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# NONPARAMETRIC ESTIMATION FOR A GENERAL REPAIR MODEL[1]

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The construction and analysis of repair models is an important area in reliability. A commonly used model is the minimal repair model. Under this model, repair restores the state of the system to its level prior to failure. Kijima introduced repair models that could be classified as "better-than-minimal." Under Kijima's models, the system, upon repair, is functionally the same as a working system of lesser age which has never experienced failure. In this paper, we present a new approach to the modeling of better-than-minimal repair models. Using this approach, we construct a general repair model that contains Kijima's models as special cases. We also study the problem of estimating the distribution of the time to first failure of a system maintained by general repair. We make use of counting processes to show strong consistency of the estimator and prove results on weak convergence. Finally, we derive a Hall–Wellner type asymptotic confidence band for the distribution of the time to first failure of the system.

**1. Introduction.** Many systems are maintained and kept going by performing repair upon each failure. The construction and analysis of plausible repair models is therefore an important area in reliability. Let $F$ be the distribution of the time to first failure of the system. The repair models we consider in this paper postulate that the distribution of the interfailure times depend in some way on $F$. Early works on repair models assume that repair restores the state of a failed system to a level equivalent to a new one each time. This is the so-called perfect repair model. Clearly, this model is inadequate to model most repair processes. It is more reasonable to expect the distribution of the remaining life to vary from one failure time to another. One such model often used in the literature is the minimal repair model. Under this model, repair restores the system to its state just before failure. Brown and Proschan (1983) introduced a model that combines both perfect and minimal repairs. Under their model, at the time of each repair either a perfect repair occurs with probability $p$ or a minimal repair occurs with probability $1 - p$.

There has been a search for repair models where the interfailure times are stochastically larger than in the case of the minimal repair model. Such models are loosely called better-than-minimal repair models. Kijima (1989) introduced two models under which a system, upon repair, is functionally the same as an identical system of lesser age. Unlike the minimal repair model, Kijima's models assume that the distribution of interfailure times depends

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on $F$ not only through the age of the system at failure but on the degree of repair as well. Further generalizations of the Kijima models can be seen in Baxter, Kijima and Tortorella (1996). In this paper we introduce a general repair model in which, instead of reducing the effective age of the system at failure, we supplement its remaining life in an appropriate way. Such a model is a better-than-minimal repair model, and contains Kijima's models as special cases. We will also study the problem of estimating $F$ based on repair data on $n$ identical systems working independently and maintained using this general repair model.

In Section 2 we describe the general repair model. Section 3 discusses stochastic processes needed to carry out the estimation process. An estimator $\hat{F}_n$ of $F$ is introduced in Section 4 and its strong consistency is proved. The estimator $\hat{F}_n$ of $F$ is the solution of the integral equation $\hat{F}_n(\cdot) = \int_0^{\cdot}(1-\hat{F}_{n-})\, d\hat{\Lambda}_n$ where $\hat{\Lambda}_n$ is the Aalen–Nelson estimator of the hazard function $\Lambda$ of $F$. In Section 5 the weak convergence of $\sqrt{n}(\hat{F}_n - F)/\bar{F}$ to a Brownian motion is established. This in turn is used to construct a confidence band of the Hall–Wellner type for $F$ in Section 6.

Estimation of the distribution of the time to first failure of a repairable system was originally done through nonparametric maximum likelihood techniques. Whitaker and Samaniego (1989) used such techniques to estimate the distribution of the time to first failure of a system maintained under the Brown–Proschan model. An alternative to the nonparametric maximum likelihood approach is the use of the theory of counting processes. Hollander, Presnell and Sethuraman (1992) rederived and extended the large sample results of Whitaker and Samaniego using such an approach, making use of the methods developed for the analysis of the censored life-data model. This paper extends their results to the general repair model discussed informally in this introduction and formally defined in Section 2.

**2. Description of the model.** Prior to describing the model, we introduce some notation. In this paper, the terms "increasing" and "decreasing" will be used loosely; a sequence $\{a_j\}_{j\geq 1}$ is said to be increasing if $a_i \leq a_j$ for $i < j$ and a function $f(\cdot)$ is said to be increasing if $f(x) \leq f(y)$ for $x < y$. Similar definitions apply to decreasing sequences and functions. When we require the condition that $a_i < a_j$ for $i < j$, we will say that the sequence $\{a_j\}$ is strictly increasing. Similarly, $x$ would be called positive if $x \geq 0$ and negative if $x \leq 0$. The notation $x \wedge y$ will be used in place of $\min(x, y)$. Similarly, $x \vee y$ will be used in place of $\max(x, y)$. An integral of the form $\int_0^t$ would denote integration on $(0, t]$. We use the Itô integral when the integrating measure is a stochastic process. The function $I(\cdot)$ will denote the indicator function, so that $I(A) = 1$ if $A$ occurs and 0 otherwise. Finally, we will adopt the convention that $0/0 = 1$.

For any function $f$, we will use $f_-(t)$ for $\lim_{s\uparrow t} f(s)$ and $f_+(t)$ for $\lim_{s\downarrow t} f(s)$. A function is cadlag if it is right continuous and has left-hand limits. It is caglad if it is left continuous with right-hand limits. Let $T < \infty$. We will denote by $D[0, T]$ the cadlag functions on $[0, T]$ and by $D_-[0, T]$ the caglad functions on $[0, T]$. Unless otherwise stated, we will assume the Skorohod topology for $D[0, T]$. The supremum norm on $[0, T]$ will be denoted by $\| \cdot \|_0^T$ and $\Rightarrow$ will denote weak convergence of probability measures on $D[0, T]$ under the Skorohod topology.

Let $G$ be a distribution function. We will denote the survival function $1 - G$ by $\bar{G}$. The cumulative hazard function, $H$, of $G$ is defined by $H(\cdot) = \int_0^{\cdot} dG/(1 - G_-)$. The distribution $G$ is said to be an increasing failure rate (IFR) distribution if $\bar{G}(t + x)/\bar{G}(t)$ is decreasing in $t$ for each $x$. Finally, all random variables, unless otherwise stated, will be assumed to be defined on the complete probability space $(\Omega, \mathscr{F}, P)$.

We will now formally define what we will refer to as the general repair model. For any distribution function $F$, $\theta \in (0, 1]$ and $a \in [0, \infty)$, consider the family of distribution functions $\bar{F}_a^\theta(x) = \bar{F}(\theta x + a)/\bar{F}(a)$, $x > 0$. The family of distributions $\{F_a^\theta\}$ are stochastically ordered in $\theta$; that is, $\theta \le \theta'$ implies that $F_a^\theta \ge_{st} F_a^{\theta'}$, for each $a$ [i.e., $F_a^\theta(t) \le F_a^{\theta'}(t)$ for every $t$]. There is a way to view the survival function $\bar{F}_a^\theta(x)$. It corresponds to the life of a functioning item of age $a$ which has been scaled by a factor of $\theta$, with lower values of $\theta$ representing longer remaining life. For this reason we will refer to $F_a^\theta(x)$ as the life distribution of an item with an effective age of $a$ and a life supplement of $\theta$. We will see that this family provides us with a rich class of distributions for the remaining life of a given system subject to repair.

Consider a system put into operation at time $S_0 = 0$ using a brand-new unit whose life distribution is $F$. Upon each failure, the system is repaired in negligible time and put back into operation. Let $\{S_j\}_{j\ge1}$ denote the sequence of failure times of the system and let $T_j = S_j - S_{j-1}$, $j \ge 1$, be the interfailure times. We further assume that the $T_j$'s are strictly positive. A repair model describes the joint distribution of the random variables $\{T_j\}$. In this paper, we describe a general repair model, based on two sequences $\{A_j\}_{j\ge1}$ and $\{\Theta_j\}_{j\ge1}$ called the effective ages and life supplements, respectively, satisfying

$$A_1 = 0,\ \Theta_1 = 1,\ A_j \ge 0,\ \Theta_j \in (0, 1] \quad \text{and} \qquad (2.1)$$
$$A_j \le A_{j-1} + \Theta_{j-1} T_{j-1} \quad \text{for } j > 1\ .$$

The model is obtained by specifying the joint distributions of the interfailure times $\{T_j\}$ as follows:

$$P(T_j \le t | A_j, \Theta_j, T_1, \dots, T_{j-1}) = F_{A_j}^{\Theta_j}(t) \quad \text{for } t > 0,\ j \ge 1. \qquad (2.2)$$

Thus for the general repair model described by (2.1) and (2.2), the distribution of $T_j$ given $A_j, \Theta_j, T_1, \dots, T_{j-1}$, which is $F_{A_j}^{\Theta_j}$, is stochastically larger than $F_{A_j}^1$; that is, is better than a working item of age $A_j$. Furthermore, from (2.1) we can see that for each $j \ge 1$ the effective age, $A_{j+1}$, of the system after the $j$th repair, is less than its effective age, $X_j =_{\text{def}} A_j + \Theta_j T_j$, just before the $j$th failure which in turn is less than the actual age $S_j$. Thus the general repair model defined by (2.1) and (2.2) can be considered as a better-than-minimal repair model and as we shall see, contains the perfect repair and minimal repair models. It will be useful to note that the survival distribution of $X_j$ given $A_j, \Theta_j, T_1, \dots, T_{j-1}$ simplifies to $\bar{F}(x)/\bar{F}(A_j)$ for $x \geq A_j$, as can be seen from (2.2).

We will illustrate this general repair model and the terms "effective ages" and "life supplements" through examples discussed in the succeeding paragraphs.

Consider the case when $\Theta_j = 1$ and $A_j = 0$ for $j \geq 1$. Then (2.1) is automatically satisfied and (2.2) reduces to

$$P(T_j \leq t | T_1, \dots, T_{j-1}) = F_0^1(t) = F(t). \tag{2.3}$$

From this we see that the $T_j$'s are independent with common distribution $F$. This corresponds to the perfect repair model.

Next, consider the case when $\Theta_j = 1$ and $A_j = S_{j-1}$ for each $j$. Clearly, $A_{j+1} = S_j = A_j + \Theta_j T_j$ and, hence, (2.1) is satisfied. Moreover, under these conditions, (2.2) reduces to

$$P(T_j \leq t | S_{j-1}) = F_{S_{j-1}}^1(t)\ . \tag{2.4}$$

Hence, we see that this case corresponds to the minimal repair model.

We will now show that Kijima's models can be derived through suitable choices of $\{A_j\}$ and $\{\Theta_j\}$ satisfying (2.1) and (2.2). Let $\{D_j\}_{j\geq 1}$ be a sequence of random variables independently distributed on $[0, 1]$ and independent of other processes. Consider the case when $\Theta_j = 1$ for each $j$ and $A_j = \sum_{i=1}^{j-1} D_i T_i$ for $j > 1$. Since $A_{j+1} = A_j + D_j T_j$ and $D_j \leq 1 = \Theta_j$ for each $j$, then (2.1) is satisfied and (2.2) reduces to

$$P(T_j \leq t | T_i, D_i, 1 \leq i \leq j-1) = F_{A_j}^1(t). \tag{2.5}$$

This is Kijima's Model I. In this model, upon the $(j-1)$th repair, the time to next failure, $T_j$, of the system has the same distribution as the time to first failure of a system whose life distribution is $F_{A_j}^1$. Hence, upon repair, the system whose actual age is $S_j$ is functionally the same as an identical system of age $A_{j+1}$ which has never experienced failure. This explains the use of the term "effective age."

Consider the case when $\Theta_j = 1$ for each $j$ and $A_j = \sum_{k=1}^{j-1}(\prod_{i=k}^{j-1} D_i)T_k$ for $j > 1$. Since $A_{j+1} = D_j(A_j + T_j)$ and $D_j \leq 1 = \Theta_j$ for each $j$, then (2.1) is satisfied and (2.2) reduces to (2.5). This is Kijima's Model II. Moreover, when $D_j$ is 1 with probability $p$ and 0 with probability $1-p$, we obtain the Brown–Proschan model.

Up to this point we have restricted the $\Theta_j$'s to be identically equal to 1. We will now describe repair models obtained through other choices of the life supplement sequence $\{\Theta_j\}$. Recall that $F_a^\theta \geq_{st} F_a^{\theta'}$ if $\theta < \theta'$. This implies that the smaller $\theta$ is the larger in the expected remaining life of the system. Hence, we can use $\theta$ as a measure of how repair supplements the expected remaining life of the system. This explains the use of the term "life supplement." If a minimal repair were performed at the time of the first failure, then $T_2$ would have the distribution $F^{1}_{T_1}$. If we want a longer expected life for $T_2$ then we can use the distribution $F^{\Theta_2}_{T_1}$ for some $\Theta_2$ satisfying $0 < \Theta_2 < 1$. Starting with the distribution $F^{\Theta_2}_{T_1}$ for $T_2$ and using minimal repair upon the second failure, the random variable $T_3$ would have the distribution $F^{1}_{A_3}$ where $A_3 = T_1 + \Theta_2 T_2$. Again, if we want a longer expected life for $T_3$ we can use the distribution $F^{\Theta_3}_{A_3}$ for some $\Theta_3$ satisfying $0 < \Theta_3 < 1$. With this in mind, we define $\Theta_1 = 1$, $A_j = \sum_{i=1}^{j-1} \Theta_i T_i$ and $0 < \Theta_j < 1$ for $j > 1$. Under these conditions, $A_{j+1} = A_j + \Theta_j T_j$ for $j \geq 1$, satisfying (2.1), and (2.2) reduces to

$$P(T_j \leq t | T_i, \Theta_i,\ 1 \leq i \leq j-1) = F^{\Theta_j}_{A_j}(t). \tag{2.6}$$

We will refer to this as the supplemented life repair model. By definition, the system enjoys a larger expected remaining life under a supplemented life repair than it would under a minimal repair.

**3. Some fundamental processes.** Observing the general repair process, as described in the previous section, will result in the following data: the sequence of failure times $0 = S_0, S_1, S_2, \ldots$ and the sequences of effective ages $A_1, A_2, \ldots$ and life supplements $\Theta_1, \Theta_2, \ldots$ which are the sequences of characteristics of the first repair, second repair and so on. We will observe the process till a fixed time $T < \infty$. The effective age $X_j$ prior to the $j$th failure can be derived from the above as $A_j + \Theta_j(S_j - S_{j-1})$ provided that $S_j \leq T$. If $S_{j-1} \leq T < S_j$ then we cannot observe $X_j$ and the effective age of the system at time $T$ is $(A_j + \Theta_j(T - S_{j-1}))$ which is also equal to $X_j \wedge (A_j + \Theta_j(T - S_{j-1}))$.

We will now see that there is a close connection between repair models and censored life-data models which enables us to use techniques developed for censored life-data models. Define the processes $N$ and $Y$ by

$$N(t) = \sum_j I(X_j \leq t, S_j \leq T)$$

and

$$Y(t) = \sum_j I(A_j < t \leq (X_j \wedge [A_j + \Theta_j(T - S_{j-1})]))$$

where $I(\cdot)$ is the usual indicator function. Let $\delta_j = I(S_j \leq T)$ and $\tilde{X}_j = X_j \wedge [A_j + \Theta_j(T - S_{j-1})]$. Then the random variables $\{(\tilde{X}_1, \delta_1), (\tilde{X}_2, \delta_2), \ldots\}$ can be viewed as observations coming from a censored life-data model. A general repair model observed during a period of length $T$ is akin to a survival study where a subject $j$ enters the study at age $A_j$ and either dies during the study at age $X_j$ or leaves the study by age $A_j + \Theta_j(T - S_{j-1})$. So that at the completion of the study only the variables $\{(\tilde{X}_1, \delta_1), (\tilde{X}_2, \delta_2), \ldots\}$ are actually observed. The random variable $N(t)$ represents the number of observed (uncensored) deaths by time $t$ and $Y(t)$ the size of the risk set at time $t$. Since the processes $N$ and $Y$ are fundamental in the estimation of the survival function in a censored life-data model, it is reasonable to expect these processes to play a similar role in estimating $F$ under general repair.

Let $\Lambda$ be the hazard function of $F$ and define the process $M = N - \int Y\, d\Lambda$. The process $M$ plays an important role in establishing the large sample properties of the Whitaker–Samaniego estimator. Based on this observation, it is reasonable to expect $M$ to play a similar role in establishing large sample properties of $\hat{F}_n$. We now find expressions for the mean and covariance functions of $M$. To simplify expressions we shall use the following notation. For $j \geq 1$, let

$$\begin{aligned}
\tilde{N}_j(t) &= I(S_j \leq t),\\
H_j^t(s) &= I(A_j + \Theta_j(s - S_{j-1}) \leq t),\\
G_j(t) &= I(S_{j-1} < t \leq S_j),\\
\Lambda_j(t) &= \Lambda(A_j + \Theta_j(t - S_{j-1})),\\
K_j^t(s) &= I(A_j < s \leq A_j + \Theta_j(t - S_{j-1}))
\end{aligned}$$

and

$$\tilde{M}_j(t) = \tilde{N}_j(t) - \int_0^t G_j\, d\Lambda_j.$$

We can now rewrite $M$ as

$$\begin{aligned}
&\sum_j \int_{S_{j-1}}^T H_j^t(s)\, d\tilde{M}_j(s)\\
&\qquad = \sum_j \int_{S_{j-1}}^T H_j^t(s)\, d\tilde{N}_j(s) - \sum_j \int_{S_{j-1}}^T H_j^t(s) G_j(s)\, d\Lambda_j(s)\\
&\qquad = \sum_j I\{X_j \leq t, S_j \leq T\}\\
&\qquad\quad - \sum_j \int_0^t I\{A_j < s \leq X_j \wedge [A_j + \Theta_j(T - S_{j-1})]\}\, d\Lambda(s)\\
&\qquad = \sum_j \tilde{N}_j(T) H_j^t(S_j) - \sum_j \int_0^t K_j^{T \wedge S_j}(s)\, d\Lambda(s)\\
&\qquad = N(t) - \int_0^t Y(s)\, d\Lambda(s) = M(t).
\end{aligned} \tag{3.1}$$

Hence to evaluate the mean and covariance functions of $M$ it is enough to evaluate those of $\int_{S_{j-1}}^T H_j^t(s)\, d\tilde{M}_j(s)$.

LEMMA 3.1. *For fixed $t$ and $t'$,*

$$E\left[\int_{S_{j-1}}^T H_j^t\, d\tilde{M}_j\right] = 0 \quad \textit{for all } j, \tag{3.2}$$

$$E\left[\int_{S_{i-1}}^T H_i^t\, d\tilde{M}_i \int_{S_{j-1}}^T H_j^{t'}\, d\tilde{M}_j\right] = 0 \quad \textit{for } i \neq j, \tag{3.3}$$

$$E\left[\int_{S_{j-1}}^{T} H_j^t \, d\tilde{M}_j \int_{S_{j-1}}^{T} H_j^{t'} \, d\tilde{M}_j\right] = E\left[\int_0^{t\wedge t'} K_j^{T\wedge S_j}(1-\Delta\Lambda)\, d\Lambda\right] \quad \textit{for all } j. \tag{3.4}$$

PROOF. For each $j$, let $\mathscr{F}_j = \sigma(A_j, \Theta_j; T_1, \dots, T_{j-1})$. The idea behind the proof is quite simple. Upon conditioning on $\mathscr{F}_j$, we are left with expressions of the form

$$\int g(s; A_j, \Theta_j, S_{j-1}) P(X_j > h(s; A_j, \Theta_j, S_{j-1})|\mathscr{F}_j)\, d\Lambda(s)$$

for some functions $g, h$. These expressions are then easily simplified using (2.2). We now carry out the calculations.

To prove (3.2), note that

$$\begin{aligned}
E\left[\int_{S_{j-1}}^{T} H_j^t G_j \, d\Lambda_j|\mathscr{F}_j\right] &= E\left[\int_0^t K_j^{T\wedge S_j}\, d\Lambda|\mathscr{F}_j\right] \\
&= \int_0^t K_j^T(s) P(X_j \geq s|\mathscr{F}_j)\, d\Lambda(s) \\
&= \int_0^t K_j^T(s)\frac{dF(s)}{\bar{F}(A_j)} \\
&= 1 - \frac{\bar{F}(t\wedge[A_j + \Theta_j(T - S_{j-1})]}{\bar{F}(A_j)} \\
&= E[\tilde{N}_j(T) H_j^t(S_j)|\mathscr{F}_j] \\
&= E\left[\int_{S_{j-1}}^{T} H_j^t \, d\tilde{N}_j|\mathscr{F}_j\right].
\end{aligned}$$

This proves (3.2). From this it follows that for $i < j$,

$$E\left[\int_{S_{i-1}}^{T} H_i^t \, d\tilde{M}_i \int_{S_{j-1}}^{T} H_j^{t'} \, d\tilde{M}_j|\mathscr{F}_j\right] = \int_{S_{i-1}}^{T} H_i^t \, d\tilde{M}_i E\left[\int_{S_{j-1}}^{T} H_j^{t'} \, d\tilde{M}_j|\mathscr{F}_j\right] = 0.$$

This proves (3.3). To prove (3.4), we assume that $t < t'$, for simplicity. By using the identity $\int_{S_{j-1}}^T H_j^t \, d\tilde{M}_j = \tilde{N}_j(T) H_j^t(S_j) - \int_0^T H_j^t G_j \, d\Lambda_j$, we obtain

$$\begin{aligned}
&E\left[\int_{S_{j-1}}^{T} H_j^t \, d\tilde{M}_j \int_{S_{j-1}}^{T} H_j^{t'} \, d\tilde{M}_j|\mathscr{F}_j\right] \\
&\quad = E[\tilde{N}_j(T) H_j^t(S_j) H_j^{t'}(S_j)|\mathscr{F}_j] \\
&\qquad - E\left[\tilde{N}_j(T) H_j^t(S_j)\int_0^T H_j^{t'} G_j \, d\Lambda_j|\mathscr{F}_j\right] \\
&\qquad - E\left[\tilde{N}_j(T) H_j^{t'}(S_j)\int_0^T H_j^{t} G_j \, d\Lambda_j|\mathscr{F}_j\right] \\
&\qquad - E\left[\int_0^T H_j^t G_j \, d\Lambda_j \int_0^T H_j^{t'} G_j \, d\Lambda_j|\mathscr{F}_j\right].
\end{aligned} \tag{3.5}$$

Now,

$$
\begin{aligned}
E[\tilde{N}_j(T)H_j^t(S_j)H_j^{t'}(S_j)|\mathscr{F}_j] &= P\{X_j \le t \wedge [A_j + \Theta_j(T - S_{j-1})]|\mathscr{F}_j\} \\
&= 1 - \frac{\bar{F}[t \wedge (A_j + \Theta_j(T - S_{j-1}))]}{\bar{F}(A_j)} \\
&= \int_0^t K_j^T(s) \frac{\bar{F}_-(s)}{\bar{F}(A_j)} \, d\Lambda(s)
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
&E[\tilde{N}_j(T)H_j^t(S_j)\int_0^T H_j^{t'} G_j \, d\Lambda_j|\mathscr{F}_j] \\
&\quad = E\left[\int_0^t K_j^T(s) I(s \le X_j \le t \wedge [A_j + \Theta_j(T - S_{j-1})]) \, d\Lambda(s)|\mathscr{F}_j\right] \\
&\quad = \int_0^t K_j^T(s) P(s \le X_j \le t \wedge [A_j + \Theta_j(T - S_{j-1})]|\mathscr{F}_j) \, d\Lambda(s) \\
&\quad = \int_0^t K_j^T(s) \frac{\bar{F}_-(s) - \bar{F}(t \wedge [A_j + \Theta_j(T - S_{j-1})])}{\bar{F}(A_j)} \, d\Lambda(s).
\end{aligned} \tag{3.7}
$$

By symmetry,

$$
\begin{aligned}
&E[\tilde{N}_j(T)H_j^{t'}(S_j)\int_0^T H_j^t G_j \, d\Lambda_j|\mathscr{F}_j] \\
&\qquad = \int_0^t K_j^T(s) \frac{\bar{F}_-(s) - \bar{F}(t' \wedge [A_j + \Theta_j(T - S_{j-1})])}{\bar{F}(A_j)} \, d\Lambda(s).
\end{aligned} \tag{3.8}
$$

Finally,

$$
\begin{aligned}
&E\left[\int_0^T H_j^t G_j \, d\Lambda_j \int_0^T H_j^{t'} G_j \, d\Lambda_j|\mathscr{F}_{j-1}\right] \\
&\qquad = E\left[\int_0^t \int_0^{t'} K_j^T(s) K_j^T(u) I(X_j \ge s \vee u) \, d\Lambda(s) \, d\Lambda(u)|\mathscr{F}_j\right] \\
&\qquad = \int_0^t \int_0^{t'} K_j^T(s) K_j^T(u) P[X_j \ge s \vee u|\mathscr{F}_j] \, d\Lambda(s) \, d\Lambda(u) \\
&\qquad = \int_0^t \int_u^{t'} K_j^T(u) K_j^T(s) \frac{\bar{F}_-(s)}{\bar{F}(A_j)} \, d\Lambda(s) \, d\Lambda(u) \\
&\qquad\quad + \int_0^{t'} \int_s^t K_j^T(u) K_j^T(s) \frac{\bar{F}_-(u)}{\bar{F}(A_j)} \, d\Lambda(u) \, d\Lambda(s) \\
&\qquad\quad + \int_0^t K_j^T(s) \frac{\bar{F}_-(s)}{\bar{F}(A_j)} \Delta\Lambda(s) \, d\Lambda(s)
\end{aligned} \tag{3.9}
$$

$$= \int_0^t K_j^T(u) \frac{F(t' \wedge [A_j + \Theta_j(T - S_{j-1})]) - F(u)}{\bar{F}(A_j)} \, d\Lambda(u)$$

$$+ \int_0^t K_j^T(s) \frac{F(t \wedge [A_j + \Theta_j(T - S_{j-1})]) - F(s)}{\bar{F}(A_j)} \, d\Lambda(s)$$

$$+ \int_0^t K_j^T(s) \frac{\bar{F}_-(s)}{\bar{F}(A_j)} \, \Delta\Lambda(s) \, d\Lambda(s).$$

Substituting (3.6), (3.7), (3.8) and (3.9) into (3.5) we get

$$\begin{aligned} E\left[\int_{S_{j-1}}^T H_j^t \, d\tilde{M}_j \int_{S_{j-1}}^T H_j^{t'} \, d\tilde{M}_j | \mathscr{F}_j\right] &= \int_0^t K_j^T(s) \frac{\bar{F}_-(s)}{\bar{F}(A_j)} [1 - \Delta\Lambda(s)] \, d\Lambda(s) \\ &= \int_0^t K_j^T(s) P[X_j \geq s | \mathscr{F}_j][1 - \Delta\Lambda(s)] \, d\Lambda(s) \\ &= E\left[\int_0^t K_j^{T \wedge S_j}(s)(1 - \Delta\Lambda(s)) \, d\Lambda(s) | \mathscr{F}_j\right]. \end{aligned}$$

This proves (3.4) and establishes Lemma 3.1. □

From this result we obtain the following expressions for the mean and covariance functions of $M$.

THEOREM 3.1.

$$EM = EN - \int EY \, d\Lambda = 0, \tag{3.10}$$

$$\operatorname{Cov}(M(t)M(t')) = \int_0^{t \wedge t'} EY(1 - \Delta\Lambda) \, d\Lambda. \tag{3.11}$$

PROOF. From (3.1) and (3.2) it follows that

$$EM = E \sum_j \int_{S_{j-1}}^T H_j^t(s) \, d\tilde{M}_j(s) = 0.$$

This proves (3.10). To prove (3.11) we assume that $t < t'$ and use equations (3.1), (3.3) and (3.4) to obtain the following:

$$\begin{aligned} E(M(t)M(t')) &= E \sum_{i,j} \int_{S_{i-1}}^T H_i^t \, d\tilde{M}_i \int_{S_{j-1}}^T H_j^{t'} \, d\tilde{M}_j \\ &= E \sum_j \int_0^t K_j^{T \wedge S_j} (1 - \Delta\Lambda) \, d\Lambda \\ &= \int_0^t EY(1 - \Delta\Lambda) \, d\Lambda. \end{aligned}$$

This completes the proof. □

**4. The estimator and its uniform consistency.** Suppose that we observe $n$ independent copies of the processes $N$ and $Y$ on a finite interval $[0, T]$. We will look at the problem of estimating $F$ based on these observations. Throughout the rest of the paper, we will assume that (i) $F(T) < 1$ and (ii) $F$ is IFR. Assumption (i) allows us to bound $EY(t)$ away from zero uniformly on $[0, T]$. This is pivotal in proving uniform consistency of the estimator. Define $\tilde{N}(t) = \sup\{j \geq 1: S_j \leq t\}$. The random variable $\tilde{N}(t)$ is the number of system failures by time $t$. Let $N^*(t) = \sup\{j \geq 1: S_j^* \leq t\}$ where $\{S_j^*\}$ is the minimal repair process based on $F$. Assumption (ii) gives us

$$P(S_k^* - S_{k-1}^* > x | S_{k-1}^* = t) \geq \frac{\bar{F}(T+x)}{\bar{F}(T)}$$

for all $k \geq 1$ and $t \in [0, T]$. It follows that the evaluation at time $t$ of a renewal process having recurrence time distribution $G(x) = 1 - \bar{F}(T+x)/\bar{F}(T)$ stochastically dominates $N^*(t)$ on $[0, T]$ which in turn stochastically dominates $\tilde{N}(t)$. (The counting process $N_1$ stochastically dominates $N_2$ on $[0, T]$ if $P(N_1(t) \geq n) \geq P(N_2(t) \geq n)$ for all $n$ and $t \in [0, T]$.) This guarantees that $\tilde{N}(T)$ has finite moments of all orders. We will need this in proving weak convergence.

Let $N_n$ and $Y_n$ be the sum of the first $n$ independent copies of $N$ and $Y$, respectively. We note that from the viewpoint of a life-data model, $dF(t)/(1 - F_{-}(t))$ represents the instantaneous failure rate at time $t$. A straightforward way of estimating this failure rate is by using the ratio of the number of observed deaths at time $t$ to the number at risk at time $t$. Hence, a natural estimator of the hazard function $\Lambda$ is given by

$$\hat{\Lambda}_n(t) = \int_0^t \frac{J_n dN_n}{Y_n},$$

where $J_n(t) = I(Y_n(t) > 0)$ for $t \in (0, T]$. The estimator $\hat{\Lambda}_n$ is referred to as the Aalen–Nelson estimator of $\Lambda$. Since $F(t) = \int_0^t (1 - F_{-})\, d\Lambda$, it is reasonable to require an estimator $\hat{F}_n$ of $F$ to satisfy

$$\hat{F}_n(t) = \int_0^t (1 - \hat{F}_{n-})\, d\hat{\Lambda}_n. \tag{4.1}$$

Equation (4.1) is sometimes referred to as a Volterra integral equation. Its solution is given by

$$\hat{\bar{F}}_n(t) = \prod_{s \leq t} (1 - d\hat{\Lambda}_n(s)) = \prod_{s \leq t} (1 - \Delta\hat{\Lambda}_n(s)),$$

where $\prod_{s \leq t}(1 - d\hat{\Lambda}_n(s))$ denotes the product integral [see Gill and Johansen (1990)].

The rest of this paper will be devoted to a study of the large sample properties of $\hat{F}_n$. Under a minimal repair assumption, the process $M$ of Section 3 turns out to be a martingale. This enabled Hollander, Presnell and Sethuraman (1992) to use martingale techniques in proving large sample results for the Whitaker–Samaniego estimator. For the general repair model, finding a suitable filtration with respect to which $M$ is a martingale proves to be a formidable task. A lot of the difficulty is due to the fact that, in general, the $X_j$'s are not monotonic. This makes the existing methods for computing a compensator inapplicable in the general case. One could try working with the failure process, say $\tilde{N}$, associated with the $S_j$'s instead of the failure process $N$ associated with the $X_j$'s. Since the $S_j$'s are strictly increasing it would not be too hard to find the compensator of $\tilde{N}$. The problem would then turn to transforming results on $\tilde{N}$ into results on $N$. Unfortunately, we found this to be an equally formidable task. However, a closer look at Theorem 3.1 reveals that although $M$ may not be a martingale with respect to the history of $N$ it nevertheless exhibits the same mean and covariance structure it would have if it were a martingale. Fortunately these features, in conjunction with techniques used by Gill (1980) for Markov renewal processes, are sufficient to prove large sample results.

Since most of the distributions considered in reliability are continuous, we will assume throughout the rest of the paper that $F$ is continuous. This will help us avoid unnecessary complications. This by no means limits our results to the continuous case. Most of our arguments carry over to the discontinuous case with very little, if any, modifications.

The process $Y(t)$ can be written as $Y_1(t) - Y_2(t)$, where $Y_1(t)$ and $Y_2$ are two left-continuous increasing processes defined by

$$Y_1(t) = \sum_j I(A_j < t,\ S_{j-1} \le T)$$

and

$$Y_2(t) = \sum_j I(\tilde{X}_j < t,\ S_{j-1} \le T),$$

where $\tilde{X}_j = X_j \wedge [A_j + \Theta_j(T - S_{j-1})]$. Let $Y_{1n}$ and $Y_{2n}$ be the sum of the first $n$ independent realizations of $Y_1$ and $Y_2$, respectively. Then the process $\hat{F}_n$ can be considered as the result of the three mappings

$$\left(\frac{N_n}{n}, \frac{Y_{1n}}{n}, \frac{Y_{2n}}{n}\right) \to \left(\frac{N_n}{n}, \frac{Y_n}{n}\right) \to \hat{\Lambda}_n \to \hat{F}_n \tag{4.2}$$

going through the spaces

$$D[0,T] \times D_{-}[0,T] \times D_{-}[0,T] \to D[0,T] \times D_{-}[0,T] \to D[0,T] \to D[0,T].$$

Under the supremum norm $\|\cdot\|$, the first mapping is clearly continuous. To show that the second mapping is continuous we use the following result.

LEMMA 4.1. *Let* $N'_n = N_n/n$, $Y'_n = Y_n/n$ *and* $\alpha = \inf_{(0,T]} EY(t)$ . *Then with probability* 1,

$$\left\|\hat{\Lambda}_n - \int \frac{dEN}{EY}\right\|_0^T \le \frac{\|Y'_n - EY\|_0^T [\ EN(T) + \|N'_n - EN\|_0^T\ ]}{\alpha(\alpha - \|Y'_n - EY\|_0^T)} + \frac{2}{\alpha}\|N'_n - EN\|_0^T \tag{4.3}$$

*for sufficiently large* $n$.

PROOF. Since $F(T) < 1$ then $P(S_1 > T) > 0$. It follows that for $t \in (0, T]$,

$$EY(t) \ge P(Y(t) > 0) \ge P(Y(T) > 0) \ge P(S_1 > T) > 0.$$

Hence, $\alpha > 0$ and

$$\begin{aligned}\hat{\Lambda}_n(t) - \int_{(0,\,t]} \frac{dEN}{EY} &= \int_{(0,\,t]} \frac{dN'_n - dEN}{EY} + \int_{(0,\,t]} \left(\frac{J_n}{Y'_n} - \frac{1}{EY}\right) dN'_n \\ &= \frac{N'_n(t) - EN(t)}{EY(t)} - \int_{(0,\,t)} (N'_n - EN)\, d\left(\frac{1}{EY}\right) \\ &\quad + \int_{(0,\,t]} \left(\frac{J_n}{Y'_n} - \frac{1}{EY}\right) dN'_n.\end{aligned}$$

This implies that

$$\begin{aligned}\left\|\hat{\Lambda}_n - \int \frac{dEN}{EY}\right\|_0^T &\le \frac{1}{\alpha}\|Y'_n - EY\|_0^T \left\| \int_{(0,\cdot]} \frac{d[(N'_n - EN) + EN]}{EY - (EY - Y'_n)} \right\|_0^T \\ &\quad + \frac{2}{\alpha}\|N'_n - EN\|_0^T \\ &\le \frac{\|Y'_n - EY\|_0^T [\ EN(T) + \|N'_n - EN\|_0^T\ ]}{\alpha(\alpha - \|Y'_n - EY\|_0^T)} \\ &\quad + \frac{2}{\alpha}\|N'_n - EN\|_0^T.\end{aligned}$$ □

Now, by the Glivenko–Cantelli theorem,

$$\max\left\{\left\|\frac{N_n}{n} - EN\right\|_0^T, \left\|\frac{Y_{1n}}{n} - EY_1\right\|_0^T, \left\|\frac{Y_{2n}}{n} - EY_2\right\|_0^T\right\} \to 0 \quad \text{a.s.}$$

From (4.3) and the continuity of the first mapping in (4.2), we get

$$\left\|\hat{\Lambda}_n - \int \frac{dEN}{EY}\right\|_0^T \to 0 \quad \text{w.p.1.}$$

But from (3.10) with $t \in (0, T]$

$$\int_0^t \frac{dEN}{EY} = \Lambda(t).$$

Hence,

$$\|\hat{\Lambda}_n - \Lambda\|_0^T \to 0 \quad \text{w.p.1 as } n \to \infty. \tag{4.4}$$

Uniform consistency now follows from (4.4) and the continuity of the product integral as shown in the next result.

THEOREM 4.1. $\|\hat{F}_n(t) - F(t)\|_0^T \to 0$ *w.p.1 as* $n \to \infty$.

PROOF. Since $\hat{\Lambda}_n(t)$ is increasing in $t$ for each $n$ and $\hat{\Lambda}_n(T) \to \Lambda(T)$ a.s. it follows that $\limsup |\hat{\Lambda}_n|((0, T]) < \infty$ w.p.1. This guarantees the continuity, under the supremum norm, of the last mapping on (4.2). [See Theorem 7 of Gill and Johansen (1990).] □

**5. Results on weak convergence.** Define $Z_n \equiv \sqrt{n}[\hat{F}_n - F]/\bar{F}$ and the process $M_n$ by

$$M_n = N_n - \int Y_n \, d\Lambda.$$

Now, by Lemma 7.2.1 of Shorack and Wellner

$$\frac{\hat{F}_n(t) - F(t)}{\bar{F}(t)} = \int_0^t \frac{1 - \hat{F}_n}{1 - F} \, d[\hat{\Lambda}_n - \Lambda]$$

$$= \int_0^t \left( \frac{\hat{\bar{F}}_{n-}}{\bar{F}} \frac{J_n}{Y_n} \right) dM_n$$

for all $t \in (0, T]$. Hence

$$Z_n(t) = \int_0^t \frac{1 - \hat{F}_{n-}}{1 - F} \frac{J_n}{Y_n/n} \, dW_n, \tag{5.1}$$

where $W_n = n^{-1/2} M_n$. Note that from the CLT, the finite-dimensional distributions of $W_n$ converge to that of a Gaussian process. This suggests that it might be possible to obtain a weak convergence result for $W_n$ and, consequently, for $Z_n$ in view of (5.1).

THEOREM 5.1. *Let $W$ be a zero mean Gaussian process with independent increments and variance function* $\mathrm{Var}(W(t)) = \int_0^t EY \, d\Lambda$. *Then* $W_n \Rightarrow W$ *on* $D[0, T]$.

PROOF. This theorem is analogous to that of Lemma 3 in Gill (1980) and is proved similarly. Note that $M_n$ is the sum of $n$ independent copies of the process $M$. That the finite-dimensional distributions of $W_n$ converge to that of $W$ follows from the CLT. It remains to prove tightness. To prove this, it suffices to show that there exist a nondecreasing right-continuous function $G$ and $\eta > 1/2$ such that

$$n^{-2}E[(M_n(u) - M_n(t))^2(M_n(t) - M_n(s))^2] \le (G(u) - G(t))^{\eta}(G(t) - G(s))^{\eta}$$

for $0 \le s \le t \le u \le T$ [see Billingsley (1968), page 133]. For notational convenience denote $\Delta_{st}H \equiv H(t) - H(s)$. Then

$$\begin{aligned} n^{-2}E(\Delta_{tu}M_n \cdot \Delta_{st}M_n)^2 &= n^{-1}E(\Delta_{tu}M \cdot \Delta_{st}M)^2 \\ &\quad + \frac{n-1}{n}E(\Delta_{tu}M)^2E(\Delta_{st}M)^2 \\ &\quad + 2\frac{n-1}{n}[E(\Delta_{tu}M \cdot \Delta_{st}M)]^2. \end{aligned} \tag{5.2}$$

Recall that $\tilde{N}(t) = \sup\{j \ge 1: S_j \le t\}$. Let $V = \tilde{N} + 1$ and $B \ge (1 - F(T))^{-1}$. Then

$$E(\Delta_{tu}M \cdot \Delta_{st}M) = 0. \tag{5.3}$$

$$\begin{aligned} E(\Delta_{tu}M)^2 &= \int_t^u EY\,d\Lambda \\ &\le BEV(T)(\Delta_{tu}F)^{\alpha} \quad \text{for any } t \le u \text{ and } 0 < \alpha < 1. \end{aligned} \tag{5.4}$$

$$\begin{aligned} |\Delta_{tu}M \cdot \Delta_{st}M| &\le (\Delta_{tu}N \cdot \Delta_{st}N) + \left(\int_t^u Y\,d\Lambda\right)\left(\int_s^t Y\,d\Lambda\right) \\ &\quad + \Delta_{tu}N\int_s^t Y\,d\Lambda + \Delta_{st}N \cdot \int_t^u Y\,d\Lambda \\ &\le [(1+B)V(T)]^2. \end{aligned} \tag{5.5}$$

From (5.3) and Hölder's inequality it follows that for $\alpha_1 = 1 - \beta_1 \in (0, 1)$,

$$\begin{aligned} &E(\Delta_{tu}N \cdot \Delta_{st}N) + E\left(\int_t^u Y\,d\Lambda \int_s^t Y\,d\Lambda\right) \\ &\quad = E\left(\Delta_{tu}N\int_s^t Y\,d\Lambda\right) + E\left(\Delta_{st}N\int_t^u Y\,d\Lambda\right) \\ &\quad \le BE(\Delta_{tu}N \cdot V(T))(\Delta_{st}F) \\ &\qquad + BE(\Delta_{st}N \cdot V(T))(\Delta_{tu}F) \\ &\quad \le B[EV(T)^{1+1/\beta_1^{-1}}]^{\beta_1}[E(\Delta_{tu}N)]^{\alpha_1}(\Delta_{st}F)^{\alpha_1} \\ &\qquad + B[EV(T)^{1+1/\beta_1^{-1}}]^{\beta_1}[E(\Delta_{st}N)]^{\alpha_1}(\Delta_{tu}F)^{\alpha_1}. \end{aligned}$$

Now $E(\Delta_{tu}N) \le B(EV(T))(\Delta_{tu}F)$ for $t \le u$. Hence,

$$E(|\Delta_{tu}M \cdot \Delta_{st}M|) \le C_1(\Delta_{tu}F \cdot \Delta_{st}F)^{\alpha_1} \tag{5.6}$$

for some constant $C_1$ provided $EV(T)^{1+1/\beta_1^{-1}} < \infty$. From (5.5), (5.6) and Hölder's inequality it follows that for some $\alpha_2 = 1 - \beta_2 \in (0,1)$ and constant $C_2$,

$$\begin{aligned} E(\Delta_{tu}M \cdot \Delta_{st}M)^2 &= E(|\Delta_{tu}M \cdot \Delta_{st}M)|^{\alpha_2}|\Delta_{tu}M \cdot \Delta_{st}M)|^{1+\beta_2}) \\ &\le (1+B)^{2+2\beta_2}(E|\Delta_{tu}M \cdot \Delta_{st}M|)^{\alpha_2}(EV(T)^{2+2\beta_2^{-1}})^{\beta_2} \\ &\le C_2(\Delta_{tu}F \cdot \Delta_{st}F)^{\alpha_1\alpha_2} \end{aligned} \tag{5.7}$$

provided $EV(T)^{2+2\beta_2^{-1}} < \infty$. Since $\tilde{N}(T)$ has finite moments we can choose $\beta_1 < 1/6$ and $\beta_2 < 2/5$ so that $\alpha_1\alpha_2 > 1/2$. Substituting (5.3), (5.4) and (5.7) into (5.2), we get

$$n^{-2}E(\Delta_{tu}M_n \cdot \Delta_{st}M_n)^2 \le C(\Delta_{tu}F \cdot \Delta_{st}F)^{\eta}$$

for some constant $C$ and $\eta > 1/2$. This completes the proof. □

From the Skorohod–Dudley–Wichura theorem [see, e.g., Shorack and Wellner (1986), Theorem 2.3.4] there exist processes $W'_n$ and $W'$ having the same distribution as $W_n$ and $W$, respectively, such that

$$\rho_S(W'_n, W') \to 0,$$

where $\rho_S$ is the Skorohod metric on $D[0,T]$. In light of this, we will assume that we actually have $\rho_S(W_n, W) \to 0$. To prove convergence results for $X_n$, however, we would need uniform convergence. If we can show that $W$ has almost surely continuous paths, then we would have uniform convergence, since under such a condition the supremum norm distance is equivalent to Skorohod distance. That $W$ has almost surely continuous paths is shown in the next theorem.

THEOREM 5.2. $\|W_n(t) - W(t)\|_0^T \to 0$ *w.p.1 as* $n \to \infty$.

PROOF. From Theorem 3.1 and Chebyshev's inequality, we get

$$P(|M(t) - M(t_0)| \ge \varepsilon) \le \frac{|\int_{t_0}^t EY\,d\Lambda|}{\varepsilon^2} \to 0 \quad \text{as } |t - t_0| \to 0$$

for each $\varepsilon > 0$ and $t_0 \in [0,T]$; that is, $M$ is stochastically continuous on $[0,T]$. This implies that $W$ has almost surely continuous sample paths [see Theorem 2 of Hahn (1978)]. Hence the result. □

REMARK. If $F$ is not continuous, then $M$ may not be stochastically continuous. Hence, Theorem 5.2 may fail to hold. Fortunately, we can construct versions of $W_n$ and $W$ so that Theorem 5.2 holds for these versions. [See Lemma 4 in Gill (1980).]

THEOREM 5.3. *Define* $Z(t) = \int_0^t (EY)^{-1}\,dW$. *Then* $\|Z_n - Z\|_0^T \to 0$ *w.p.1.*

PROOF. Let $H_n = (\hat{\bar{F}}_{n-}/\bar{F})[nJ_n/Y_n]^{-1}$. It follows from (5.1) that for $t \in (0, T]$, $Z_n(t) = \int_0^t H_n\, dW_n$. The result now follows since $\|H_n - (EY)^{-1}\|_0^T \to 0$ w.p.1 and $\|W_n - W\|_0^T \to 0$ w.p.1 [see Lemma 5 of Gill (1980)]. □

Now since the supremum norm distance is larger than the Skorohod distance, $\rho_S$, it follows that $\rho_S(Z_n, Z) \to 0$ w.p.1. This gives us the following result.

THEOREM 5.4. *$Z_n \equiv \sqrt{n}(\hat{F}_n - F)/\bar{F} \Rightarrow Z$ on $D[0, T]$.*

The proof is analogous to that of Corollary 2.3.1 of Shorack and Wellner (1986).

COROLLARY 5.1. *Let $B$ denote Brownian motion on $[0, \infty)$ and*

$$C(t) = \int_0^t \frac{1}{EY} \frac{dF}{1 - F}.$$

*Then $\sqrt{n}(\hat{F}_n - F)/\bar{F} \Rightarrow B(C)$ on $D[0, T]$.*

PROOF. Note that $\{W(t)\}_{t \in [0,T]}$ is a square integrable martingale with respect to $\mathscr{F}_t = \sigma(W(s)\colon s \leq t)$. Furthermore, it is easily checked that $\langle W \rangle_t = \int_0^t EY\, d\Lambda$. Hence

$$\begin{aligned} \operatorname{Cov}(Z(s), Z(t)) &= E[Z(s \wedge t)]^2 \\ &= E \int_0^{s \wedge t} (EY)^{-2}\, d\langle W \rangle \\ &= \int_0^{s \wedge t} (EY)^{-1}\, d\Lambda = C(s \wedge t). \end{aligned}$$

This proves the result. □

COROLLARY 5.2. *Let $B^\circ$ denote a Brownian bridge on $[0, 1]$ and $K = C/(1+C)$. Then*

$$\sqrt{n} \frac{\bar{K}}{\bar{F}} (\hat{F}_n - F) \Rightarrow B^\circ(K) \quad \text{on } D[0, T].$$

PROOF. Let

$$B^\circ(t) = (1 - t) B\left(\frac{t}{1 - t}\right) \quad \text{for } t \in [0, K(T)].$$

The result follows immediately. □

**6. A Hall–Wellner type confidence band.** In this section we construct a confidence band for $F$ similar to the bands of Hall and Wellner (1980) for the censored life-data model. We then present results of some simulation studies of the coverage probabilities of the band using repair models discussed in Section 2.

For $t$ in $[0, T]$, define $L_n(t) = I(\hat{F}_n(t) < 1)$ and let

$$\hat{C}_n(t) = \int_0^t \frac{J_n L_n}{Y_n/n} \frac{d\hat{F}_n}{1 - \hat{F}_n}.$$

Define

$$\hat{K}_n(t) = \frac{\hat{C}_n(t)}{1 + \hat{C}_n(t)}.$$

For $t$ such that $\hat{F}_n(t) = 1$, set $\hat{K}_n(t) = 1$. Then the result of Corollary 5.2 suggests a confidence band for $F$ of the form

$$\hat{F}_n \pm n^{-1/2} \lambda_\alpha \hat{\bar{F}}_n / \hat{\bar{K}}_n, \tag{6.1}$$

where $\lambda_\alpha$ is such that $P\{\sup_{t\in[0,1]} |B^\circ(t)| \le \lambda_\alpha\} = 1 - \alpha$. This gives our band an asymptotic confidence level of at least $100(1-\alpha)\%$. To justify this band we need the following result.

LEMMA 6.1. $\|\hat{K}_n - K\|_0^T \Rightarrow 0$ *w.p.*1.

PROOF. For $t \in [0, T]$,

$$|\hat{C}_n(t) - C(t)| \le \left| \int_0^t \left[ \frac{J_n L_n}{(Y_n/n)(1 - \hat{F}_n)} - \frac{1}{EY(1 - F)} \right] d\hat{F}_n \right| + \left| \int_0^t \frac{1}{EY(1 - F)} (d\hat{F}_n - dF) \right|.$$

Note that $\alpha \equiv \inf_{t\in(0,T]} EY(t) > 0$ (Lemma 4.1) and that $\sup_{t\in(0,T]} F(t) \le F(T) < 1$. These together with the uniform consistency of $Y_n/n$ and $\hat{F}_n$ imply that $J_n L_n = 1$ on $(0, T]$ with probability 1 for sufficiently large $n$. Hence, with probability 1,

$$|\hat{C}_n(t) - C(t)| \le \left( \frac{2}{\alpha \bar{F}(T)} \right)^2 \|(Y_n/n)(1 - \hat{F}_n) - EY(1 - F)\|_0^T + \left\| \frac{1}{EY(1 - F)} \right\|_0^T \|\hat{F}_n - F\|_0^T$$

for sufficiently large $n$. Since $Y_n/n$ and $\hat{F}_n$ are uniformly consistent then so is $\hat{C}_n$. The result follows. □

This leads to the next theorem which justifies our confidence band.

THEOREM 6.1. $\sqrt{n}(\hat{\bar{K}}_n/\hat{\bar{F}}_n)(\hat{F}_n - F) \Rightarrow B^\circ(K)$ *on* $D[0, T]$ .

PROOF. The result follows from the identity

$$\sqrt{n}\frac{\hat{\bar{K}}_n}{\hat{\bar{F}}_n}(\hat{F}_n - F) = \sqrt{n}\frac{\bar{K}}{\bar{F}}(\hat{F}_n - F) + \left[\frac{\hat{\bar{K}}_n}{\hat{\bar{F}}_n} - \frac{\bar{K}}{\bar{F}}\right][\sqrt{n}(\hat{F}_n - F)]. \qquad \square$$

REMARKS. (i) In practice, it may be that the data obtained lead to $\hat{F}_n(t_0) = 1$ for some $0 < t_0 < T$. When this happens, the data obtained give us a confidence band only on the interval $[0, \sigma)$ where $\sigma = \inf\{t \in [0, T]: \hat{F}_n(t) = 1\}$.

(ii) Let $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be the distinct ordered values of the $X$'s whose corresponding failure times are within $[0, T]$. Also, let $\delta_j$ be the number of observations with value $X_{(j)}$. Then for computational purposes we note that

$$\hat{\bar{F}}_n(t) = \prod_{X_{(j)} \le t}\left(1 - \frac{\delta_j}{Y_n(X_{(j)})}\right)$$

and

$$\hat{C}_n(t) = n\sum_{X_{(j)} \le t}\frac{\hat{F}_n(X_{(j)}) - \hat{F}_n(X_{(j-1)})}{Y_n(X_{(j)})\hat{\bar{F}}_n(X_{(j)})}.$$

We now consider simulation studies of our confidence bands. For our first repair model, we use Kijima's Model II where the $D_j$'s are taken to be uniformly distributed on $(0, 1)$. We call this Model A. We compute the coverage probabilities of the confidence bands for Model A with gamma and Weibull distributions, using simulation studies. The results of the simulation studies are shown in Table 1 for the gamma distribution and in Table 3 for the Weibull distribution. For the gamma distribution we chose $T = 10$ and for the Weibull we chose $T = 2$.

TABLE 1
*Coverage probabilities of* $100p\%$ *confidence bands for gamma(α) under Model A*

| | $p = 0.90$ | | | $p = 0.95$ | | | $p = 0.99$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha = 3.0$ | $\alpha = 5.0$ | $\alpha = 7.0$ | $\alpha = 3.0$ | $\alpha = 5.0$ | $\alpha = 7.0$ | $\alpha = 3.0$ | $\alpha = 5.0$ | $\alpha = 7.0$ |
| 10 | 0.8808 | 0.8990 | 0.9080 | 0.9242 | 0.9356 | 0.9520 | 0.9704 | 0.9766 | 0.9844 |
| 20 | 0.8916 | 0.9080 | 0.9062 | 0.9352 | 0.9514 | 0.9522 | 0.9794 | 0.9848 | 0.9888 |
| 30 | 0.8900 | 0.9056 | 0.9074 | 0.9366 | 0.9494 | 0.9516 | 0.9808 | 0.9852 | 0.9870 |
| 50 | 0.8922 | 0.9000 | 0.9080 | 0.9394 | 0.9478 | 0.9552 | 0.9852 | 0.9878 | 0.9882 |
| 100 | 0.8988 | 0.9058 | 0.9002 | 0.9464 | 0.9504 | 0.9474 | 0.9864 | 0.9874 | 0.9888 |
| 200 | 0.8960 | 0.9020 | 0.9024 | 0.9462 | 0.9500 | 0.9512 | 0.9862 | 0.9906 | 0.9908 |

TABLE 2
*Coverage probabilities of* $100p\%$ *confidence bands for gamma(α) under Model B*

| | **p = 0.90** | | | **p = 0.95** | | | **p = 0.99** | | |
|---|---|---|---|---|---|---|---|---|---|
| ***n*** | **α = 3.0** | **α = 5.0** | **α = 7.0** | **α = 3.0** | **α = 5.0** | **α = 7.0** | **α = 3.0** | **α = 5.0** | **α = 7.0** |
| 10 | 0.7406 | 0.7936 | 0.8478 | 0.7944 | 0.8448 | 0.8974 | 0.8732 | 0.9134 | 0.9506 |
| 20 | 0.8146 | 0.8442 | 0.8838 | 0.8716 | 0.9024 | 0.9318 | 0.9364 | 0.9602 | 0.9730 |
| 30 | 0.8454 | 0.8722 | 0.8914 | 0.8988 | 0.9158 | 0.9368 | 0.9568 | 0.9714 | 0.9780 |
| 50 | 0.8650 | 0.8826 | 0.9036 | 0.9204 | 0.9338 | 0.9430 | 0.9750 | 0.9814 | 0.9854 |
| 100 | 0.8878 | 0.8920 | 0.8934 | 0.9390 | 0.9444 | 0.9480 | 0.9850 | 0.9852 | 0.9898 |
| 200 | 0.8986 | 0.8932 | 0.9096 | 0.9410 | 0.9492 | 0.9500 | 0.9856 | 0.9896 | 0.9878 |

For our second repair model, we use the supplemented-life repair model with $\Theta_j = \prod_{i=1}^{j-1} D_i$ where the $D_j$'s are uniformly distributed on $(0.8, 1)$. We refer to this as Model B. We compute the coverage probabilities of the confidence bands for Model B. In the context of our interpretation of the supplemented-life repair model, restricting the $D_j$'s to be at least 0.8 restricts the increase in the expected remaining life to be at most 25% of the original. This seems to be a reasonable assumption; hence the choice of the interval. The results of the simulation are shown in Table 2 for the gamma distribution and in Table 4 for the Weibull distribution. We use the same value for $T$ as in Model A.

All the results are based on 5,000 iterations of the simulation. To generate a sample for the gamma distribution, we make use of an algorithm by Dagpunar (1978) on sampling variates from a truncated gamma distribution. For the Weibull case, let $Z_1, Z_2, \dots, Z_r$ be a random sample from a standard exponential distribution. It is not difficult to see that setting $X_j = (A_j^\alpha + Z_j)^{1/\alpha}$ produces the desired repair process for a Weibull with parameter $\alpha$ under Model A and setting $X_j = (Z_1 + Z_2 + \cdots + Z_j)^{1/\alpha}$ produces the desired repair process for Model B. To generate the exponentials we used the function REXP given by Marsaglia and Tsang (1984).

In general, the simulation seems to indicate that the band performs well in both cases with as low a sample size as 20 needed to get within 2% of

TABLE 3
*Coverage probabilities of* $100p\%$ *confidence bands for Weibull(α) under Model A*

| | **p = 0.90** | | | **p = 0.95** | | | **p = 0.99** | | |
|---|---|---|---|---|---|---|---|---|---|
| ***n*** | **α = 1.0** | **α = 1.5** | **α = 2.0** | **α = 1.0** | **α = 1.5** | **α = 2.0** | **α = 1.0** | **α = 1.5** | **α = 2.0** |
| 10 | 0.9044 | 0.8934 | 0.9018 | 0.9452 | 0.9400 | 0.9406 | 0.9792 | 0.9774 | 0.9770 |
| 20 | 0.8996 | 0.8942 | 0.9034 | 0.9444 | 0.9386 | 0.9464 | 0.9832 | 0.9800 | 0.9824 |
| 30 | 0.9038 | 0.9010 | 0.9042 | 0.9460 | 0.9474 | 0.9458 | 0.9872 | 0.9862 | 0.9840 |
| 50 | 0.8944 | 0.8986 | 0.9018 | 0.9506 | 0.9462 | 0.9466 | 0.9870 | 0.9872 | 0.9868 |
| 100 | 0.9056 | 0.9042 | 0.8998 | 0.9482 | 0.9474 | 0.9472 | 0.9890 | 0.9874 | 0.9880 |
| 200 | 0.9094 | 0.9016 | 0.9114 | 0.9522 | 0.9502 | 0.9506 | 0.9878 | 0.9892 | 0.9896 |

TABLE 4
*Coverage probabilities of* $100p\%$ *confidence bands for Weibull*($\alpha$) *under Model B*

| | $p = 0.90$ | | | $p = 0.95$ | | | $p = 0.99$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha = 1.0$ | $\alpha = 1.5$ | $\alpha = 2.0$ | $\alpha = 1.0$ | $\alpha = 1.5$ | $\alpha = 2.0$ | $\alpha = 1.0$ | $\alpha = 1.5$ | $\alpha = 2.0$ |
| 10 | 0.8590 | 0.8136 | 0.7776 | 0.9050 | 0.8678 | 0.8344 | 0.9526 | 0.9318 | 0.9064 |
| 20 | 0.8816 | 0.8562 | 0.8398 | 0.9254 | 0.9110 | 0.8950 | 0.9732 | 0.9634 | 0.9524 |
| 30 | 0.8876 | 0.8710 | 0.8608 | 0.9340 | 0.9212 | 0.9088 | 0.9766 | 0.9712 | 0.9638 |
| 50 | 0.8972 | 0.8858 | 0.8736 | 0.9416 | 0.9342 | 0.9250 | 0.9838 | 0.9766 | 0.9780 |
| 100 | 0.9056 | 0.8980 | 0.8868 | 0.9510 | 0.9420 | 0.9398 | 0.9864 | 0.9868 | 0.9842 |
| 200 | 0.9104 | 0.9110 | 0.8938 | 0.9546 | 0.9468 | 0.9470 | 0.9888 | 0.9888 | 0.9886 |

the desired confidence level in Model A. In Model B, a sample size of 50 is sufficient in most cases to attain the same accuracy. The discrepancy is due mainly to the fact that, in view of the way the data are generated, there are more failure times per sample under Model A than there are under Model B. In most instances, the band performed better as $F(T)$ moved further away from 1 (i.e, large values of $\alpha$ for gamma and small values for Weibull). This is expected because of the reliance of our large sample results on the assumption that $F(T) < 1$. In the gamma case, this could also be attributed to the low efficiency exhibited by the data generating process when obtaining variates close to the tails of the distribution. Finally, the results seem to indicate that a larger sample size is needed to attain the confidence level of 99% than to attain either a 90% or 95% confidence level.

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