

ASYMPTOTIC DISTRIBUTIONS FOR QUADRATIC FORMS WITH APPLICATIONS TO TESTS OF FIT

BY T. DE WET AND J. H. VENTER

Potchefstroom University, South Africa

Let Z_1, Z_2, \dots , be independent and identically distributed random variables and $\{c_{ijn}\}$ real numbers; put $T_n = \sum_{i,j=1}^n c_{ijn} Z_i Z_j$. This paper gives conditions under which the distribution of $T_n - ET_n$ converges to the distribution of $\sum \gamma_m (Y_m^2 - 1)$ with $\{\gamma_m\}$ a real sequence and Y_1, Y_2, \dots independent $N(0, 1)$ random variables. The results are applied to the calculation of the asymptotic distributions of test criteria of the form $Q_n^W = \sum [F_0(X_{kn}) - k/n + 1]^2 W(k/n + 1)$ for testing the hypothesis that $X_{1n}, X_{2n}, \dots, X_{nn}$ are the order statistics of an independent sample from the distribution function F_0 ; here W is a weight function.

1. Introduction. Let Z_1, Z_2, \dots be independent and identically distributed random variables with $EZ_i = 0$, $\text{Var}(Z_i) = 1$ and $EZ_i^4 = \mu_4 < \infty$. Let $\{c_{ijn}, i, j = 1, \dots, n\}$ be real numbers with $c_{ijn} = c_{jin}$ for all i and j and put

$$(1) \quad T_n = \sum_{i,j=1}^n c_{ijn} Z_i Z_j.$$

In Section 2 of this paper we give some results on the characterization of the asymptotic distribution of T_n as $n \rightarrow \infty$ under suitable assumptions on the c_{ijn} 's.

Let $X_{1n}, X_{2n}, \dots, X_{nn}$ be the order statistics of an independent sample from the distribution F and suppose $H_0: F = F_0$ is to be tested. In Sections 3, 4 and 5 we show how the results of Section 2 can be used to characterize the asymptotic distributions under H_0 of statistics of the form

$$(2) \quad Q_n^W = \sum_{k=1}^n (F_0(X_{kn}) - k/n + 1)^2 W(k/n + 1)$$

where W is a weight function on $(0, 1)$. These statistics are closely related to the generalized Cramér-von Mises-Smirnov statistics (see e.g. [1] and [3]).

Studies on asymptotic distributions of quadratic forms have been reported by Varberg [11] but he considers only the case where the c_{ijn} 's do not depend on n . There are points of contact between the present work and that of Schach [9], [10] but the overlap is negligible.

2. Main results. Throughout this paper \sum_i, \sum_j and $\sum_{i,j}$ will denote summations in which i, j run through the indices 1 to n ; limits will be taken with respect to $n \rightarrow \infty$. Straightforward calculations show that

$$(3) \quad ET_n = \sum_i c_{iin} \text{Var}(T_n) = (\mu_4 - 3) \sum_i c_{iin}^2 + 2 \sum_{i,j} c_{ijn}^2.$$

THEOREM 1. Let $\{b_{imn}; i = 1, \dots, n, m = 1, 2, \dots\}$ and $\{\gamma_m, m = 1, 2, \dots\}$

Received March, 1972; revised July 31, 1972.

AMS 1970 subject classifications. Primary 62E20; Secondary 60F99.

Key words and phrases. Asymptotic distributions, quadratic forms, tests of fit, Cramér-von Mises-Smirnov statistics.

be real numbers (not all zero); let $\{M_n\}$ be a sequence of integers with $M_n \rightarrow \infty$, put

$$(4) \quad t_n = \sum_{m=1}^{M_n} |\gamma_m|$$

and assume that

$$(B1) \quad \max_{1 \leq i \leq n} |b_{imn}| \rightarrow 0 \quad \text{for each } m,$$

$$(B2) \quad B_n = \max_{1 \leq m, k \leq M_n} |\sum_i b_{imn} b_{ikn} - \delta_{mk}| = o(t_n^{-1})$$

$$(B3) \quad D_n = \max_{1 \leq m, k \leq M_n} \sum_i b_{imn}^2 b_{ikn}^2 = o(t_n^{-2})$$

$$(C) \quad \sum_{i,j} c_{ij}^2 \rightarrow \Gamma = \sum_1^\infty \gamma_m^2 < \infty$$

$$(BC) \quad C_n = \max_{1 \leq m \leq M_n} |\sum_{i,j} c_{ij} b_{imn} b_{jmn} - \gamma_m| = o(t_n^{-1}).$$

Then

$$(5) \quad D(T_n - ET_n) \rightarrow D(\sum_1^\infty \gamma_m (Y_m^2 - 1))$$

with Y_1, Y_2, \dots independent $N(0, 1)$ random variables.

PROOF. Put

$$(6) \quad c_{ijn}^* = \sum_{m=1}^{M_n} \gamma_m b_{imn} b_{jmn}, \quad T_n^* = \sum_{i,j} c_{ijn}^* Z_i Z_j.$$

We then show that $T_n^* - ET_n^*$ has the asymptotic distribution shown in (5) and that

$$(7) \quad \sum_{i,j} (c_{ijn} - c_{ijn}^*)^2 \rightarrow 0.$$

From this and (3) it would follow that $\text{Var}(T_n - T_n^*) \rightarrow 0$ and therefore that $T_n - ET_n$ has the same asymptotic distribution as $T_n^* - ET_n^*$ completing proof of the result.

From (6) we get

$$T_n^* - ET_n^* = \sum_{m=1}^{M_n} \gamma_m (Y_{mn}^2 - S_{mn}^2)$$

with $Y_{mn} = \sum_i b_{imn} Z_i, S_{mn}^2 = \sum_i b_{imn}^2$.

Put

$$X_{kn} = \sum_{m=1}^k \gamma_m (Y_{mn}^2 - S_{mn}^2).$$

By standard central limit theory, $D((Y_{in}, \dots, Y_{kn})) \rightarrow N(0, I)$; by (B2) $S_{mn}^2 \rightarrow 1$; hence $D(X_{kn}) \rightarrow D(X_k)$ with $X_k = \sum_1^k \gamma_m (Y_m^2 - 1)$. Also $D(X_k) \rightarrow D(X)$ as $k \rightarrow \infty$ with X the random variable in the asymptotic distribution of (5). In view of Theorem 4.2 of [2] and Chebychev's inequality it is now sufficient to show that

$$(8) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} E|Y_n - X_{kn}|^2 = 0$$

where $Y_n = T_n^* - ET_n^*$ in order to establish that $D(T_n^* - ET_n^*) \rightarrow D(X)$.

But

$$Y_n - X_{kn} = \sum_{m=k+1}^{M_n} \gamma_m (Y_{mn}^2 - S_{mn}^2).$$

Squaring and taking expectations we get

$$E|Y_n - X_{kn}|^2 = \sum_{m,r=k+1}^{M_n} \gamma_m \gamma_r A_{mnr}$$

with

$$A_{mrn} = E(Y_{mn}^2 - S_{mn}^2)(Y_{rn}^2 - S_{rn}^2) \\ = (\mu_4 - 3) \sum_i b_{imn}^2 b_{irn}^2 + 2(\sum_i b_{imn} b_{irn})^2$$

after some algebraic manipulations.

With $\Delta_{mr} = \sum_i b_{imn} b_{irn}$ we have

$$A_{mrn} \leq |\mu_4 - 3|D_n + 2(\Delta_{mr} - \delta_{mr})^2 + 4\delta_{mr}(\Delta_{mr} - \delta_{mr}) + 2\delta_{mr}^2 \\ \leq |\mu_4 - 3|D_n + 2B_n^2 + 4\delta_{mr}B_n + 2\delta_{mr}.$$

Hence

$$E|Y_n - X_{kn}|^2 \leq |\mu_4 - 3|D_n t_n^2 + 2B_n^2 t_n^2 + (4B_n + 2) \sum_{k+1}^{M_n} \gamma_m^2.$$

By (B2), (C) and (B3) we get (8).

It remains to show (7). Substituting for c_{ijn}^* and squaring we get

$$(9) \quad \sum_{i,j} (c_{ijn} - c_{ijn}^*)^2 = \sum_{i,j} c_{ijn}^2 - 2 \sum_{m=1}^{M_n} \gamma_m \sum_{i,j} c_{ijn} b_{imn} b_{jmn} \\ + \sum_{m,k=1}^{M_n} \gamma_m \gamma_k (\sum_i b_{imn} b_{ikn})^2 \\ = \sum_{i,j} c_{ijn}^2 - 2 \sum_m \gamma_m (\sum_{i,j} c_{ijn} b_{imn} b_{jmn} - \gamma_m) - \sum_m \gamma_m^2 \\ + \sum_{m,k} \gamma_m \gamma_k (\Delta_{mk} - \delta_{mk})^2 + 2 \sum_m \gamma_m^2 (\Delta_{mm} - 1).$$

The first and third terms together tend to 0 by (C); the second term is in absolute value less than $C_n t_n = o(1)$ by (BC); the fourth term is in absolute value less than $B_n^2 t_n^2 = o(1)$ by (B2). The fifth term is in absolute value less than $B_n \Gamma = o(1)$. (7) follows and the theorem is proved.

In the theorem above it is not assumed that $\sum |\gamma_m| < \infty$; the need for not doing this will be apparent in our applications. However, with such an assumption together with suitable boundedness assumptions on the b_{imn} 's simpler requirements suffice to prove the same result.

THEOREM 2. *Let $\{b_{imn}\}$ satisfy*

$$(B4) \quad \sum_i b_{imn} b_{ikn} \rightarrow \delta_{mk}$$

$$(B5) \quad n^{\frac{1}{2}} |b_{imn}| \leq b < \infty \quad \text{for all } i, m, n.$$

Let $\{\gamma_m\}$ be a real sequence for which (C) holds as well as

$$(C1) \quad \sum |\gamma_m| < \infty$$

$$(BC1) \quad \sum_{i,j} c_{ijn} b'_{imn} b'_{jmn} \rightarrow \gamma_m \quad \text{for each } m.$$

Then (5) holds.

PROOF. The proof of this result is much like that of the first one and is omitted.

REMARK. One source of applications for these theorems is that in which

$$(10) \quad c_{ijn} \cong n^{-1} c(i/n + 1, j/n + 1)$$

with c a measurable square-integrable function on the unit square, symmetric in its arguments. Let $\{g_m\}$ and $\{\gamma_m\}$ denote the eigenfunctions and eigenvalues

of c and suppose $\{g_m\}$ is an orthonormal system on $(0, 1)$. If we then select

$$(11) \quad b_{imn} = n^{-1/2} g_m(i/n + 1)$$

we can often expect that the conditions of Theorems 1 or 2 would apply; e.g. the sum in (B2) is $\sum_i b_{imn} b_{ikn} = n^{-1} \sum g_m(i/n + 1) g_k(i/n + 1)$ which approximates $\int_0^1 g_m(x) g_k(x) dx = \delta_{mk}$, etc.

If $\sum |\gamma_m| < \infty$ then $\int_0^1 c(x, x) dx = \sum \gamma_m$ and we can also expect that $ET_n = \sum c_{iin} \rightarrow \sum \gamma_m$ which would entail the conclusion $D(T_n) \rightarrow D(\sum \gamma_m Y_m^2)$ in addition to (5).

3. Application to certain tests of fit. Consider the test of fit problem mentioned in Section 1 and assume that H_0 is true and that F_0 is a continuous distribution function. Then we may write $F_0(X_{kn}) = S_k/S_{n+1}$ where $S_k = Z_1^* + \dots + Z_k^*$ with $\{Z_i^*\}$ independent random variables each exponentially distributed with expectation one. Putting $Z_i = Z_i^* - 1$ and introducing the function

$$(12) \quad \begin{aligned} \psi(x, y) &= 1 - y && \text{for } x \leq y \\ &= -y && \text{for } x > y \end{aligned}$$

it then follows that

$$(13) \quad Q_n^W = (n + 1/S_{n+1})^2 T_{n+1}$$

with T_n as in (1) and

$$(14) \quad c_{ijn} = n^{-2} \sum_{k=1}^{n-1} W(k/n) \psi(i/n, k/n) \psi(j/n, k/n) .$$

Since $S_{n+1}/n + 1 \rightarrow 1$ with probability one we can expect the asymptotic distribution of Q_n^W to be the same as that of T_n .

Note that here $c_{ijn} \cong n^{-1} c_w(i/n + 1, j/n + 1)$ with

$$(15) \quad c_w(x, y) = \int_0^1 W(z) \psi(x, z) \psi(y, z) dz$$

assuming W to be such that this integral exists. Hence if c_w is also quadratically integrable we can expect to be able to apply the theorems of the previous paragraph in the way indicated in the remark above.

For this purpose it would be necessary to calculate the eigenvalues and eigenfunctions of c_w . General results to this effect are given in the next paragraph and some special cases are considered in the fifth paragraph.

4. The kernel c_w . Throughout this paragraph we assume that W is such that integral (15) defining c_w exists and is finite for $0 < x, y < 1$ and moreover that

$$(16) \quad W(u) > 0 \quad \text{for } 0 < u < 1$$

$$(17) \quad \int_0^1 \int_0^1 c_w^2(x, y) dx dy < \infty .$$

Then c_w has eigenfunctions $\{g_m\}$ and eigenvalues $\{\gamma_m\}$ with $\gamma_1 > \gamma_2 > \dots > 0$ satisfying

$$(18) \quad \gamma_m g_m(x) = \int_0^1 c_w(x, y) g_m(y) dy .$$

Noting that $\int_0^1 c_w(x, y) dx = 0$ for all y , integration of (18) shows that

$$(19) \quad \int_0^1 g_m(x) dx = 0 \quad \text{for all } m .$$

Writing $G_m(x) = \int_0^x g_m(t) dt$ and substituting for c_w , (18) yields

$$(20) \quad \begin{aligned} \gamma_m g_m(x) &= \int_0^1 W(u)\phi(x, u)G_m(u) du \\ &= -\int_0^x uW(u)G_m(u) du + \int_x^1 (1-u)W(u)G_m(u) du \end{aligned}$$

from which it follows that g_m is differentiable for $0 < x < 1$ satisfying the differential equation

$$(21) \quad P(x)g_m'(x) + \lambda_m G_m(x) = 0$$

with

$$(22) \quad P(x) = 1/W(x), \quad \lambda_m = 1/\gamma_m .$$

Also

$$(23) \quad P(x)g_m'(x) \rightarrow 0 \quad \text{as } x \downarrow 0 \quad \text{and } x \uparrow 1 .$$

Differentiating (21) we find that g_m satisfies the Sturm-Liouville-type equation

$$(24) \quad dP(x)g_m'(x)/dx + \lambda_m g_m(x) = 0$$

subject to (23). Conversely, it is easily seen that if g_m satisfies (24) and (23) and is such that $xg_m(x) \rightarrow 0$ as $x \downarrow 0$ and $(1-x)g_m(x) \rightarrow 0$ as $x \uparrow 1$ then g_m also satisfies (18) and is an eigenfunction of c_w with eigenvalue $\gamma_m = 1/\lambda_m$. The usefulness of this remark can be extended somewhat as follows.

Consider the differential equation

$$(25) \quad dp(u)z'/du + \lambda_m r(u)z = 0$$

with $a < u < b$ for some real numbers a and b and p, r real functions on this interval such that

$$(26) \quad p(u) > 0 \quad \text{and} \quad r(u) > 0 .$$

Put

$$(27) \quad \rho = 1/\int_a^b r(t) dt, \quad U(u) = \rho \int_a^u r(t) dt$$

and introduce the new variable $x = U(u)$. Then (25) becomes (24) with

$$(28) \quad P(x) = \rho^2 p(U^{-1}(x))r(U^{-1}(x)) .$$

Our remark now applies to this equation and the following theorem follows.

THEOREM 3. *If $h_m(u)$ is a function on (a, b) and λ_m a constant satisfying (25) as well as*

$$(29) \quad p(u)h_m'(u) \rightarrow 0 \quad \text{as } u \uparrow b \quad \text{and } u \downarrow a ,$$

$$(30) \quad U(u)h_m(u) \rightarrow 0 \quad \text{as } u \downarrow a, \quad (1-U(u))h_m(u) \rightarrow 0 \quad \text{as } u \uparrow b ,$$

then $g_m(x) = h_m(U^{-1}(x))$ is an eigenfunction and $\gamma_m = 1/\lambda_m$ an eigenvalue of c_w with

$$(31) \quad W(x) = 1/P(x) = 1/\rho^2 p(U^{-1}(x))r(U^{-1}(x)) .$$

5. Weight functions associated with the classical orthogonal polynomials. The classical orthogonal polynomials ([5] page 163) satisfy the differential equation (25) ([5] page 167 eq. (3)); using the so-called differentiation formula it is straightforward to verify that (29) is satisfied in each case; (30) is also easily verified in each case. Theorem 3 therefore applies and with each of the sequences of classical orthogonal polynomials we can associate a weight function W and to each of the corresponding Q_n^W 's we can try to apply Theorems 1 or 2 as follows.

5.1. Jacobi weight functions. In case of the Jacobi polynomials $a = -1, b = 1$ and $p(u) = (1 - u)^{\alpha+1}(1 + u)^{\beta+1}, r(u) = (1 - u)^\alpha(1 + u)^\beta$ with $\alpha > -1, \beta > -1$ and $\rho = 2^{-\alpha-\beta-1}/B(\alpha + 1, \beta + 1)$ and $U(u) = F_{\beta,\alpha}((1 + u)/2)$ with $F_{\beta,\alpha}$ the beta distribution function with parameters $\beta + 1, \alpha + 1$; the corresponding weight-function $W_{\alpha,\beta}$ can be determined from (31); the eigenvalues are

$$(32) \quad \gamma_m = 1/m(m + \alpha + \beta + 1), \quad m = 1, 2, \dots$$

and the eigenfunctions follow from Theorem 3 with h_m the m th Jacobi polynomial (suitably normalized).

Consider now the statistic $Q_n^{\alpha,\beta} = Q_n^{W_{\alpha,\beta}}$. As in (15) let $T_n^{\alpha,\beta}$ be the corresponding quadratic form. With $b_{imn} = n^{-\frac{1}{2}}h_m(U^{-1}(i/n + 1))$ we wish to apply Theorem 2. Since h_m is bounded and continuous, the sum in (B4) converges to $\int_0^1 h_m(U^{-1}(x))h_k(U^{-1}(x)) dx = \rho \int_{-1}^1 h_m(u)h_k(u)r(u) du = \delta_{mk}$ so that (B4) holds. The other conditions can be verified similarly. Also

$$\begin{aligned} ET_n^{\alpha,\beta} &= \sum c_{iin} = n^{-2} \sum_{i=1}^n \sum_{k=1}^{n-1} W_{\alpha,\beta}(k/n)\phi^2(i/n, k/n) \\ &= n^{-1} \sum_{k=1}^{n-1} W_{\alpha,\beta}(k/n)k/n(1 - k/n) \end{aligned}$$

and it is easy but somewhat tedious to show that this tends to $\int_0^1 W_{\alpha,\beta}(x)x(1 - x) dx = \sum \gamma_m$ as $n \rightarrow \infty$. From Theorem 2 we conclude that $D(T_n^{\alpha,\beta}) \rightarrow D(\sum Y_m^2/m(m + \alpha + \beta + 1))$ and by (15) $Q_n^{\alpha,\beta}$ has the same asymptotic distribution.

Some special cases of this result are noteworthy:

5.1.1. For $\alpha = \beta = -\frac{1}{2}$ we get $W_{-\frac{1}{2},-\frac{1}{2}}(x) = 1$ for $0 < x < 1, \gamma_m = 1/m^2$ and the asymptotic distribution is that of the ordinary Cramér-von Mises-Smirnov statistic W_n^2 ([1]); this is to be expected in view of the fact that ([8] page 100) $W_n^2 = 1/12n + \sum (F_c(X_{jn}) - (2j - 1)/2n)^2 = Q_n^{-\frac{1}{2},-\frac{1}{2}} + R_n$ with $R_n = o_p(1)$ under H_0 as can readily be shown.

5.1.2. For $\alpha = \beta = 0$ we get $W_{0,0}(x) = 1/x(1 - x)$ for $0 < x < 1, \gamma_m = 1/m(m + 1)$ and the corresponding asymptotic distribution is the same as that of the Cramér-von Mises-Smirnov statistic with this weight function as is to be expected ([1]).

5.1.3. Other cases for which the weight function can be obtained in explicit form are $W_{0,\beta}$ and $W_{\alpha,0}$.

5.2. Hermitian weight function. In this case $a = -\infty, b = +\infty, r(u) = \phi(u), p(u) = \phi(u)$ with $\phi(u)$ the $N(0, 1)$ density function (we deviate slightly from the

notation of [5] here). Hence $U(u) = \Phi(u)$, the $N(0, 1)$ distribution function and

$$(33) \quad W(x) = h(x) = 1/\phi(\Phi^{-1}(x))^2, \quad \gamma_m = 1/m,$$

and the eigenfunctions are $h_m(\Phi^{-1}(x))$ with h_m now the m th Hermitian polynomial (suitably normalized).

Consider now the corresponding statistic Q_n^h ; as in (15) let T_n^h be the corresponding quadratic form. As $\sum |\gamma_m| = \infty$ here Theorem 2 does not apply; in order to apply Theorem 1 a convenient choice of $M_n = [|\Phi^{-1}(1/n + 1)|]$ with $[x]$ the greatest integer function. Then $t_n \sim \log M_n$ increases very slowly with n . Verification of Conditions (B1), (B2), (B3), (C) and (BC) amounts to getting sufficiently good estimates of the differences between the various sums and corresponding integrals; the details of these calculations are quite involved and are omitted (see [4]). It follows that $D(T_n^h - ET_n^h) \rightarrow D(\sum (Y_m^2 - 1)/m)$. As before $ET_n^h = n^{-1} \sum_{k=1}^{n-1} h(k/n)(k/n)(1 - k/n) = o(\log n)$ as is easily shown. With S_{n+1} as in (15) $(n + 1/S_{n+1})^2 = 1 + O_p(n^{-\frac{1}{2}})$; hence from (15) $Q_n^h - ET_{n+1}^h$ and $T_{n+1}^h - ET_{n+1}^h$ have the same asymptotic distribution. Also $EQ_n^h = (n + 1/n + 2)ET_{n+1}^h$; therefore

$$(34) \quad D(Q_n^h - EQ_n^h) \rightarrow D(\sum_{i=1}^{\infty} (Y_m^2 - 1)/m).$$

The characteristic function of this limiting distribution can be written in various forms such as

$$\prod_{k=1}^{\infty} e^{-it/k}(1 - 2it/k)^{-\frac{1}{2}} = (-2it\Gamma(-2it))^{\frac{1}{2}}e^{-itc} = r(t)e^{i\theta(t)}$$

with Γ the gamma function, c Euler's constant, $r(t) = (\sinh 2\pi t/2\pi t)^{-\frac{1}{2}}$ and

$$\theta(t) = \frac{1}{2} \sum_{k=1}^{\infty} (\tan^{-1}(2t/k) - 2t/k).$$

Inversion of this characteristic function has been done and the distribution is tabulated in [4].

We remark that this weight function h places relatively more weight on the extreme order statistics than the Jacobi functions in the sense that $h(x)/W_{\alpha,\beta}(x) \rightarrow \infty$ as $x \uparrow 1$ and $x \downarrow 0$.

5.3. Laguerre weight functions. In this case $a = 0$, $b = \infty$ and $r(u) = e^{-u}u^\alpha$, $p(u) = e^{-u}u^{\alpha+1}$, $\alpha > -1$. Hence $\rho = 1/\Gamma(\alpha + 1)$ and $U(u) = F_\alpha(u)$ with F_α the gamma distribution function with parameter $\alpha + 1$; if f_α denotes the corresponding density, then

$$(35) \quad W(x) \equiv W_\alpha(x) = 1/F_\alpha^{-1}(x)f_\alpha(F_\alpha^{-1}(x))^2, \quad \gamma_m = 1/m$$

and the eigenfunctions are expressible in terms of suitably normalized Laguerre functions.

The same analysis as that used in the previous case can be applied and the conclusion is that the asymptotic distribution of $Q_n^\alpha - EQ_n^\alpha$ is the same as that in (34) regardless of the value of α .

The special case $\alpha = 0$ is of some interest; in this case

$$W_0(x) = -1/(1-x)^2 \log(1-x).$$

Now $W_0(x)/h(x) \rightarrow 2$ as $x \uparrow 1$ while $W_0(x)/W_{0,0}(x) \rightarrow 1$ as $x \downarrow 0$; hence this weight function places weight similar to the Hermitian function on the upper order statistics and relatively smaller weight on the lower order statistics.

6. Discussion.

I. For a stochastic process approach by means of which characterizations similar to the above can be found see [1], [3], [6], [7]. This seems to break down for the statistics Q_n^w if W places too much weight on extreme order statistics to which our approach still applies. Our method however is usable only for statistics which can be expressed in terms of suitable quadratic forms; this limitation does not apply to the stochastic process approach.

II. The methods outlined here have also been applied to statistics other than Q_n^w and have been found useful in studies of the asymptotic power of certain tests of fit. These results will be reported in a forthcoming paper.

REFERENCES

- [1] ANDERSON, T. W. and DARLING, D. A. (1952). Asymptotic theory of certain goodness of fit criteria based on stochastic processes. *Ann. Math. Statist.* **23** 193-212.
- [2] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.
- [3] DARLING, D. A. (1957). The Kolmogorov-Smirnov, Cramér-von Mises tests. *Ann. Math. Statist.* **28** 823-838.
- [4] DE WET, T. (1972). Die asymptotiese verdeling van kwadratische vorme in onafhanklike stogastiese veranderlikes met toepassing op sekere passingstoette. D. Sc. dissertation, Potchefstroom Univ.
- [5] ERDÉLYI, A., et al. (1953). *Higher Transcendental Functions 2*. McGraw-Hill, New York.
- [6] KAC, M., KIEFER, J. and WOLFOWITZ, J. (1955). On tests of normality and other tests of goodness of fit based on distance methods. *Ann. Math. Statist.* **26** 189-211.
- [7] KAC, M. and SIEGERT, A. J. F. (1947). An explicit representation of a stationary Gaussian process. *Ann. Math. Statist.* **18** 438-442.
- [8] PARZEN, E. (1964). *Stochastic Processes*. Holden Day, San Francisco.
- [9] SCHACH, S. (1969). The asymptotic distribution of some non-linear functions of the two-sample rank vector. *Ann. Math. Statist.* **40** 1011-1020.
- [10] SCHACH, S. (1970). The asymptotic distribution of a class of non-parametric test statistics. *Metrika* **15** 48-58.
- [11] VARBERG, D. E. (1966). Convergence of quadratic forms in independent random variables. *Ann. Math. Statist.* **37** 567-575.

DEPARTEMENT STATISTIEK
POTCHEFSTROOMSE UNIVERSITEIT
POTCHEFSTROOM, SUID-AFRIKA