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A FAMILY OF ADMISSIBLE MINIMAX ESTIMATORS OF THE MEAN OF A MULTIVARIATE NORMAL DISTRIBUTION[1]

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Let the p-component vector X be normally distributed with mean ξ and covariance $\sigma^2 I$ where I denotes the identity matrix and σ is known. For estimating ξ with quadratic loss, it is known that X is minimax but inadmissible for $p \geqq 3$. We obtain a family of estimators which dominate X and are admissible. These estimators are, therefore, both minimax and admissible.

0. Summary. Let the p-component vector X be normally distributed with mean ξ and covariance $\sigma^2 I$ where I denotes the identity matrix and σ is known. For estimating ξ with quadratic loss, it is known that X is minimax but inadmissible for $p \geqq 3$. We obtain a family of estimators which dominate X and are admissible. These estimators are, therefore, both minimax and admissible.

1. Introduction. Let the p-component vector X be normally distributed with mean ξ and covariance $\sigma^2 I$, and let the loss be quadratic, given by

$$L(\hat{\xi}, \xi) = ||\hat{\xi} - \xi||^2/\sigma^2 \tag{1.1}$$

where $\hat{\xi}$ represents an estimate of ξ and $||x||$ denotes the length of a vector x. For estimating ξ, Stein [8] showed that X is inadmissible when $p \geqq 3$. An estimator which dominates X, was given by James and Stein [6] for the case when σ is unknown and an independent estimate of σ^2 is available, which is distributed as $\sigma^2\chi_n^2$ (chi-square with n degrees of freedom). The estimator was improved upon by Baranchik [3]. Alam and Thompson [2] have considered a family of estimators that dominate X. Baranchik [4] has shown that X is dominated by a general class of estimators, including the estimators given in [2], [3], and [6]. We extend this class for the case when σ is known. The estimators in a subclass of the extended class are shown to be admissible. As X is minimax, these estimators are both minimax and admissible. In an unpublished paper Baranchik has also obtained admissible minimax estimators.

On the problem of estimating the mean of a multivariate normal distribution, two other papers have appeared recently, which should be mentioned. Strawderman [9] gives a family of minimax and proper Bayes estimators of ξ for $p \geqq 6$. This family is different from the family of estimators given in this paper. Strawderman and Cohen [10] give a necessary and sufficient condition for an estimator of the form $\delta(x) = h(||x||^2)x$ to be generalized Bayes. The

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generalized Bayes estimators are shown to be admissible under additional assumptions.

Suppose that σ is known and that $p \geqq 3$. Let

$$\delta(X) = X\phi(||X||^2/\sigma^2) \tag{1.2}$$

represent an estimator of ξ where $\phi(y)$ is a real-valued function of y. Let

$$f_t(y) = y^{t+1}(1 - \phi(y)), \qquad y \geqq 0 \tag{1.3}$$

and

$$\phi_0(y) = \frac{2\nu}{p} M\left(\nu + 1, \frac{p}{2} + 1, \frac{y}{2}\right) \Big/ M\left(\nu, \frac{p}{2}, \frac{y}{2}\right), \qquad \nu_p \leqq \nu < 1 \tag{1.4}$$

where

$$\nu_p = \tfrac{1}{4}(2p + 5 - (4p^2 + 8p - 7)^{\frac{1}{2}}) \tag{1.5}$$

and

$$M(a, b, y) = 1 + \frac{(a)_1 y}{(b)_1} + \frac{(a)_2 y^2}{(b)_2 2!} + \cdots + \frac{(a)_n y^n}{(b)_n n!} + \cdots \tag{1.6}$$

denotes the confluent hypergeometric function, $(a)_0 = 1$ and

$$(a)_n = a(a + 1) \cdots (a + n - 1).$$

Let $\eta(X) = \delta(X)$ for $\phi = \phi_0$. The main results of this paper are contained in Theorem 1.1 and Theorem 1.2, given below.

THEOREM 1.1. *$\delta(X)$ dominates X if* (i) *$f_t(y)$ is a monotone non-decreasing function of y and* (ii) *$0 \leqq y^{-t}f_t(y) < 2p - 4t - 4$ for some value of $t \geqq 0$.*

The class of estimators $\delta(X)$ for which (i) and (ii) hold for the particular value $t = 0$, is the class of estimators given by Baranchik [4] for the case when σ is known.

THEOREM 1.2. *$\eta(X)$ dominates X and is admissible.*

2. Main results. First we prove Theorem 1.1. Let $\theta = ||\xi||^2/2\sigma^2$, and let $R_\delta(\theta)$ denote the risk of $\delta(X)$. Directly, as also from the computation leading to (1.10) in [4], we have

$$\begin{aligned} R_\delta(\theta) &= \sigma^{-2}E(||X\phi(||X||^2/\sigma^2) - \xi||^2) \\ &= \sum_{K=0}^{\infty} \frac{\theta^K e^{-\theta}}{K!} E\{Y\phi^2(Y) - 4K\phi(Y)\} + 2\theta \\ &= \sum_{K=0}^{\infty} \frac{\theta^K e^{-\theta}}{K!} E\{((Y\phi(Y) - 2k)^2 - 4k^2)Y^{-1}\} \end{aligned} \tag{2.1}$$

where E denotes expectation, and Y is distributed as χ^2_{p+2K}. It is clear from (2.1) that we may assume that $\phi(y) \geqq 0$, for $R_\delta(\theta)$ is not increased by substituting $|\phi(y)|$ for $\phi(y)$.

Suppose that $0 \leqq y^{-t}f_t(y) < C$ for some positive number C and a fixed value of t $(0 \leqq t < p/2 - 1)$. Substituting $1 - y^{-t-1}f_t(y)$ for $\phi(y)$ in the second line on the right-hand side of (2.1) we have after simplification

$$(2.2) \qquad \begin{aligned} R_\delta(\theta) &= p + \textstyle\sum_{K=0}^{\infty} \frac{\theta^K e^{-\theta}}{K!} E\{f_t(Y)(Y^{-2t-1}f_t(Y) + 4KY^{-t-1} - 2Y^{-t})\} \\ &\leqq p + \textstyle\sum_{K=0}^{\infty} \frac{\theta^K e^{-\theta}}{K!} E[f_t(Y)\{CY^{-t-1} + 4KY^{-t-1} - 2Y^{-t}\}] . \end{aligned}$$

Let $h(Y)$ denote the quantity inside the braces in the second line on the right-hand side of (2.2). Then $h(y) \geqq (\leqq) 0$ for $y \leqq (\geqq) y_0 = 2K + C/2$. As $f_t(y)$ is non-decreasing in y by the condition (i), we have

$$(2.3) \qquad \begin{aligned} R_\delta(\theta) &\leqq p + \textstyle\sum_{K=0}^{\infty} \frac{\theta^K e^{-\theta}}{K!} f_t(y_0) E(CY^{-t-1} + 4KY^{-t-1} - 2Y^{-t}) \\ &= p + \textstyle\sum_{K=0}^{\infty} \frac{\theta^K e^{-\theta}}{K!} f_t(y_0) \Big\{2^{-t-1}C\Gamma\left(\frac{p}{2} + K - t - 1\right) \\ &\qquad + 2^{-t+1}K\Gamma\left(\frac{p}{2} + K - t - 1\right) \\ &\qquad - 2^{-t+1}\Gamma\left(\frac{p}{2} + K - t\right)\Big\}\Big/\Gamma\left(\frac{p}{2} + K\right) \\ &= p + \textstyle\sum_{K=0}^{\infty} \frac{\theta^K e^{-\theta}}{K!} f_t(y_0)(C - 2p + 4t + 4) \\ &\qquad \times \Gamma\left(\frac{p}{2} + K - t - 1\right)\Big/2^{t+1}\Gamma\left(\frac{p}{2} + K\right) \\ &= p \end{aligned}$$

for $C = 2p - 4t - 4$. Clearly, strict inequality holds in (2.2) and therefore in (2.3), unless $f_t(y) = 0$ and thus $\delta(x) = x$, almost everywhere. Theorem 1.1 follows from (2.3).

Next we prove Theorem 1.2. First we obtain a Bayes solution of the functional ϕ. Consider a prior distribution g_λ on θ with density

$$(2.4) \qquad g_\lambda(\theta) = \frac{\lambda^\nu}{\Gamma(\nu)} \theta^{\nu-1}e^{-\lambda\theta} , \qquad \lambda > 0, \nu_p \leqq \nu < 1$$

where ν_p is given by (1.5). From (2.1) we have that the average risk of $\delta(X)$ with respect to g_λ is given by

$$(2.5) \qquad \begin{aligned} r_\lambda &= \textstyle\int_0^\infty R_\delta(\theta)g_\lambda(\theta)\, d\theta \\ &= \textstyle\sum_{K=0}^{\infty} \frac{\Gamma(K+\nu)\lambda^\nu}{(1+\lambda)^{K+\nu}\Gamma(\nu)K!} \{((Y\phi(Y) - 2K)^2 - 4K^2)Y^{-1}\} \\ &= \textstyle\int_0^\infty \sum_{K=0}^{\infty} \frac{\Gamma(K+\nu)\lambda^\nu}{(1+\lambda)^{K+\nu}\Gamma(\nu)K!} ((y\phi(y) - 2K)^2 - 4K^2)y^{-1} \\ &\qquad \times \frac{2^{-\frac{1}{2}p-K}y^{\frac{1}{2}p+K-1}e^{-\frac{1}{2}y}\, dy}{\Gamma(\frac{1}{2}p + K)} + \frac{2\nu}{\lambda} . \end{aligned}$$

From (2.5) we see that the functional ϕ minimizing r_λ is given by

$$\frac{y\phi(y)}{2} = \sum_{K=1}^{\infty} \frac{K\Gamma(K+\nu)y^K}{2^K(1+\lambda)^K\Gamma(\frac{1}{2}p+K)K!} \Big/ \sum_{K=0}^{\infty} \frac{\Gamma(K+\nu)y^K}{2^K(1+\lambda)^K\Gamma(\frac{1}{2}p+K)K!} \tag{2.6}$$
$$= \frac{\nu y}{p(1+\lambda)} M\left(\nu+1, \frac{p}{2}+1, \frac{y}{2(1+\lambda)}\right) \Big/ M\left(\nu, \frac{p}{2}, \frac{y}{2(1+\lambda)}\right).$$

Let

$$\phi_\lambda(y) = \frac{2\nu}{p(1+\lambda)} M\left(\nu+1, \frac{p}{2}+1, \frac{y}{2(1+\lambda)}\right) \Big/ M\left(\nu, \frac{p}{2}, \frac{y}{2(1+\lambda)}\right). \tag{2.7}$$

Setting $\lambda = 0$ in (2.7) we return to (1.4).

We show that $\phi = \phi_0$ satisfies the conditions (i) and (ii) of Theorem 1.1. Let $y \geqq 0$. From the formulas (13.4.3), (13.4.4), (13.4.8) and (13.5.1) given by Abramowitz and Stegun [1] we have the following relations which will be used in the sequel. (2.11) below, is obtained from (13.5.1) of [1], letting z be real and positive. For large $z > 0$ the first part on the right-hand side of (13.5.1) can be disregarded, as it is equal to $O(z^{-a})$ while the second part is equal to $O(e^z z^{a-b})$. We have

$$yM(a, b+1, y) = bM(a, b, y) - bM(a-1, b, y), \tag{2.8}$$

$$aM(a+1, b, y) = (1+a-b)M(a, b, y) + (b-1)M(a, b-1, y), \tag{2.9}$$

$$M'(a, b, y) = \partial M(a, b, y)/\partial y = \frac{a}{b} M(a+1, b+1, y), \tag{2.10}$$

and for large y

$$M(a, b, y) = \frac{\Gamma(b)}{\Gamma(a)} e^y y^{a-b} \left\{1 + (1-a)(b-a)y^{-1} + (1-a)_2(b-a)_2 \frac{y^{-2}}{2} + O(y^{-3})\right\}. \tag{2.11}$$

Applying (2.8), (2.9) and (2.11), we get from (1.4)

$$\phi_0(y) = 1 - \frac{p-2\nu}{p} M\left(\nu, \frac{p}{2}+1, \frac{y}{2}\right) \Big/ M\left(\nu, \frac{p}{2}, \frac{y}{2}\right)$$
$$= 1 - \frac{p-2\nu}{y}\left\{1 - M\left(\nu-1, \frac{p}{2}, \frac{y}{2}\right) \Big/ M\left(\nu, \frac{p}{2}, \frac{y}{2}\right)\right\} \tag{2.12}$$
$$= 1 - \left(\frac{p-2\nu}{y}\right)\left\{1 + \frac{2(1-\nu)}{y} + O(y^{-2})\right\}.$$

Let $U_\nu(y) = M(\nu, \frac{1}{2}p, \frac{1}{2}y)$, $V_\nu(y) = M(\nu, \frac{1}{2}p+1, \frac{1}{2}y)$,

$$g(y) = U_{\nu-1}(y)/U_\nu(y) = -2(1-\nu)/y + O(y^{-2}) \tag{2.13}$$

and

$$(2.14)\quad \begin{aligned} h(y) &= U'_{\nu-1}(y)/U_\nu'(y) \\ &= -(1-\nu)V_\nu(y)/\nu V_{\nu+1}(y) \quad \text{by (2.11)} \\ &= -2(1-\nu)/y + O(y^{-2}) \end{aligned}$$

where prime denotes derivative with respect to y. By Lemma 2.1, given at the end of this section, $V_\nu(y)/V_{\nu+1}(y)$ is non-increasing in y. Then from (2.14) we have that $h(y)$ is non-decreasing in y. Therefore, as $U_\nu(y) > 0$ and $U_\nu'(y) > 0$, $g(y)$ has at most two extrema in the domain $y > 0$, for the extreme value of $g(y)$ is given by $g(y) = h(y)$. As $g(0) = 1$ and $g(\infty) = -0$, it has exactly one extremum (minimum) at y_0, say. That is, as y varies from 0 to ∞, $g(y)$ first decreases then increases to zero. The minimum value of $g(y)$ is given by

$$(2.15)\quad \begin{aligned} g(y_0) &= h(y_0) \\ &\geqq h(0) \\ &= -(1-\nu)/\nu \quad \text{by (2.14)}\,. \end{aligned}$$

From (2.12) and (2.15) we have that

$$(2.16)\quad \begin{aligned} y(1-\phi_0(y)) &= (p-2\nu)(1-g(y)) \\ &\leqq (p-2\nu)/\nu\,. \end{aligned}$$

Thus, $\phi = \phi_0$ satisfies the condition (ii) of Theorem 1.1 for $0 \leqq t < \frac{1}{2}(p-1)-p/4\nu$.

Now we show that $g_t(y) = y^{1+t}(1-\phi_0(y)) = (p-2\nu)y^{1+t}V_\nu(y)/pU_\nu(y)$ is non-decreasing in y for $y \leqq 0$ and $t \geqq (1-\nu)(p-2\nu+4)/\nu$. Let $Z(y) = (\frac{1}{2}y)^{1+t}V_\nu(y)$. Applying (2.8) and (2.10) we have

$$(2.17)\quad \begin{aligned} Z'(y) &= \tfrac{1}{2}(1+t)\left(\frac{y}{2}\right)^t V_\nu(y) + \left(\frac{y}{2}\right)^{1+t} \frac{\nu}{p+2} M\left(\nu+1, \frac{p}{2}+2, \frac{y}{2}\right) \\ &= \frac{1}{2}\left(\frac{y}{2}\right)^t \{(1+t-\nu)V_\nu(y) + \nu V_{\nu+1}(y)\}\,, \end{aligned}$$

$$(2.18)\quad \frac{Z'(y)}{U'_\nu(y)} = \frac{p}{2}\left(\frac{y}{2}\right)^t \{1 + (1+t-\nu)V_\nu(y)/\nu V_{\nu+1}(y)\}\,,$$

and

$$(2.19)\quad \frac{Z(y)}{U_\nu(y)} = \frac{p}{2}\left(\frac{y}{2}\right)^t \{1 - U_{\nu-1}(y)/U_\nu(y)\}\,.$$

From (2.12) and (2.19) we have that $g_t(y) = 2^{1+t}(p-2\nu)Z(y)/pU_\nu(y)$. From (2.18) and (2.19) we have that $g_t(y)$ is non-decreasing in y when

$$(2.20)\quad (1+t-\nu)V_\nu(y)/\nu V_{\nu+1}(y) + U_{\nu-1}(y)/U_\nu(y) \geqq 0$$

or

$$(2.21)\quad \alpha V_\nu(y)U_\nu(y) + U_{\nu-1}(y)V_{\nu+1}(y) \geqq 0$$

where $\alpha = (1+t-\nu)/\nu$.

Using the series expansion for the confluent hypergeometric function, given by (1.6), and writing c for $p/2$ and x for $y/2$, we have

$$(2.22)\quad \begin{aligned} U_\nu(y)V_\nu(y) &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{\gamma=0}^{n}\binom{n}{\gamma}\frac{(\nu)_\gamma(\nu)_{n-\gamma}}{(c)_\gamma(c+1)_{n-\gamma}} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{\gamma=0}^{n}\binom{n}{\gamma}\frac{(\nu)_\gamma(\nu)_{n-\gamma}\,c}{(c)_\gamma(c)_{n-\gamma}(c+n-\gamma)} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{\gamma=0}^{[\frac{1}{2}n]}\binom{n}{\gamma}\frac{(\nu)_\gamma(\nu)_{n-\gamma}\,c(2c+n)q_\gamma}{(c)_\gamma(c)_{n-\gamma}(c+\gamma)(c+n-\gamma)} \end{aligned}$$

where $[x]$ denotes the largest integer less than or equal to x, $q_\gamma = 1$ for $\gamma < \frac{1}{2}n$ and $q_\gamma = \frac{1}{2}$ for $\gamma = \frac{1}{2}n$. Similarly,

$$(2.23)\quad \begin{aligned} &U_{\nu-1}(y)V_{\nu+1}(y) \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{\gamma=0}^{[\frac{1}{2}n]}\binom{n}{\gamma}\frac{(\nu)_\gamma(\nu)_{n-\gamma}\,c(\nu-1)q_\gamma}{(c)_\gamma(c)_{n-\gamma}\,\nu} \\ &\quad\times\left\{\frac{\nu+n-\gamma}{(\nu+\gamma-1)(c+n-\gamma)}+\frac{\nu+\gamma}{(\nu+n-\gamma-1)(c+\gamma)}\right\}. \end{aligned}$$

From (2.22) and (2.23) we have that the left-hand side of (2.21) is nonnegative when for each $n = 0, 1, \cdots$

$$(2.24)\quad \begin{aligned} &\frac{\alpha(2c+n)}{(c+\gamma)(c+n-\gamma)}+\frac{(\nu-1)}{\nu} \\ &\quad\times\left\{\frac{\nu+n-\gamma}{(\nu+\gamma-1)(c+n-\gamma)}+\frac{\nu+\gamma}{(\nu+n-\gamma-1)(c+\gamma)}\right\}\geqq 0 \end{aligned}$$

for $\gamma = 0, 1, \cdots, [\frac{1}{2}n]$.

Let L denote the quantity on the left-hand side of (2.24). We have

$$(2.25)\quad \begin{aligned} L &= \frac{t(2c+n)}{\nu(c+\gamma)(c+n-\gamma)}+\frac{\nu-1}{\nu}\left\{\frac{\nu+n-\gamma}{(\nu+\gamma-1)(c+n-\gamma)}\right. \\ &\quad\left.+\frac{\nu+\gamma}{(\nu+n-\gamma-1)(c+\gamma)}-\frac{2c+n}{(c+\gamma)(c+n-\gamma)}\right\}. \end{aligned}$$

The quantity inside the braces on the right-hand side can be written as

$$\begin{aligned} &\left(\frac{\nu+n-\gamma}{(\nu+\gamma-1)(c+n-\gamma)}-\frac{1}{c+n-\gamma}\right) \\ &\quad+\left(\frac{\nu+\gamma}{(\nu+n-\gamma-1)(c+\gamma)}-\frac{1}{c+\gamma}\right) \\ &\qquad=\frac{n-2\gamma+1}{(\nu+\gamma-1)(c+n-\gamma)}+\frac{2\gamma-n+1}{(\nu+n-\gamma-1)(c+\gamma)} \\ &\qquad=(n-2\gamma)\left(\frac{1}{(\nu+\gamma-1)(c+n-\gamma)}-\frac{1}{(\nu+n-\gamma-1)(c+\gamma)}\right) \\ &\qquad\quad+\frac{1}{(\nu+\gamma-1)(c+n-\gamma)}+\frac{1}{(\nu+n-\gamma-1)(c+\gamma)} \end{aligned}$$

$$= \frac{(n-2\gamma)^2(c-\nu+1)}{(\nu+\gamma-1)(c+n-\gamma)(\nu+n-\gamma-1)(c+\gamma)} + \frac{1}{(\nu+\gamma-1)(c+n-\gamma)} + \frac{1}{(\nu+n-\gamma-1)(c+\gamma)}$$

$$= \frac{1}{(c+\gamma)(c+n-\gamma)} \left\{ \frac{(n-2\gamma)^2(c-\nu+1)}{(\nu+\gamma-1)(\nu+n-\gamma-1)} + \left(\frac{c+\gamma}{\nu+\gamma-1} + \frac{c+n-\gamma}{\nu+n-\gamma-1} \right) \right\}$$

$$\leqq \frac{1}{(c+\gamma)(c+n-\gamma)} \left\{ \frac{(2c+n)(c-\nu+1)}{\nu} + \frac{2c+n}{\nu} \right\}$$

$$= (2c+n)(c-\nu+2)/\nu(c+\gamma)(c+n-\gamma) .$$

Thus $L \geqq 0$, and therefore (2.21) holds for

$$t \geqq (1-\nu)(c-\nu+2)/\nu . \tag{2.26}$$

From (2.26) and the conclusion following (2.16), we have that the conditions (i) and (ii) of Theorem 1.1 are satisfied for $\phi = \phi_0$ when

$$\frac{1-\nu}{2\nu}(p-2\nu+4) \leqq t < \frac{p-1}{2} - \frac{p}{4\nu} . \tag{2.27}$$

A nonnegative value of t satisfying (2.27) exists when $\nu_p \leqq \nu < 1$. Therefore, $\delta(X)$ dominates X, by Theorem 1.1. The admissibility of $\delta(X)$ is shown in the following section, thus completing the proof of Theorem 1.2. The lemma cited above, is given below.

LEMMA 2.1. *Let $h(y) = (\sum_{i=0}^{\infty} d_i y^i)/(\sum_{i=0}^{\infty} a_i y^i)$ where a_i, b_i are nonnegtive, and $\sum a_i y^i$ and $\sum b_i y^i$ converge for all $y > 0$. If the sequence $\{b_i/a_i\}$ is monotone non-decreasing (non-increasing) then $h(y)$ is monotone non-decreasing (non-increasing) in y.*

The lemma can be shown by differentiating $h(y)$ (see Lehmann [6], Problem 4 (i) page 312).

3. Admissibility of $\eta(X)$. Stein [7] has shown that for the loss given by (1.1), an estimator which is admissible in the class of estimators $\delta(X)$ given by (1.2), is admissible in the class of all estimators. Therefore, to prove the admissibility of $\eta(X)$ we need only to show that $\eta(X)$ is admissible in the class of estimators $\delta(X)$.

Let $\eta_\lambda(X)$ denote the estimator $\delta(X)$ for $\phi = \phi_\lambda$, given by (2.7). Then $\eta_0(X) = \eta(X)$. To show that $\eta(X)$ is admissible in the class of estimators $\delta(X)$, and hence admissible in the class of all estimators, it is sufficient to show that

$$\lim_{\lambda\to\infty} P(\lambda) = 0 \tag{3.1}$$

where $P(\lambda) = \lambda^{-\nu} \int_0^\infty (R_\eta(\theta) - R_{\eta_\lambda}(\theta))g_\lambda(\theta)\, d\theta$. For, suppose that an estimator $\delta(X)$ dominates $\eta(x)$. We have

$$\int_0^\infty \lambda^{-\nu}(R_\eta(\theta) - R_\delta(\theta))g_\lambda(\theta)\, d\theta = P(\lambda) + \int_0^\infty \lambda^{-\nu}(R_{\eta_\lambda}(\theta) - R_\delta(\theta))g_\lambda(\theta)\, d\theta . \tag{3.2}$$

Let $\lambda \to 0$. The quantity on the left-hand side of (3.2) is positive as $R_\eta(\theta) - R_\delta(\theta) \geqq 0$ with inequality for at least one value of θ and hence for a neighborhood of θ, by continuity. On the other hand, the integral on the right-hand side of (3.2) is non-positive for $\eta_\lambda(X)$ minimizes the average risk given by (2.5). Then (3.1) contradicts (3.2).

From (2.1) we have

$$(3.3) \qquad R_\eta(\theta) - R_{\eta_\lambda}(\theta) = \sum_{K=0}^\infty \frac{\theta^K e^{-\theta}}{K!} E\{Y(\phi_0^2(Y) - \phi_\lambda^2(Y)) - 4K(\phi_0(Y) - \phi_\lambda(Y))\}.$$

Multiplying both sides of (3.3) by $\lambda^{-\nu} g_\lambda(\theta)$ and integrating with respect to θ we get

$$(3.4) \qquad \begin{aligned} P(\lambda) &= \sum_{K=0}^\infty \frac{\Gamma(K+\nu)}{(1+\lambda)^{K+\nu}\Gamma(\nu)K!} E\{Y(\phi_0^2(Y) - \phi_\lambda^2(Y)) \\ &\qquad - 4K(\phi_0(Y) - \phi_\lambda(Y))\} \\ &= \int_0^\infty \frac{y^{\frac{1}{2}p-1}e^{-\frac{1}{2}y}}{(1+\lambda)^\nu 2^{\frac{1}{2}p}\Gamma(\frac{1}{2}p)} \Big\{y(\phi_0^2(y) - \phi_\lambda^2(\lambda))M\Big(\nu, \frac{p}{2}, \frac{y}{2(1+\lambda)}\Big) \\ &\qquad - \frac{4\nu y}{p(1+\lambda)}(\phi_0(y) - \phi_\lambda(y))M\Big(\nu+1, \frac{p}{2}+1, \frac{y}{2(1+\lambda)}\Big)\Big\} dy. \end{aligned}$$

From (2.7) applying (2.11), we have

$$(3.5) \qquad \phi_\lambda(y) = \frac{1}{1+\lambda} - \frac{p-2\nu}{y} + (1+\lambda)O(y^{-2})$$

corresponding to the asymptotic expression for $\phi_0(y)$, given by (2.12). Let $Q(y)$ denote the integrand on the right-hand side of (3.4). It is seen that $Q(y) \to 0$ as $\lambda \to 0$. From the asymptotic expressions for $\phi_0(y)$ and $\phi_\lambda(y)$ and the confluent hypergeometric function, we have for large y

$$(3.6) \qquad Q(y) = \frac{y^\nu e^{-\lambda y/2(1+\lambda)}\lambda}{2^\nu(1+\lambda)^{2\nu-\frac{1}{2}p}\Gamma(\nu)} \Big\{\frac{\lambda}{(1+\lambda)^2} - \frac{2p-4\nu}{y(1+\lambda)} + O(y^{-2})\Big\}.$$

From (3.6) we obtain that (3.1) holds for $0 < \nu < 1$. Therefore, $\eta(X)$ is admissible.

The admissibility of $\eta(X)$ through the use of (3.1) can be proved also from a result of Brown ([5], Theorem 5.6.1). Another proof of the admissibility can be obtained from Corollary 6.3.2 of Brown [5] which shows that $\eta(X)$ is admissible if $\eta(X)$ is a generalized Bayes estimator, and for some positive number L

$$(3.7) \qquad x'(\eta(x) - x) \leqq (2-p)\sigma^2 \qquad \text{for} \quad ||x|| > L$$

where $x'z$ denote the inner product of the vectors x and z. That (3.7) holds is verified easily from the asymptotic expression for $\phi_0(y)$, given by (2.12), and noting that $0 < \nu < 1$.

For the computation of $\eta(X)$ in application, tables of the confluent hypergeometric function should be used (see Abramowitz and Stegun [1] for reference to the tables).

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