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ON THE RELATIONSHIP BETWEEN TWO ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTION OF SAMPLE MEAN AND ITS APPLICATIONS[1]

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Although the cumulative distribution function may be differentiated to obtain the corresponding density function, whether or not a similar relationship exists between their asymptotic expansions remains a question. We provide a rigorous argument to prove that Lugannani and Rice's asymptotic expansion for the cumulative distribution function of the mean of a sample of i.i.d. observations may be differentiated to obtain Daniels's asymptotic expansion for the corresponding density function. We then apply this result to study the relationship between the truncated versions of the two series, which establishes the derivative of a truncated Lugannani and Rice series as an alternative asymptotic approximation for the density function. This alternative approximation in general does not need to be renormalized. Numerical examples demonstrating its accuracy are included.

1. Introduction. Daniels (1954) introduced the method of saddlepoint approximation into statistics, and derived an asymptotic expansion for the density function $f_n(\bar{x})$, where $\bar{X} = \sum X_i/n$ is the mean of a sample of n independently, identically distributed random variables. Lugannani and Rice (1980) used a closely related method to obtain an asymptotic expansion for the tail probability of $\bar{X}$, $Q_n(\bar{x})$, which leads to an asymptotic expansion for the cumulative distribution function $F_n(\bar{x})$. These two expansions are remarkably accurate even for very small n. The question of the relationship between them was first raised by Lugannani and Rice in their 1980 paper, in which they conjectured, "the integration of Daniels' series and our series for $Q_n(\bar{x})$ both give approximation to $Q_n(\bar{x})$ that are in error by the same order of magnitude." Since then various authors have provided numerical evidence supporting this conjecture [e.g., Daniels (1983) and Field and Ronchetti (1990)].

In the present paper, we study the relationship between these two expansions by focusing on the derivatives of the entire and truncated Lugannani and Rice expansion for $F_n(\bar{x})$. In Section 2 we introduce the notation and formally differentiate Lugannani and Rice's expansion for $F_n(\bar{x})$. The resulting expansion resembles Daniels's expansion for $f_n(\bar{x})$. In Section 3 we justify with a rigorous proof that, under a uniform validity assumption concerning the expansions, we can indeed differentiate Lugannani and Rice's expansion

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for $F_n(\bar{x})$ to obtain Daniels's expansion for $f_n(\bar{x})$. This result is then applied in Section 4 to investigate the relationship between the truncated versions of these two expansions, which establishes the derivative of a truncated Lugannani and Rice expansion as an alternative asymptotic approximation to $f_n(\bar{x})$ and provides an answer to Lugannani and Rice's conjecture.

We assume that X_i has a continuous density function $f(x)$ defined on an interval on the real line, and we refer to this interval as the support of X_i. As the existence of the two expansions depends on the existence of the saddlepoint, we shall only be concerned with those $\bar{x}$ values that have saddlepoints. These $\bar{x}$ values usually form an interval. We refer to this interval as the domain of $\bar{x}$. For more discussion concerning saddlepoints, see Daniels (1954).

2. Formal differentiation. The notation used in this paper is consistent with that in Daniels (1987, 1954). Daniels's series for $f_n(\bar{x})$ and Lugannani and Rice's series for $F_n(\bar{x})$ are, respectively,

$$f_n(\bar{x}) \sim g_n(\bar{x}) \sum_{r=0}^{\infty} \frac{a_r(\bar{x})}{n^r}, \tag{1}$$

$$F_n(\bar{x}) = 1 - Q_n(\bar{x}) \sim \Phi(\hat{W} n^{1/2}) - \phi(\hat{W} n^{1/2}) \sum_{r=0}^{\infty} \frac{b_r(\bar{x})}{n^{r+1/2}}, \tag{2}$$

where $\hat{W} = \operatorname{sgn}(\hat{T})\{2[\hat{T}K'(\hat{T}) - K(\hat{T})]\}^{1/2}$, $K(T)$ is the cumulant generating function of X_i, $\hat{T}$ is the saddlepoint satisfying $K'(\hat{T}) = \bar{x}$, Φ and ϕ are the cumulative distribution function and probability density function of the standard normal distribution, respectively, and $g_n(\bar{x})$ is the saddlepoint approximation of $f_n(\bar{x})$ given by

$$g_n(\bar{x}) = \left(\frac{n}{2\pi K''(\hat{T})}\right)^{1/2} e^{n[K(\hat{T}) - \hat{T}\bar{x}]}. \tag{3}$$

The $a_r(\bar{x})$'s and $b_r(\bar{x})$'s are coefficients in (3.3) in Daniels (1954) and (4.5) in Daniels (1987), respectively. For brevity we write $K^{(r)}(\hat{T})$, $a_r(\bar{x})$ and $b_r(\bar{x})$ as $\hat{K}^{(r)}$, a_r and b_r, respectively. We shall also use $\lambda_r = K^{(r)}(\hat{T})/[K''(\hat{T})]^{r/2}$. By differentiating (2) formally and collecting terms according to the powers of n, we obtain

$$\begin{aligned} f_n(\bar{x}) &= \frac{dF_n(\bar{x})}{d\bar{x}} \\ &\sim \phi(\hat{W}n^{1/2}) \frac{d\hat{T}}{d\bar{x}} \left[\frac{d(\hat{W}n^{1/2})}{d\hat{T}} + \hat{W}n^{1/2} \frac{d(\hat{W}n^{1/2})}{d\hat{T}} \sum_{r=0}^{\infty} \frac{b_r}{n^{r+1/2}} \right. \\ &\qquad \left. - \sum_{r=0}^{\infty} \frac{db_r}{d\hat{T}} \frac{1}{n^{r+1/2}} \right] \end{aligned} \tag{4}$$

$$= g_n(\bar{x}) \sum_{r=0}^{\infty} \frac{1}{n^r} \left[\hat{T}(\hat{K}'')^{1/2} b_r - (\hat{K}'')^{-1/2} \frac{db_{r-1}}{d\hat{T}} \right]$$

$$= g_n(\bar{x}) \sum_{r=0}^{\infty} \frac{1}{n^r} c_r(\bar{x}),$$

where $b_{-1} = -\hat{W}$ and

$$c_r(\bar{x}) = \hat{T}(\hat{K}'')^{1/2} b_r - (\hat{K}'')^{-1/2} \frac{db_{r-1}}{d\hat{T}}. \tag{5}$$

The formal expansion (4) is of the same form as the Daniels expansion (1). This suggests that they may be identical, that is, $c_r = a_r$ for $r = 0, 1, \dots$. It is not difficult to show that $c_0 = 1$ and $c_1 = \frac{1}{8}\lambda_4 - \frac{5}{24}\lambda_3^2$, which indeed match a_0 and a_1. If this formal expansion can be proven valid, then by the uniqueness of asymptotic expansion with respect to the asymptotic sequence $\{1/n^r\}$, it is the Daniels expansion (1). Consequently, $c_r = a_r$ for $r = 0, 1, \dots$. We now discuss the validity of (4).

3. The main theorem. When a function of two variables $f(s, t)$ has an asymptotic power series in variable s, it is not always true that formally differentiating this series with respect to t will result in an asymptotic power series for the partial derivative $f_t(s, t)$. Wasow [(1965), pages 43–48] discussed conditions under which this is true. However, these conditions are in general not satisfied here. We now state and prove the main result of this paper, which presents a sufficient condition under which the formal expansion (4) is indeed valid.

THEOREM 1. *Let $D_{\bar{x}}$ be a bounded closed interval in the interior of the domain of $\bar{x}$. If the Daniels expansion* (1) *and the Lugannani–Rice expansion* (2) *are both uniformly valid in $D_{\bar{x}}$, then* (4) *is an asymptotic expansion for $f_n(\bar{x})$ uniformly valid in $D_{\bar{x}}$.*

When (1) and (2) are written as

$$f_n(\bar{x}) = g_n(\bar{x}) \left\{ 1 + \frac{a_1(\bar{x})}{n} + \cdots + \frac{a_m(\bar{x})}{n^m} + O\left(\frac{1}{n^{m+1}}\right) \right\},$$

$$F_n(\bar{x}) = \Phi(\hat{W} n^{1/2}) - \phi(\hat{W} n^{1/2}) \left\{ \frac{b_0(\bar{x})}{n^{3/2}} + \cdots + \frac{b_m(\bar{x})}{n^{m+1/2}} + O\left(\frac{1}{n^{m+3/2}}\right) \right\},$$

where $m = 0, 1, \dots$, the uniform validity condition in the theorem means that the constants associated with the two $O(\cdot)$'s are independent of the $\bar{x}$'s in $D_{\bar{x}}$. See, for example, Wong (1989) for a general discussion on uniform validity. To prove this theorem, we first establish conditions, through Lemmas 1 and 2 below, under which an asymptotic power series of $f(s, t)$ in s may be differentiated with respect to t to obtain an asymptotic power series of $f_t(s, t)$.

LEMMA 1 [Wasow (1965), Lemma 9.1, page 45]. *Let* $f(s,t)$ *be bounded in* $D_s \times D_t$, *where* $0 \in D_s$, *and let* $h_r(t)$ *be bounded in* D_t *for* $r = 0, 1, \dots$. *Then*

$$f(s,t) \sim \sum_{r=0}^{\infty} h_r(t) s^r \quad \text{as } s \to 0 \tag{6}$$

uniformly for $t \in D_t$ *iff for every* m *the function* $E_m(s,t)$ *defined by the relation*

$$f(s,t) = \sum_{r=0}^{m} h_r(t) s^r + E_m(s,t) s^{m+1} \tag{7}$$

is bounded in $D_s \times D_t$.

PROOF. See Wasow [(1965), page 45]. □

LEMMA 2. *Assume* $\partial f(s,t)/\partial t$ *is integrable in* t *and bounded in* $D_s \times D_t$, *where* $0 \in D_s$ *and* D_t *is a bounded closed interval on the real line. If*

$$f(s,t) \sim \sum_{r=0}^{\infty} h_r(t) s^r \quad \text{as } s \to 0 \tag{8}$$

and

$$\frac{\partial f(s,t)}{\partial t} \sim \sum_{r=0}^{\infty} l_r(t) s^r \quad \text{as } s \to 0, \tag{9}$$

where (9) *is uniformly valid with respect to* $t \in D_t$ *and the* $l_r(t)$*'s are continuous functions of* t, *then the* $h_r(t)$*'s are differentiable in* D_t *and*

$$l_r(t) = \frac{dh_r(t)}{dt} \quad \text{for } r = 0, 1, \dots . \tag{10}$$

PROOF. Since $\partial f(s,t)/\partial t$ and the $l_r(t)$'s are bounded, Lemma 1 implies that for every m the function $E_m(s,t)$ defined by the relation

$$\frac{\partial f(s,t)}{\partial t} = \sum_{r=0}^{m} l_r(t) s^r + E_m(s,t) s^{m+1} \tag{11}$$

is bounded in $D_s \times D_t$. For t_0 and t in D_t,

$$\int_{t_0}^{t} \frac{\partial f(s,v)}{\partial v}\, dv = \sum_{r=0}^{m} \left[\int_{t_0}^{t} l_r(v)\, dv \right] s^r + \left[\int_{t_0}^{t} E_m(s,v)\, dv \right] s^{m+1}. \tag{12}$$

Since $\int_{t_0}^{t} l_r(v)\, dv$, $r = 0, 1, \dots$, are bounded in D_t, and

$$\int_{t_0}^{t} \frac{\partial f(s,v)}{\partial v}\, dv \quad \text{and} \quad \int_{t_0}^{t} E_m(s,v)\, dv, \qquad m = 0, 1, \dots,$$

are bounded in $D_s \times D_t$, by Lemma 1,

$$f(s,t) - f(s,t_0) = \int_{t_0}^{t} \frac{\partial f(s,v)}{\partial v}\, dv \sim \sum_{r=0}^{\infty} \left[\int_{t_0}^{t} l_r(v)\, dv \right] s^r \tag{13}$$

uniformly in D_t. By the uniqueness of asymptotic expansion with respect to a given asymptotic sequence,

$$h_r(t) - h_r(t_0) = \int_{t_0}^{t} l_r(v)\,dv \quad \text{for } r = 0, 1, \ldots. \tag{14}$$

Since l_r is continuous, (14) implies that $h'_r(t_0)$ exists and equals $l_r(t_0)$, $r = 0, 1, \ldots$, for any t_0 in D_t. □

Nonetheless, since $f_n(\bar{x})$ in (1) is not bounded when n approaches infinity, and Lugannani and Rice's series (2) is not a standard power series, Lemma 2 cannot be directly applied to (1) and (2) to prove the theorem. We now focus on a new function $I(n, \bar{x})$, defined below, and show that this function and its partial derivative have asymptotic power series expansions. Lemma 2 is then used to establish the relationship between these two power series, which leads to the theorem.

LEMMA 3. *Let D_n be the set of positive integers, let $D_{\bar{x}}$ be a bounded closed interval in the domain of $\bar{x}$ and let $I(n, \bar{x})$ be defined by the relation*

$$F_n(\bar{x}) = \Phi(\hat{W}n^{1/2}) - \frac{\phi(\hat{W}n^{1/2})b_0}{n^{1/2}} - \phi(\hat{W}n^{1/2})I(n, \bar{x}). \tag{15}$$

Assume that (1) *and* (2) *are uniformly valid in $D_{\bar{x}}$. Then the following hold*:

(i) $$I(n, \bar{x}) \sim \sum_{r=1}^{\infty} \frac{b_r}{n^{r+1/2}} \quad \textit{uniformly for } \bar{x} \in D_{\bar{x}},$$

(ii) $$\frac{\partial I(n, \bar{x})}{\partial \bar{x}} \textit{ is continuous in } \bar{x} \textit{ and bounded in } D_n \times D_{\bar{x}},$$

(iii) $$\frac{\partial I(n, \bar{x})}{\partial \bar{x}} \sim \sum_{r=1}^{\infty} h_r(\bar{x}) \frac{1}{n^{r+1/2}} \quad \textit{uniformly with respect to } \bar{x} \in D_{\bar{x}},$$

where $h_r(\bar{x}) = db_r/d\bar{x}$ for $r = 1, 2, \ldots$.

PROOF. Part (i) is readily obtained upon substituting the series (2) for $F_n(\bar{x})$ in (15). To show (ii) is true, differentiate both sides of (15) with respect to $\bar{x}$. We obtain

$$\begin{aligned} f_n(\bar{x}) = g_n(\bar{x})\Bigg[1 - \frac{(\hat{K}'')^{-1/2} db_0/d\hat{T}}{n} &+ n^{1/2}\hat{T}(\hat{K}'')^{1/2} I(n, \bar{x}) \\ &- n^{-1/2}(\hat{K}'')^{1/2} \frac{\partial I(n, \bar{x})}{\partial \bar{x}}\Bigg]. \end{aligned} \tag{16}$$

Define $D_1(n, \bar{x})$ and $L_1(n, \bar{x})$ by the following relations

$$\frac{f_n(\bar{x})}{g_n(\bar{x})} = 1 + \frac{a_1}{n} + \frac{D_1(n, \bar{x})}{n^2}, \tag{17}$$

$$I(n, \bar{x}) = \frac{b_1}{n^{1+1/2}} + \frac{L_1(n, \bar{x})}{n^{2+1/2}}. \tag{18}$$

For any finite n, $D_1(n, \bar{x})$ and $L_1(n, \bar{x})$ are both bounded due to the boundedness of other terms in (17) and (18). When n approaches infinity, by the uniform validity assumption of (1) and (2), they remain bounded on $D_n \times D_{\bar{x}}$. They are also continuous in $\bar{x}$ since $f_n(\bar{x})/g_n(\bar{x})$ and $I(n, \bar{x})$, and a_1 and b_1 are all continuous in $\bar{x}$.

Substitute (17) and (18) into (16), and use the identity $a_1 = \hat{T}(\hat{K}'')^{1/2}b_1 - (\hat{K}'')^{-1/2}db_0/d\hat{T}$, to obtain

$$\frac{\partial I(n, \bar{x})}{\partial \bar{x}} = \hat{T}\left[\frac{L_1(n, \bar{x})}{n^{1+1/2}}\right] - (\hat{K}'')^{-1/2}\left[\frac{D_1(n, \bar{x})}{n^{1+1/2}}\right]. \tag{19}$$

Thus $\partial I(n, \bar{x})/\partial \bar{x}$ is continuous in $\bar{x}$ and bounded in $D_n \times D_{\bar{x}}$.

Finally, (iii) may be obtained by substituting (1) for $f_n(\bar{x})$ and the asymptotic series in (i) for $I(n, \bar{x})$ into (16), and collecting terms according to the powers of n. The h_r's are functions of a_r's, b_r's, $\hat{K}''$ and $\hat{T}$, and thus are continuous in $\bar{x}$. Part (ii) and Lemma 2 then imply that $h_r(\bar{x}) = db_r/d\bar{x}$ for $r = 1, 2, \ldots$. □

PROOF OF THEOREM 1. By substituting the uniformly valid expansions of $I(n, \bar{x})$ and $\partial I(n, \bar{x})/\partial \bar{x}$ in Lemma 3 into (16) and collecting terms according to the powers of n, we obtain (4). Thus (4) is indeed a uniformly valid asymptotic expansion of $f_n(\bar{x})$ for $\bar{x}$ in $D_{\bar{x}}$. □

With (4) being a valid expansion for $f_n(\bar{x})$, the uniqueness of asymptotic expansion with respect to the asymptotic sequence $\{1/n^r\}$ then implies that expansion (4) coincides with that of Daniels at each $\bar{x} \in D_{\bar{x}}$. We say that we can differentiate the Lugannani–Rice expansion (2) to obtain the Daniels expansion (1) in the sense that the formal derivative of (2), (4), is indeed (1). To conclude, we note that the uniform validity condition required by the theorem is satisfied by all commonly used continuous densities. For four important classes of densities, Daniels (1954) and Jensen (1988) showed that (1) is uniformly valid in the entire domain. Routledge and Tsao (1995) showed that (1) is, for all practical purposes, always uniformly valid on any compact subset in the interior of the domain. The uniform validity of (2) has been addressed by Lugannani and Rice (1980) and Daniels (1987) in a formal manner. Tsao (1996) contains a detailed proof that (2) is in general uniformly valid on any compact subset in the interior. We now discuss two applications of the above result under the assumption that the uniform validity condition in the theorem is satisfied.

4. Applications.

4.1. *The derivative of a truncated Lugannani–Rice series as an asymptotic approximation to the density function.* Denote the sum of the first $m + 2$ terms, including $\Phi(\hat{W}n^{1/2})$, of (2) by $F_n^{(m)}(\bar{x})$, and the sum of the first $m + 1$ terms in (1) by $f_n^{(m)}(\bar{x})$. By differentiating $F_n^{(m)}(\bar{x})$ and then using the equation

$c_r = a_r$, $r = 0, 1, \ldots,$ we obtain

$$\frac{dF_n^{(m)}(\bar{x})}{d\bar{x}} = g_n(\bar{x})\left[\sum_{r=0}^{m} \frac{a_r}{n^r} - \frac{(K'')^{1/2}}{n^{m+1}} \frac{db_m}{d\bar{x}}\right]. \tag{20}$$

Since $f_n^{(m)}(\bar{x}) = g_n(\bar{x}) \sum_{r=0}^{m} a_r/n^r$ satisfies

$$f_n(\bar{x}) = g_n(\bar{x})\left[\sum_{r=0}^{m} \frac{a_r}{n^r} + O\left(\frac{1}{n^{m+1}}\right)\right], \tag{21}$$

it follows that

$$f_n(\bar{x}) = g_n(\bar{x})\left[\sum_{r=0}^{m} \frac{a_r}{n^r} - \frac{(K'')^{1/2}}{n^{m+1}} \frac{db_m}{d\bar{x}} + O\left(\frac{1}{n^{m+1}}\right)\right]. \tag{22}$$

Equations (20) and (22) imply that the derivative of $F_n^{(m)}(\bar{x})$ is an asymptotic approximation for $f_n(\bar{x})$ and that its error is of the same order as that of $f_n^{(m)}(\bar{x})$.

We give prominence to the derivative of $F_n^{(0)}(\bar{x})$ given below:

$$\frac{dF_n^{(0)}(\bar{x})}{d\bar{x}} = g_n(\bar{x})\left[1 - \frac{1}{n}\left(\frac{|\hat{T}|(\hat{K}'')^{1/2}}{[2(\hat{T}\hat{K}' - \hat{K})]^{3/2}} - \frac{1}{\hat{T}^2 \hat{K}''} - \frac{\hat{K}^{(3)}}{2\hat{T}(\hat{K}'')^2}\right)\right]. \tag{23}$$

We shall refer to the right-hand side of (23) as the *adjusted saddlepoint approximation*, and the second term in the square brackets as the adjustment term. It is clear from (22) that the relative error of the adjusted saddlepoint approximation is $O(1/n)$, the same as that of the saddlepoint approximation. Also, computationally it requires little effort beyond that needed for computing the saddlepoint approximation.

The advantage of the adjusted saddlepoint approximation is that it in general does not need to be numerically renormalized since $F_n^{(0)}(\bar{x})$ generally approaches 0 (1) when $\bar{x}$ approaches the lower (upper) end of its domain. However, it also raises the following concerns: (1) it could be negative when the adjustment term is greater than 1; (2) the adjustment term may compromise the accuracy of the original saddlepoint approximation. Nevertheless, our experience with the adjusted saddlepoint approximation has yet to validate these concerns. The first problem can only emerge when $F_n^{(0)}(\bar{x})$ is a decreasing function of $\bar{x}$. We have not found any example where this happens. Based on examples that we looked at, the adjusted saddlepoint approximation is actually more accurate than the original and is often substantially more accurate near the mean. The following tables further illustrate this point. They contain, along with the exact values, the values of the saddlepoint approximation (spa) and of the adjusted saddlepoint approximation (aspa). Table 1 is for the case where the underlying distribution is gamma with both parameters equal to 2, and a sample size of 3. Table 2 is for the case where the underlying distribution is uniform$[-1, 1]$, and a sample size of 5. Renormalization, although it can be quite involved, will in general improve the accuracy of the

TABLE 1
Approximations to the density function for the mean of three independent observations from a gamma(2, 2) distribution

$\bar{x}$	0.5	1.5	2.5	3.5	4.5
spa	0.6133652	0.3694524	0.0117770	0.0001570	1.3673e − 06
aspa	0.6049278	0.3644115	0.0116238	0.0001551	1.3510e − 06
exact	0.6049129	0.3643613	0.0116147	0.0001548	1.3484e − 06

saddlepoint approximation. The renormalized spa values for the uniform case may be found in Field and Ronchetti (1990) and are indeed more accurate than the unrenormalized ones. However, even compared with these renormalized values, aspa values are still more accurate.

When $\bar{x}$ is near the mean μ, the individual terms making up the adjustment term are seen to be large. The adjustment term, as given in (23), is undefined at $\bar{x} = \mu$ where $\hat{T} = 0$. One may thus be concerned with the accuracy of adjusted saddlepoint approximation near μ. We now show that as $\bar{x}$ approaches μ the adjustment term has a finite limit and that this limit equals $-a_1/n$, the second term in the Daniels expansion (1). Using the identity $a_1 = c_1$, the adjustment term may be expressed in terms of a_1 as

$$\frac{1}{n}(\hat{K}'')^{-1/2}\frac{db_0}{d\hat{T}} = \frac{1}{n}\big[\hat{T}(\hat{K}'')^{1/2}b_1 - a_1\big]. \tag{24}$$

As $\bar{x}$ approaches μ, $\hat{T}$ approaches 0 and b_1 has a finite limit. See Daniels [(1987), (4.8)] for a proof of the latter point. Thus the right-hand side of (24) approaches $-a_1/n$. It follows that, in the neighborhood of μ where $\hat{T}(\hat{K}'')^{1/2}b_1 = O(1/n)$, the adjusted saddlepoint approximation satisfies

$$\begin{aligned}\frac{dF_n^{(0)}(\bar{x})}{d\bar{x}} &= g_n(\bar{x})\left\{1 - \frac{1}{n}\big[\hat{T}(\hat{K}'')^{1/2}b_1 - a_1\big]\right\}\\ &= g_n(\bar{x})\left\{1 + \frac{a_1}{n} + O\left(\frac{1}{n^2}\right)\right\}.\end{aligned}$$

This means that near the mean the adjusted saddlepoint approximation is a second-order approximation and agrees with the examples where it is seen to be more accurate than the original saddlepoint approximation.

TABLE 2
Approximations to the density function for the mean of five independent observations from a uniform[−1, 1] distribution

$\bar{x}$	0.1	0.3	0.5	0.7	0.9
spa	1.4461734	0.8411568	0.2628890	0.0340165	0.0004137
aspa	1.4022376	0.8128860	0.2521337	0.0323275	0.0004034
exact	1.4021810	0.8121745	0.2522786	0.0329590	0.0004069

To conclude, when asymptotic approximations to both $F_n(\bar{x})$ and $f_n(\bar{x})$ are sought, the adjusted saddlepoint approximation is a particularly attractive alternative to saddlepoint approximation since the corresponding approximation to $F_n(\bar{x})$, $F_n^{(0)}(\bar{x})$, is easily available.

4.2. *Lugannani and Rice's conjecture.* We examine this conjecture in a simple setting where the domain of $\bar{x}$ coincides with the support of X_i, and $F_n^{(m)}(\bar{x})$ approaches 0 (1) when $\bar{x}$ approaches the lower (upper) endpoint of its domain. Since

$$F_n(\bar{x}) - F_n^{(m)}(\bar{x}) = \phi(\hat{W}n^{1/2})O(1/n^{m+3/2}), \tag{25}$$

the conjecture may be formulated as

$$F_n(\bar{x}) - \int_{s_l}^{\bar{x}} f_n^{(m)}(y)\,dy = \phi(\hat{W}n^{1/2})O(1/n^{m+3/2}), \tag{26}$$

where s_l is the lower end of the support. We now rewrite the derivative of $F_n^{(m)}(y)$, given by (20), at a point y in the domain as

$$\frac{dF_n^{(m)}(y)}{dy} = f_n^{(m)}(y) - \frac{1}{n^{m+1/2}}\phi(Wn^{1/2})\frac{db_m}{dT}\frac{dT}{dy}, \tag{27}$$

where $T = T(y)$ is the saddlepoint corresponding to y and $W = \operatorname{sgn}(T)\{2[TK'(T) - K(T)]\}^{1/2}$. By integrating both sides of (27) from s_l to $\bar{x}$, we obtain

$$F_n^{(m)}(\bar{x}) = \int_{s_l}^{\bar{x}} f_n^{(m)}(y)\,dy - \frac{1}{n^{m+1/2}}R_n^{(m)}(\bar{x}), \tag{28}$$

where

$$R_n^{(m)}(\bar{x}) = \int_{T(s_l)}^{\hat{T}} \phi(Wn^{1/2})\frac{db_m}{dT}\,dT. \tag{29}$$

Equations (25) and (28) lead to the difference between $F_n(\bar{x})$ and integrated $f_n^{(m)}(\bar{x})$,

$$F_n(\bar{x}) - \int_{s_l}^{\bar{x}} f_n^{(m)}(y)\,dy = \phi(\hat{W}n^{1/2})O\left(\frac{1}{n^{m+3/2}}\right) - \frac{1}{n^{m+1/2}}R_n^{(m)}(\bar{x}). \tag{30}$$

Since $-W^2(T)$ has only one extremum at $T = 0$ and this extremum is a maximum, we use Laplace's method to expand $R_n^{(m)}(\bar{x})$ and obtain

$$R_n^{(m)}(\bar{x}) = \begin{cases} \phi(\hat{W}n^{1/2})O(1/n), & \text{if } \bar{x} < E(\bar{X}), \\ O(1/n^{1/2}), & \text{if } \bar{x} \geq E(\bar{X}). \end{cases} \tag{31}$$

It follows from (30) and (31) that

$$F_n(\bar{x}) - \int_{s_l}^{\bar{x}} f_n^{(m)}(y)dy = \begin{cases} \phi(\hat{W}n^{1/2})O(1/n^{m+3/2}), & \text{if } \bar{x} < E(\bar{X}), \\ O(1/n^{m+1}), & \text{if } \bar{x} \geq E(\bar{X}). \end{cases} \tag{32}$$

For $\bar{x} > E(\bar{X})$, $F_n(\bar{x})$ may be more accurately approximated by subtracting from 1 the integral of $f_n^{(m)}(y)$ over $(\bar{x}, s_u)$, where s_u is the upper end of the support. By essentially repeating the above procedure, we obtain

$$F_n(\bar{x}) - \left(1 - \int_{\bar{x}}^{s_u} f_n^{(m)}(y)\,dy\right) = \phi(\hat{W}n^{1/2})O(1/n^{m+3/2}) \quad \text{if } \bar{x} > E(\bar{X}). \tag{33}$$

It follows from (32) and (33) that when $\bar{x} \neq E(\bar{X})$ the error for the integrated truncated Daniels series is $\phi(\hat{W}n^{1/2})O(1/n^{m+3/2})$, the same as that of Lugannani and Rice. At the mean, it is $O(1/n^{m+1})$, but that of Lugannani and Rice is $O(1/n^{m+3/2})$. Thus Lugannani and Rice's conjecture, as formulated in (26), is correct everywhere, except at the mean.

In particular, for $m = 0$, this result implies that the first approximation given by Lugannani and Rice's expansion [i.e., $QA(1)$ in Lugannani and Rice (1980) or (4.9) in Daniels (1987)] is asymptotically at least as accurate as the integrated saddlepoint approximation. This suggests that one should use the first approximation instead of the integrated saddlepoint approximation since, unlike the latter, it does not require numerical integration.

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