The Annals of Statistics
1973, Vol. 1, No. 2, 388-391

MONOTONICITY OF THE POWER FUNCTIONS OF SOME TESTS IN GENERAL MANOVA MODELS[1]

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In this paper we give further reduction for a general MANOVA problem, and show that the general MANOVA problem can be treated as the MANOVA problem with the conditional setup. The last result is used to prove that the power functions of some tests in the general MANOVA models are monotone increasing functions of each of the noncentrality parameters, and, in particular, the likelihood-ratio test has the monotonicity property.

1. Introduction. The general MANOVA model which has been introduced by Potthoff and Roy [5] is the following: Let $Y: N \times p$ be a random matrix with independent rows distributed as $N_p(\cdot, \Sigma)$ and $EY = A_1 \Xi A_2$, where $A_1: N \times k$ and $A_2: q \times p$ are known matrices of ranks k and q, respectively, and Ξ is unknown. The hypothesis to be tested is $H_0: A_3 \Xi A_4 = 0$ against $H_1: A_3 \Xi A_4 \neq 0$, where $A_3: u \times k$, $A_4: q \times v$ are known matrices of ranks u and v, respectively. It is assumed that $n = N - k - p + q \geqq v$. The canonical reduction for this problem has been obtained by Gleser and Olkin [3] as follows: $Z: N \times p$ is a random matrix with independent rows distributed as $N_p(\cdot, \Sigma)$ and

$$EZ = E\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix}\begin{matrix} u \\ k-u \\ N-k \end{matrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} & 0 \\ \Theta_{21} & \Theta_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.1}$$
$$\qquad q-v \quad\; v \quad\; p-q$$

The hypothesis is $H_0: \Theta_{12} = 0$; $H_1: \Theta_{12} \neq 0$. The likelihood-ratio test is an increasing function of $\Lambda = |S_e|/|S_h + S_e|$ (refer to Gleser and Olkin [3]), where the matrices S_e and S_h are defined by $X'FX$ and $X'HX$, respectively, with $X' = (Z_{12}', Z_{32}')$,

$$F = \begin{bmatrix} 0 & 0 \\ 0 & I - Z_{33}(Z_{33}'Z_{33})^{-1}Z_{33}' \end{bmatrix}, \tag{1.2}$$
$$H = (I, -Z_{13}(Z_{33}'Z_{33})^{-1}Z_{33}')'(I + Z_{13}(Z_{33}'Z_{33})^{-1}Z_{13}')^{-1} \times (I, -Z_{13}(Z_{33}'Z_{33})^{-1}Z_{33}') .$$

In this paper we consider the test procedures based on the characteristic roots of $S_h S_e^{-1}$. The power function of any such test depends only on the parameters

Received March 4, 1972; revised June 8, 1972.

[1] This research was supported in part by the Sakko-kai Foundation.

AMS 1970 *subject classifications*. Primary 62H15; Secondary 62H10.

Key words and phrases. Power function, general MANOVA problem, test procedures, multivariate normal distribution, Wishart distribution.

$\lambda_1, \lambda_2, \cdots, \lambda_t$, where $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_t$ are the possibly nonzero roots of $\Sigma_{22\cdot 3}^{-1}\Theta_{12}'\Theta_{12}$ and $t = \min(u, v)$, where

$$(1.3) \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}, \qquad \Sigma_{22\cdot 3} = \Sigma_{22} - \Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32},$$

$\Sigma_{11}: (q - v) \times (q - v)$ and $\Sigma_{22}: v \times v$. We shall derive that the general MANOVA problem can be treated as the usual MANOVA problem with the conditional setup. This result is used to show that the power functions of some tests are monotone increasing functions of each λ_i, and, in particular, Λ has the monotonicity property.

2. Further reduction and distribution. Let us consider the transformation,

$$(2.1) \qquad \tilde{Z} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} = \begin{bmatrix} L' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_{12} & Z_{13} \\ Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} \Sigma_{22\cdot 3}^{-\frac{1}{2}} & 0 \\ 0 & \Sigma_{33}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} M' & 0 \\ 0 & I \end{bmatrix},$$

where L, M are defined as follows: There exist orthogonal L, M such that $\Theta_{12}\Sigma_{22\cdot 3}^{-\frac{1}{2}} = LD_\lambda^{\frac{1}{2}}M$, where

$$D_\lambda = \left[\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & 0 \\ & & \lambda_t & \\ \hline & 0 & & 0 \end{array}\right].$$

Then the rows of $\tilde{Z}$ are independent normal with the same covariance matrix,

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix},$$

where $\Gamma_{22} = I$, $\Gamma_{11\cdot 2} = \Gamma_{11} - \Gamma_{12}\Gamma_{22}^{-1}\Gamma_{21} = I$. Furthermore

$$(2.2) \qquad E\begin{bmatrix} \tilde{Z}_{12} \\ \tilde{Z}_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad E\left[\begin{matrix} \tilde{Z}_{11} \\ \tilde{Z}_{21} \end{matrix} \middle| \tilde{Z}_{12}, \tilde{Z}_{22}\right] = \begin{bmatrix} D_\lambda^{\frac{1}{2}} + \tilde{Z}_{12}\beta \\ \tilde{Z}_{22}\beta \end{bmatrix},$$

where $\beta = \Gamma_{22}^{-1}\Gamma_{21}$. Under such a transformation, the roots of $S_h S_e^{-1}$ remain unchanged, and, in particular, Λ remains invariant. Thus, one may use the notation in the canonical reduction, and replace Θ_{12} and

$$\begin{bmatrix} \Sigma_{22} & \Sigma_{23} \\ \Sigma_{32} & \Sigma_{33} \end{bmatrix}$$

by $D_\lambda^{\frac{1}{2}}$ and Γ, respectively, if one considers the tests based on the roots of $S_h S_e^{-1}$. In the following we shall do that. Then, given Z_{13} and Z_{33}, the rows of X are independently distributed as $N_v(\cdot, I)$. Noting that $F^2 = F$, $H^2 = H$ and $FH = 0$, and using (2.2), we have,

THEOREM 1. *Given Z_{13} and Z_{33},*

(i) *S_e has a central Wishart distribution $W_v(n, I)$,*

(ii) *S_h has a noncentral (possibly singular) Wishart distribution $W_v(u, I, \Delta)$ with the noncentrality parameter given by*

$$\Delta = D_\lambda^{\frac{1}{2}\prime}[I + Z_{13}(Z'_{33}Z_{33})^{-1}Z'_{13}]^{-1}D_\lambda^{\frac{1}{2}} ,$$

(iii) *S_e and S_h are independent.*

(iv) *The distribution of $Z_{13}(Z'_{33}Z_{33}^{-1})^{-1}Z'_{13}$ is free from any parameter.*

Conditionally, the distribution of roots of $S_h S_e^{-1}$ will involve the roots of Δ, given by δ_i's.

3. Monotonicity of power functions. Let ϕ be any test based on the roots $c_1 \geqq c_2 \geqq \cdots \geqq c_v$ of $S_h S_e^{-1}$. Then we may write the corresponding power and the conditional power, given Z_{13} and Z_{33}, as $\beta_\phi(D_\lambda)$ and $\beta_\phi(D_\delta \mid Z_{13}, Z_{33})$, respectively, where $D_\delta = \text{diag}(\delta_1, \cdots, \delta_t)$. By using a method similar to that in the proof of Theorem 1 of Anderson and Das Gupta [1], we have the following:

THEOREM 2. *If $\beta_\phi(D_\delta \mid Z_{13}, Z_{33})$ increases monotonically in each δ_i, then $\beta_\phi(D_\lambda)$ increases monotonically in each λ_i.*

Some sufficient conditions for $\beta_\phi(D_\delta \mid Z_{13}, Z_{33})$ to have the monotonicity in δ_i's have been obtained by Das Gupta, Anderson and Mudholkar [2]. Therefore, for example, the test procedures having the following acceptance regions have the power functions which are monotonically increasing in each λ_i:

(i) $$c_1 \leqq \mu ,$$

(ii) $$\textstyle\sum_{k=1}^{v} a_k W_k \leqq \nu , \qquad (a_k\text{'s} \geqq 0) ,$$

where μ and ν are constants, W_k $(k = 1, 2, \cdots, v)$ are the sums of all different products of $d_1, \cdots, d_v$ taken k at a time and $d_i = 1 + c_i$ $(i = 1, 2, \cdots, v)$. The likelihood-ratio test has the acceptance region of the form $W_v \leqq \nu$ which is a special case of (ii).

Note. Khatri [4] has also discussed the monotonicity of the power functions of the same tests treated in this paper by deriving a conditional model. However, his final results on monotonicity depend on random variables, and consequently are monotone in the conditional sense. There is no mention of the monotonicity in each λ_i from his formulation; further reductions are required.

Acknowledgment. The author wishes to express his gratitude to the referee for some helpful suggestions.

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