, \cdots, M_m^{(d)}$ satisfy (a) and (b), then

$$P\{\textstyle\bigcup_1^m M_\alpha^{(d)}\} = \sum_1^m P\{M_\alpha^{(d)}\} \, .$$

This shows that the collection of d-manifolds satisfying (a) and (b) is at most countable.

If $v_1, \cdots, v_\beta$ are vectors in V^p, $S(v_1, \cdots, v_\beta)$ will denote the span of $v_1, \cdots, v_\beta$. The following result yields Theorem 2.1.

THEOREM 4.2. *For every j, $1 \leqq j \leqq p$, for every $k \geqq j$, and for every set of independent random vectors $X_1, \cdots, X_k$ in V^p, the following are equivalent:*

(i) $P\{\dim S(X_1, \cdots, X_k) \leqq j - 1\} > 0$.

(ii) *For some s, $1 \leqq s \leqq j$, the following assertion, $A(s)$, holds: there exist $k - j + s$ vectors $\{X_{i_\alpha} \mid \alpha = 1, \cdots, k - j + s\}$ and there exists $M^{(s-1)}$ such that $P\{X_{i_\alpha} \in M^{(s-1)}, \alpha = 1, \cdots, k - j + s\} > 0$.*

PROOF. Clearly (ii) implies (i). We prove that (i) implies (ii) by induction on p. This is easily verified for $p = 1$. For $p \geqq 2$, assume the result is true for all p', $1 \leqq p' \leqq p - 1$.

To prove the result for dimension p, we use a secondary induction on j. The result is clear for $j = 1$. For $2 \leqq j \leqq p$, assume (i) implies (ii) for all j', $1 \leqq j' \leqq j - 1$. Now fix $k \geqq j$ and $X_1, \cdots, X_k$ in V^p. Assume that

$$P\{\dim S(X_1, \cdots, X_k) \leqq j - 1\} > 0 \tag{4.1}$$

but that $A(1), \cdots, A(j-1)$ do not hold; we will show that $A(j)$ obtains. Since $A(1)$ does not hold, there is some X_i, say X_1, such that $P\{X_1 = 0\} = 0$. Using the secondary induction hypothesis for $j' \equiv j - 1$ and $k' \equiv k - 1$, not $A(1), \cdots,$ not $A(j-1)$ implies

$$P\{\dim S(X_2, \cdots, X_k) \leqq j - 2\} = 0$$

so

$$P\{\dim S(X_2, \cdots, X_k) \geqq j - 1\} = 1 . \tag{4.2}$$

Combining (4.1) and (4.2) gives

$$P\{\dim S(X_2, \cdots, X_k) = j - 1, \dim S(X_1, \cdots, X_k) \leqq j - 1\} > 0$$

so

$$P\{X_1 \in S(X_2, \cdots, X_k), \dim S(X_2, \cdots, X_k) = j - 1\} > 0 . \tag{4.3}$$

Now (4.3) can be written as

$$0 < \int_C P\{X_1 \in S(X_2, \cdots, X_k) \mid X_2, \cdots, X_k\} \, dQ \tag{4.4}$$

where Q is the probability distribution of $(X_2, \cdots, X_k)$ and $C = D \cap E$, where D and E are the following events:

$$D = \{\dim S(X_2, \cdots, X_k) = j - 1\}$$
$$E = \{P\{X_1 \in S(X_2, \cdots, X_k) \mid X_2, \cdots, X_k\} > 0\} .$$

By Lemma 4.1 with $Z = X_1$,

$$C = \bigcup_{d=1}^{j-1} \bigcup_{i=1}^{\infty} [\{M_i^{(d)} \subseteqq S(X_2, \cdots, X_k)\} \cap D] .$$

From (4.4), there exists a d and an i such that

$$P\{X_1 \in M_i^{(d)}\} P[\{M_i^{(d)} \subseteqq S(X_2, \cdots, X_k)\} \cap D] > 0 . \tag{4.5}$$

If $d = j - 1$, then $A(j)$ obtains with $M^{(j-1)} = M_i^{(d)}$. If $1 \leqq d \leqq j - 2$, then we have

$$P[\{M_i^{(d)} \subseteqq S(X_2, \cdots, X_k)\} \cap D] > 0 . \tag{4.6}$$

Let Π denote the orthogonal projection onto $(M_i^{(d)})^\perp$. Clearly,

$$\Pi(S(X_2, \cdots, X_k)) = S(\Pi X_2, \cdots, \Pi X_k) ,$$

so (4.6) implies that

$$P\{\dim S(\Pi X_2, \cdots, \Pi X_k) \leqq j - 1 - d\} > 0 . \tag{4.7}$$

Set $p' = p - d$, $j' = j - d$ and $k' = k - 1$. Since $\Pi X_\alpha \in (M_i^{(d)})^\perp$ and $\dim (M_i^{(d)})^\perp = p' \leqq p - 1$, we can apply our original induction hypothesis.

Hence for some s', $1 \leqq s' \leqq j'$, there exists $k' - j' + s'$ vectors $\{\Pi X_{i_\alpha} \mid \alpha = 1, \cdots, k' - j' + s'\}$ and a manifold $M^{(s'-1)} \subset (M_i^{(d)})^\perp$ such that

$$P\{\Pi X_{i_\alpha} \in M^{(s'-1)}, \alpha = 1, \cdots, k' - j' + s'\} > 0 .$$

Now, let $M_0 = M^{(s'-1)} \oplus M_i^{(d)} \subset V^p$ so

$$P\{X_{i_\alpha} \in M_0, \alpha = 1, \cdots, k' - j' + s'\} > 0 . \tag{4.8}$$

Combining (4.8) and (4.5), we have

$$P\{X_1 \in M_0, X_{i_\alpha} \in M_0, \alpha = 1, \cdots, k' - j' + s'\} > 0 . \tag{4.9}$$

Since we have assumed that $A(1), \cdots, A(j-1)$ do not hold, it follows that $1 + k' - j' + s' = k$ and $\dim(M_0) = j - 1$. Thus $A(j)$ obtains, completing the proof.

Theorem (2.1) follows by setting $j = p$ and $k = n$ in Theorem 4.2.

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