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METHOD OF DIFFERENCES IN THE CONSTRUCTION OF $L_2(s)$ DESIGNS

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For the notation and terminology used in this paper we refer the reader to Raghavarao (1971). The powerful method of constructing designs by the method differences is not so far used for $L_2(s)$ designs. The purpose of this note is to develop this method.

1. Main results. Let M be a module of s elements $0, 1, 2, \cdots, s-1$. To each element u of M we associate s symbols designated by $u_1, u_2, \cdots, u_s$. Let the $L_2(s)$ association scheme be the square array

$$\begin{array}{cccc} 0_1 & 1_1 & \cdots & (s-1)_1 \\ 0_2 & 1_2 & \cdots & (s-1)_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0_s & 1_s & \cdots & (s-1)_s \, . \end{array}$$

Analogous to the module theorems of Bose (1939), the following can be obtained.

THEOREM. *Let it be possible to obtain t sets each containing k symbols satisfying*:

(i) *There are r symbols of each class (clearly $rs = kt$).*

(ii) *Of the $tk(k-1)$ differences, every nonzero pure difference and every zero mixed difference occurs exactly λ_1 times and every nonzero mixed difference occurs exactly λ_2 times (clearly $tk(k-1) = 2\lambda_1 s(s-1) + \lambda_2 s(s-1)^2$).*

Then by developing the t initial sets to modulus s, $L_2(s)$ design with parameters

$$v = s^2 \, , \qquad b = st \, , \qquad r, k, \lambda_1, \lambda_2$$

can be constructed.

As an example, let us construct the LS_1 design of the Tables (1954) with parameters

$$v = 9 = b \, , \qquad r = 4 = k \, , \qquad \lambda_1 = 1 \, , \qquad \lambda_2 = 2 \, . \tag{2.1}$$

The three initial sets $(0_1, 2_1, 1_2, 1_3)$; $(0_2, 2_2, 1_3, 1_1)$; $(0_3, 2_3, 1_1, 1_2)$ satisfy the requirements of the theorem, and the completed plan, by developing these initial sets, will be

$$\begin{array}{lll} (0_1, 2_1, 1_2, 1_3)\, ; & (0_2, 2_2, 1_3, 1_1)\, ; & (0_3, 2_3, 1_1, 1_2)\, ; \\ (1_1, 0_1, 2_2, 2_3)\, ; & (1_2, 0_2, 2_3, 2_1)\, ; & (1_3, 0_3, 2_1, 2_2)\, ; \\ (2_1, 1_1, 0_2, 0_3)\, ; & (2_2, 1_2, 0_3, 0_1)\, ; & (2_3, 1_3, 0_1, 0_2)\, . \end{array} \tag{2.2}$$

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The initial sets

$$
(2.3)\qquad
\begin{array}{lll}
(0_1, 1_1, 3_1, & 0_2, 1_2, 3_2, & 0_4, 1_4, 3_4) \\
(0_2, 1_2, 3_2, & 0_3, 1_3, 3_3, & 0_5, 1_5, 3_5) \\
(0_3, 1_3, 3_3, & 0_4, 1_4, 3_4, & 0_6, 1_6, 3_6) \\
(0_4, 1_4, 3_4, & 0_5, 1_5, 3_5, & 0_7, 1_7, 3_7) \\
(0_5, 1_5, 3_5, & 0_6, 1_6, 3_6, & 0_1, 1_1, 3_1) \\
(0_6, 1_6, 3_6, & 0_7, 1_7, 3_7, & 0_2, 1_2, 3_2) \\
(0_7, 1_7, 3_7, & 0_1, 1_1, 3_1, & 0_3, 1_3, 3_3)
\end{array}
$$

when developed (mod 7) give the solution of $L_2(s)$ design with parameters

$$(2.4)\qquad v = 49 = b\,, \qquad r = 9 = k\,, \qquad \lambda_1 = 3\,, \qquad \lambda_2 = 1\,.$$

It is to be added here that the parametric combinations (2.4) were listed by Chang and Liu (1964), as those whose solution is unknown. However, the solutions for the parametric combination were obtained by Vartak (1955) and Archbold and Johnson (1956). Vartak's solution as the Kronecker product of two BIB designs with parameters

$$(2.5)\qquad v^* = b^* = 7\,, \qquad r^* = k^* = 3\,, \qquad \lambda^+ = 1$$

is clearly isomorphic to our solution by the method of differences. It is unknown whether Archbold and Johnson's solution is isomorphic or not to our solution.

2. Concluding remarks. This method can easily be extended to the construction of rectangular designs.

REFERENCES

[1] ARCHBOLD, J. W. and JOHNSON, N. L. (1956). A method of constructing partially balanced incomplete block designs. *Ann. Math. Statist.* **27** 624–632.

[2] CHANG, L-C. and LIU, W-R. (1964). Incomplete block designs with square parameters for $k \leqq 10$ and $r \leqq 10$. *Sci. Sinica* **13** 1493–1495.

[3] RAGHAVARAO, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments.* Wiley, New York.

[4] VARTAK, M. N. (1955). On an application of the Kronecker product of matrices to statistical designs. *Ann. Math, Statist.* **26** 420–438.

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