The Annals of Statistics
2003, Vol. 31, No. 3, 1012–1026

HIDDEN PROJECTION PROPERTIES OF SOME NONREGULAR FRACTIONAL FACTORIAL DESIGNS AND THEIR APPLICATIONS[1]

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In factor screening, often only a few factors among a large pool of potential factors are active. Under such assumption of effect sparsity, in choosing a design for factor screening, it is important to consider projections of the design onto small subsets of factors. Cheng showed that as long as the run size of a two-level orthogonal array of strength two is not a multiple of 8, its projection onto any four factors allows the estimation of all the main effects and two-factor interactions when the higher-order interactions are negligible. This result applies, for example, to all Plackett–Burman designs whose run sizes are not multiples of 8. It is shown here that the same hidden projection property also holds for Paley designs of sizes greater than 8, even when their run sizes are multiples of 8. A key result is that such designs do not have defining words of length three or four. Applications of this result to the construction of $E(s^2)$-optimal supersaturated designs are also discussed. In particular, certain designs constructed by using Wu's method are shown to be $E(s^2)$-optimal. The article concludes with some three-level designs with good projection properties.

1. Introduction. In the initial stage of experimentation, one may have to consider a large number of potentially important factors while only a few of these factors are active. In designing experiments for factor screening, it is important to consider projections of the design onto small subsets of factors. Box and Tyssedal (1996) defined a design to be of projectivity p if in every subset of p factors, a complete factorial (possibly with some combinations replicated) is produced. When there are no more than p active factors, no matter what these factors are, the projection of a design of projectivity p onto the active factors allows all the factorial effects to be estimated. This important concept was also discussed on page 363 of Constantine (1987), and appeared in another context: p-projectivity is the same as p-covering in designs for circuit testing; see, for example, Sloane (1993).

Projectivity can be viewed as an extension of the concept of strength of an orthogonal array. Recall that an orthogonal array of size N, m constraints, s levels and strength t, denoted by $OA(N, s^m, t)$, is an $N \times m$ matrix $\mathbf{X}$ of s symbols such that all the ordered t-tuples of the symbols occur equally often as row vectors

Received March 2002; revised August 2002.

[1]Supported in part by the NSF and the NSA.

AMS 2000 subject classification. 62K15.

Key words and phrases. $E(s^2)$-optimality, Hadamard matrix, orthogonal array, Paley design, Plackett–Burman design, supersaturated design.

of any $N \times t$ submatrix of $\mathbf{X}$. Each $OA(N, s^m, t)$ defines an N-run factorial design for m s-level factors, with the symbols representing factor levels, columns corresponding to factors and rows representing factor-level combinations. Such a design has projectivity t since in every subset of t factors, all the factor-level combinations are replicated the same number of times.

An important class of orthogonal arrays are the so-called regular fractional factorial designs which are constructed by using defining relations. Each effect that appears in the defining relation is called a defining word and the length of the shortest defining word is called the resolution of the design. It is well known that a regular fractional factorial design of resolution R is an orthogonal array of strength $R - 1$. The projection property of such a design can be studied via its defining relation in a straightforward manner. For example, since the strength is $R - 1$, the design has projectivity $R - 1$. However, it cannot have projectivity greater than $R - 1$; projecting the design onto the R factors that appear in a defining word of length R does not produce a complete factorial.

On the other hand, it is possible for a nonregular orthogonal array of maximum strength t to have projectivity greater than t. Lin and Draper (1992) and Box and Bisgaard (1993) discovered that certain small Plackett–Burman designs (which are orthogonal arrays of strength two) have projectivity three. For example, each of the 165 projections of a 12-run Plackett–Burman design onto three factors consists of a complete 2^3 design and a half-replicate of 2^3. The 12-run Plackett–Burman design does not have projectivity four, but in its projection onto any four factors, all the main effects and two-factor interactions are estimable if the higher-order interactions are negligible [see Lin and Draper (1993) and Wang and Wu (1995)]. A regular design needs resolution at least five to have such a hidden projection property.

Cheng (1995) showed that a key to the hidden projection property described above is the absence of defining words of length three or four. While this concept will be precisely defined for nonregular designs as well at the end of this section, roughly speaking, a design (regular or nonregular) has a defining word of length k if it causes a main effect to be completely aliased with a $(k - 1)$-factor interaction. If a design has a defining word of length three or four, then certain two-factor interactions are completely aliased with main effects or other two-factor interactions. As a result, when the design is projected onto the factors involved, not all the main effects and two-factor interactions can be estimated. Thus a necessary condition for a design to have the property that the main effects and two-factor interactions are estimable in all four-factor projections is that it has no defining word of length three or four. Cheng (1995) showed that for two-level orthogonal arrays of strength two, this condition is also sufficient. For a two-level orthogonal array of strength three, the main effects and two-factor interactions are estimable in all five-factor projections if and only if the design has no defining word of length four [Cheng (1998a)].

The desirability to have no defining words of certain (short) lengths also arises in other situations. For example, Lin (1993) proposed a method of constructing two-level supersaturated designs from Hadamard matrices. Nguyen (1996) and Cheng (1997) showed that his method produces optimal designs under the $E(s^2)$-criterion proposed by Booth and Cox (1962), provided that the resulting designs do not have completely aliased factors. It can easily be seen that this is the case if the method is applied to Hadamard matrices that are equivalent to orthogonal arrays without defining words of length three. Wu (1993) proposed another method of constructing supersaturated designs from Hadamard matrices. Similarly, his method produces designs without completely aliased factors if it is applied to Hadamard matrices that are equivalent to orthogonal arrays without defining words of length three or four.

It is clear that if N is not a multiple of 8, then any $OA(N, 2^m, 2)$ with $m \geq 4$ has no defining word of length three or four. This leads to Cheng's (1995) result that such a design has projectivity three, and in all its four-factor projections, the main effects and two-factor interactions are estimable. If the conjecture that a Hadamard matrix exists for every order that is a multiple of 4 is true, then for every N that is a multiple of 8, one can always construct a strength-two orthogonal array of size N that has defining words of length three or four; see Cheng (1995). Such designs do not have the desirable projection properties mentioned above. However, this does not mean that when N is a multiple of 8, there are no OA(N, 2^m, 2)'s with good hidden projection properties. Cheng (1998b) reported an $OA(24, 2^{23}, 2)$ that has projectivity three and the four-factor projection property described earlier. The same paper also presented an $OA(32, 2^{31}, 2)$ with the stronger property that its projection onto any six factors allows the estimation of all the main effects and two-factor interactions. Both of these arrays are members of a family of designs called Paley designs.

One objective of this paper is to show in Section 2 that a Paley design of size greater than 8 has no defining words of length three or four, and therefore has desirable hidden projection properties. Section 3 contains some applications of this result to supersaturated designs. In particular, we show that certain supersaturated designs obtained by using Wu's (1993) method are $E(s^2)$-optimal. Note that Wu (1993) contains no optimality results. Section 4 considers s-level designs with good projection properties where $s > 2$. For example, Cheng and Wu (2001) reported an $OA(27, 3^8, 2)$ with the property that in all its four-factor projections, the quadratic model is estimable. We find an $OA(27, 3^8, 2)$ such that the quadratic model is estimable in all the five-factor projections. If one does not insist on using an orthogonal array, then an even smaller design with the aforementioned projection property can be constructed.

Throughout this paper, an N-run design with m two-level factors is represented by an $N \times m$ matrix $[d_{ij}]$, where d_{ij} is the level of the jth factor in the ith run. If a factor has two levels, then they are represented by 1 and -1. For two vectors $\mathbf{x} = (x_1, \ldots, x_N)^T$ and $\mathbf{y} = (y_1, \ldots, y_N)^T$, the Hadamard product of $\mathbf{x}$ and $\mathbf{y}$ is defined as $\mathbf{x} \odot \mathbf{y} = (x_1 y_1, \ldots, x_N y_N)^T$. An N-run, two-level design $\mathbf{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_m]$ is said to have a defining word of length k if there exist k columns of $\mathbf{D}$, say columns $j_1, \ldots, j_k$, such that $\mathbf{d}_{j_1} \odot \cdots \odot \mathbf{d}_{j_k} = \mathbf{1}_N$ or $-\mathbf{1}_N$, where $\mathbf{1}_N$ is the $N \times 1$ column of ones. Two columns $\mathbf{d}_i$ and $\mathbf{d}_j$ of $\mathbf{D}$ are said to be completely aliased if $\mathbf{d}_i = \mathbf{d}_j$ or $\mathbf{d}_i = -\mathbf{d}_j$. Finally, a finite field with q elements is denoted by $GF(q)$.

2. Paley designs. Paley matrices are a family of Hadamard matrices constructed by Paley (1933). Suppose N is a multiple of 4 such that $N - 1$ is an odd prime power. Let $q = N - 1$ and let $\alpha_1 = 0, \alpha_2, \ldots, \alpha_q$ denote the elements of $GF(q)$. Define a function $\chi : GF(q) \to \{0, 1, -1\}$ by

$$\chi(\beta) = \begin{cases} 1, & \text{if } \beta = y^2 \text{ for some } y \in GF(q), \\ 0, & \text{if } \beta = 0, \\ -1, & \text{otherwise.} \end{cases}$$

Let $\mathbf{A}$ be the $q \times q$ matrix $[a_{ij}]$, where $a_{ij} = \chi(\alpha_i - \alpha_j)$ for $i, j = 1, 2, \ldots, q$, and

$$\mathbf{P}_N = \begin{bmatrix} 1 & -\mathbf{1}_q^T \\ \mathbf{1}_q & \mathbf{A} + \mathbf{I}_q \end{bmatrix}, \tag{2.1}$$

where $\mathbf{I}_q$ is the identity matrix of order q. Then $\mathbf{P}_N$ is a Hadamard matrix. For a proof see Hedayat, Sloane and Stufken (1999). The matrix $\mathbf{P}_N$ is known as a Paley matrix of the first kind.

It is well known that a Hadamard matrix of order N is equivalent to an $OA(N, 2^{N-1}, 2)$. Without loss of generality, we may assume that all the entries of the first column of a Hadamard matrix are equal to 1. Then an $OA(N, 2^{N-1}, 2)$ can be obtained by deleting the first column. We shall call the orthogonal array obtained by deleting the first column of $\mathbf{P}_N$ a Paley design, and denote it by $\mathbf{D}_N$. For $N = 12$, the Paley design $\mathbf{D}_{12}$ is the 12-run Plackett–Burman design.

Projection properties of a design can be revealed by its J-characteristics as defined by Deng and Tang (1999). Suppose $\mathbf{D}$ is an N-run design with m two-level factors. Then for $1 \le k \le m$ and any k-subset $s = \{h_1, \ldots, h_k\}$, let

$$j_k(s) = \sum_{i=1}^{N} d_{ih_1} d_{ih_2} \cdots d_{ih_k}.$$

The value $j_k(s)$ is called the J-characteristic of $\mathbf{D}$ associated with the factors in s. Clearly, a design $\mathbf{D}$ has a defining word of length k if and only if there is a k-subset s of $\{1, \ldots, m\}$ such that $|j_k(s)| = N$, and it is an orthogonal array with strength t if and only if $j_k(s) = 0$ for all k-subsets s of $\{1, \ldots, m\}$ with $k \le t$. Thus showing the lack of defining words of length k in a design $\mathbf{D}$ of size N is equivalent to showing that $|j_k(s)| < N$ for all k-subsets s of $\{1, \ldots, m\}$.

A result from number theory (Riemann hypothesis for curves over finite fields) can be used to establish the following bounds on $j_k(s)$ for Paley designs:

THEOREM 2.1. *Let $\mathbf{D}_N$ be a Paley design of size N, where $q = N - 1$ is a prime power. Then for any k-subset s of $\{1, \dots, N-1\}$, $|j_k(s)| \le k+1+(k-1) \times (N-1)^{1/2}$ if k is odd, and $|j_k(s)+1| \le k+1+(k-2)(N-1)^{1/2}$ if k is even.*

PROOF. Let $s = \{h_1, \dots, h_k\}$. Then by (2.1),

$$\begin{aligned} j_k(s) &= (-1)^k + \sum_{1 \le i \le q, i \ne h_1, \dots, h_k} \chi(\alpha_i - \alpha_{h_1}) \cdots \chi(\alpha_i - \alpha_{h_k}) \\ &\quad + \sum_{j=1}^{k} \prod_{1 \le j' \le k, j' \ne j} \chi(\alpha_{h_j} - \alpha_{h_{j'}}) \\ &= \sum_{1 \le i \le q, i \ne h_1, \dots, h_k} \chi(\alpha_i - \alpha_{h_1}) \cdots \chi(\alpha_i - \alpha_{h_k}) + e, \end{aligned} \tag{2.2}$$

where $|e| \le k+1$.

Now write (2.2) as

$$j_k(s) = \sum_{y \in GF(q), y \ne \alpha_{h_1}, \dots, \alpha_{h_k}} \chi(y - \alpha_{h_1}) \cdots \chi(y - \alpha_{h_k}) + e. \tag{2.3}$$

We have $(y - \alpha_{h_1}) \cdots (y - \alpha_{h_k}) = y^k + a_1 y^{k-1} + a_2 y^{k-2} + \cdots + a_k$ for some $a_1, \dots, a_k \in GF(q)$. Since $\chi(xy) = \chi(x)\chi(y)$ for all x and y, by (2.3),

$$j_k(s) = \sum_{y \in GF(q), y \ne \alpha_{h_1}, \dots, \alpha_{h_k}} \chi(y^k + a_1 y^{k-1} + a_2 y^{k-2} + \cdots + a_k) + e. \tag{2.4}$$

Let N_q be the number of solutions (z, y), $z, y \in GF(q)$, of the equation

$$z^2 = y^k + a_1 y^{k-1} + a_2 y^{k-2} + \cdots + a_k. \tag{2.5}$$

Since $\alpha_{h_1}, \dots, \alpha_{h_k}$ are the zeros of $y^k + a_1 y^{k-1} + a_2 y^{k-2} + \cdots + a_k = (y - \alpha_{h_1}) \cdots (y - \alpha_{h_k})$, there are $N_q - k$ solutions (z, y) of (2.5) with $z \ne 0$. Also, every pair of solutions of (2.5) of the form (z, y) and $(-z, y)$, where $z \ne 0$, corresponds to one value of $y^k + a_1 y^{k-1} + a_2 y^{k-2} + \cdots + a_k$. Therefore, for $y \in GF(q)$, $y \ne \alpha_{h_1}, \dots, \alpha_{h_k}$,

$$\begin{aligned} &\chi(y^k + a_1 y^{k-1} + a_2 y^{k-2} + \cdots + a_k) \\ &\qquad = \begin{cases} 1, & (N_q - k)/2 \text{ times}, \\ -1, & q - k - (N_q - k)/2 \text{ times}. \end{cases} \end{aligned} \tag{2.6}$$

By (2.4) and (2.6),

$$j_k(s) = (N_q - k)/2 - \{q - k - (N_q - k)/2\} + e = N_q - q + e. \tag{2.7}$$

On the other hand, since $\alpha_{h_1}, \dots, \alpha_{h_k}$ are distinct, the polynomial $y^k + a_1 y^{k-1} + a_2 y^{k-2} + \cdots + a_k = (y - \alpha_{h_1}) \cdots (y - \alpha_{h_k})$ has no double roots. Thus by a result of Hasse (1936) and Weil (1948) as quoted in Stark (1973), we have

$$|N_q - q| \le (k-1)q^{1/2} \qquad \text{for odd } k \tag{2.8}$$

and

$$|N_q - q + 1| \le (k-2)q^{1/2} \qquad \text{for even } k. \tag{2.9}$$

Since $q = N-1$, by (2.7), (2.8) and (2.9), we have $|j_k(s) - e| \le (k-1)(N-1)^{1/2}$ for odd k, and $|j_k(s) - e + 1| \le (k-2)(N-1)^{1/2}$ for even k. This and $|e| \le k+1$ imply that $|j_k(s)| \le k+1+(k-1)(N-1)^{1/2}$ for odd k, and $|j_k(s)+1| \le k+1+(k-2)(N-1)^{1/2}$ for even k. □

Theorem 2.1 can be used to investigate the existence of defining words of length k in a Paley design $\mathbf{D}_N$. Suppose there is a defining word of length k in $\mathbf{D}_N$. Then there is a k-subset s of $\{1, \dots, N-1\}$ such that $|j_k(s)| = N$. By Theorem 2.1,

$$N \le \begin{cases} k+1+(k-1)(N-1)^{1/2}, & \text{if } k \text{ is odd,} \\ k+2+(k-2)(N-1)^{1/2}, & \text{if } k \text{ is even.} \end{cases} \tag{2.10}$$

This inequality cannot hold when N is sufficiently large. Thus for each k, there is a positive integer $N(k)$ such that there is no defining word of length k in any Paley design $\mathbf{D}_N$ with $N > N(k)$. For example, for $k = 3$ or 4, (2.10) does not hold for all multiples of 4 that are greater than 8 or 12, respectively. Since it is already known that $\mathbf{D}_{12}$ has no defining words of length four, this proves the following result.

THEOREM 2.2. *Let $\mathbf{D}_N$ be a Paley design of size N, $N \ge 12$. Then $\mathbf{D}_N$ has no defining word of length three or four.*

Note that the Paley design of size 8 is a regular design of resolution three, thus Theorem 2.2 does not hold for $N = 8$.

Combining the results in Cheng (1995) with Theorem 2.2, we conclude the following projection property of Paley designs.

COROLLARY 2.3. *Let $\mathbf{D}_N$ be a Paley design of size N, $N \ge 12$. Then $\mathbf{D}_N$ has projectivity three, and in its projection onto any four factors, the main effects and two-factor interactions are estimable under the assumption that the higher-order interactions are negligible.*

Theorem 2.1 can also be used to narrow down the possible values of $j_k(s)$. The following result is useful for this purpose.

LEMMA 2.4. *Suppose* $\mathbf{X}$ *is an* $OA(N, 2^m, t)$ *with* $m \geq t+2$, *where* t *is an even positive integer. Let* $N = 2^t\lambda$. *Then for* $k = t+1$ *and* $t+2$, $j_k(s)$ *is an odd multiple of* 2^t *if* λ *is odd, and is an even multiple of* 2^t *if* λ *is even.*

PROOF. Let $s = \{i_1, \dots, i_{t+1}\}$ be a $(t+1)$-subset of $\{1, \dots, m\}$ and let $\mathbf{Y}$ be the $N \times (t+1)$ submatrix of $\mathbf{X}$ consisting of columns $i_1, \dots, i_{t+1}$. Then by Theorem 2.1 of Cheng (1995), there exist two integers α and β such that each vector $(x_1, \dots, x_{t+1})$ with $x_1 \cdots x_{t+1} = 1$, where $x_i = 1$ or -1, appears α times as row vectors of $\mathbf{Y}$, and those with $x_1 \cdots x_{t+1} = -1$ appear β times. Then $j_{t+1}(s) = 2^t(\alpha - \beta)$ and $2^t(\alpha+\beta) = N = 2^t\lambda$. This implies that $j_{t+1}(s) = 2^t(2\alpha - \lambda)$. Thus $j_{t+1}(s)$ is an odd multiple of 2^t if λ is odd, and is an even multiple of 2^t if λ is even. The result for $j_{t+2}(s)$ follows the same argument by applying Corollary 3.1 of Cheng (1995). $\square$

As an application of Theorem 2.1 and Lemma 2.4, consider, for example, Paley designs of order 24. By Theorem 2.1, we have $-16 < j_4(s) < 16$ for all 4-subsets s of $\{1, \dots, 23\}$. On the other hand, by Lemma 2.4, $j_4(s)$ must be a multiple of 8. Therefore the only possible values of $j_4(s)$ are 0, 8 and -8.

3. Supersaturated designs. Supersaturated two-level designs are those with the number of factors m greater than or equal to the run size N. Lin (1993) proposed a method of constructing supersaturated designs from Hadamard matrices. Let $\mathbf{H}$ be an $n \times n$ Hadamard matrix. Without loss of generality, $\mathbf{H}$ can be written as

$$\mathbf{H} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{H}^h \\ \mathbf{1} & -\mathbf{1} & * \end{bmatrix},$$

where $\mathbf{1}$ is the $n/2 \times 1$ vector of 1's. If no two columns of $\mathbf{H}^h$ are completely aliased, then $\mathbf{H}^h$ defines a supersaturated design with $n-2$ factors and $n/2$ runs. Nguyen (1996) and Cheng (1997) showed that this design is optimal under the $E(s^2)$-criterion proposed by Booth and Cox (1962). Note that it is essential that no two columns of a supersaturated design are completely aliased so that there are no redundant factors. In Lin's construction, if certain redundant factors must be removed to produce a legitimate design, then the resulting design may no longer be $E(s^2)$-optimal.

It can be seen that no two columns of $\mathbf{H}^h$ are completely aliased if the orthogonal array

$$\mathbf{H}' = \begin{bmatrix} \mathbf{1} & \mathbf{H}^h \\ -\mathbf{1} & * \end{bmatrix}$$

has no defining word of length three. This is because if $\mathbf{H}'$ has no defining word of length three, then it has projectivity at least three. Projecting $\mathbf{H}'$ onto its first column and two other arbitrary columns, we see that $\mathbf{H}^h$ must have projectivity at least two, and therefore has no completely aliased columns. As shown in Theorem 2.2, Paley designs of sizes ≥ 12 have no defining words of length three. Thus when applied to a Paley matrix of order greater than or equal to 12, Lin's (1993) method always yields an $E(s^2)$-optimal supersaturated design with no completely aliased factors.

Wu (1993) proposed another method of constructing supersaturated designs, also based on Hadamard matrices. Let $\mathbf{X}$ be an $OA(N, 2^{N-1}, 2)$ obtained by deleting a column of 1's from an $N \times N$ Hadamard matrix. Supplementing $\mathbf{X}$ by f pairwise Hadamard products of its columns, where $f \leq \binom{N-1}{2}$, a supersaturated design with $N - 1 + f$ factors is obtained if no two of the f added columns are completely aliased and none of which is completely aliased with a column of $\mathbf{X}$. It is easy to see that this is the case if $\mathbf{X}$ has no defining word of length three or four. Again this holds for Paley designs of sizes $N \geq 12$.

Wu's (1993) method can be used to construct an N-run design with up to $N-1+\binom{N-1}{2}$ factors. These designs, however, may not be $E(s^2)$-optimal. Indeed, Wu (1993) reports no optimality result. We shall show in the following that in certain cases, including when $f = \binom{N-1}{2}$, the designs constructed by Wu's (1993) method are $E(s^2)$-optimal. Thus, for example, the design obtained by adding to the 12-run Plackett–Burman design all the $\binom{11}{2} = 55$ pairwise Hadamard products of its columns is an $E(s^2)$-optimal design for 66 factors in 12 runs.

THEOREM 3.1. *Let $\mathbf{X}$ be an $OA(N, 2^{N-1}, 2)$ obtained by deleting a column of 1's from an $N \times N$ Hadamard matrix, and let $\mathbf{D}^{\text{ext}}$ be the design obtained by adding to $\mathbf{X}$ all the $\binom{N-1}{2}$ pairwise Hadamard products of its columns. If $\mathbf{D}^{\text{ext}}$ contains no completely aliased columns, then it is an $E(s^2)$-optimal supersaturated design with N runs and $N - 1 + \binom{N-1}{2}$ factors.*

PROOF. Let the $N - 1$ columns of $\mathbf{X}$ be $\mathbf{c}_1, \dots, \mathbf{c}_{N-1}$, $\mathbf{c}_0 = \mathbf{1}_N$, and $\mathbf{H} = [\mathbf{c}_0 \vdots \mathbf{c}_1 \vdots \cdots \vdots \mathbf{c}_{N-1}]$. Then $\mathbf{H}$ is a Hadamard matrix. Define $\mathbf{M}$ to be the matrix

$$\mathbf{M} = [\mathbf{c}_0 \odot \mathbf{H} \vdots \mathbf{c}_1 \odot \mathbf{H} \vdots \cdots \vdots \mathbf{c}_{N-1} \odot \mathbf{H}], \tag{3.1}$$

where the jth column of $\mathbf{c}_i \odot \mathbf{H}$ is $\mathbf{c}_i \odot \mathbf{c}_j$, $0 \leq j \leq N - 1$. Let $\mathbf{c}_{ij} = \mathbf{c}_i \odot \mathbf{c}_j$. We observe that the columns of $\mathbf{M}$ are all the $\mathbf{c}_{ij}$'s with $0 \leq i, j \leq N - 1$. Partition the columns of $\mathbf{M}$ as $\mathbf{M} = [\mathbf{M}_1 \vdots \mathbf{M}_2 \vdots \mathbf{M}_3]$, where the columns of $\mathbf{M}_1$ are the $\mathbf{c}_{ij}$'s with $0 \leq i < j \leq N - 1$, the columns of $\mathbf{M}_2$ are those with $0 \leq j < i \leq N - 1$, and $\mathbf{M}_3$ consists of the remaining columns, that is, the $\mathbf{c}_{ii}$'s with $0 \leq i \leq N - 1$. Then since $\mathbf{c}_{ij} = \mathbf{c}_{ji}$, we see that the columns of $\mathbf{M}_1$ coincide with those of $\mathbf{M}_2$. In fact, after proper reordering of the columns, we have $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{D}^{\text{ext}}$. By Theorem 2.1 of Cheng (1997), $\mathbf{D}^{\text{ext}}$ is $E(s^2)$-optimal if the row vectors of $[\mathbf{D}^{\text{ext}} \vdots \mathbf{J}_{N,N/2}]$ are mutually orthogonal, where $\mathbf{J}_{N,N/2}$ is the $N \times N/2$ matrix of ones.

Now since $\mathbf{c}_{ii} = c_i \odot c_i = \mathbf{1}_N$ for all $0 \le i \le N-1$, we have $\mathbf{M}_3 = \mathbf{J}_{N,N}$. Then it follows from $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{D}^{\text{ext}}$ that subject to reordering of columns, we can write $\mathbf{M}$ as $\mathbf{M} = [\mathbf{M}_1' \vdots \mathbf{M}_2']$, where $\mathbf{M}_1' = \mathbf{M}_2' = [\mathbf{D}^{\text{ext}} \vdots \mathbf{J}_{N,N/2}]$. It is clear that the rows of $[\mathbf{D}^{\text{ext}} \vdots \mathbf{J}_{N,N/2}]$ are mutually orthogonal if and only if the rows of M are mutually orthogonal, since $\mathbf{MM}^T = 2[\mathbf{D}^{\text{ext}} \vdots \mathbf{J}_{N,N/2}][\mathbf{D}^{\text{ext}} \vdots \mathbf{J}_{N,N/2}]^T$. Therefore the proof is finished if we can show that the row vectors of $\mathbf{M}$ are mutually orthogonal. This is true because each $\mathbf{c}_i \odot \mathbf{H}$ in (3.1) is an $N \times N$ Hadamard matrix. $\square$

COROLLARY 3.2. *Let* $\mathbf{X}$ *be an* $OA(N, 2^{N-1}, 2)$ *obtained by deleting a column of* 1*'s from an* $N \times N$ *Hadamard matrix, and let* $\mathbf{S}$ *be an* $N \times \binom{N-1}{2}$ *matrix whose columns consist of all* $\mathbf{c}_i \odot \mathbf{c}_j$, $1 \le i < j \le N-1$, *where* $\mathbf{c}_1, \dots, \mathbf{c}_{N-1}$ *are the columns of* $\mathbf{X}$. *If* $\mathbf{S}$ *contains no completely aliased columns, then it is an* $E(s^2)$*-optimal supersaturated design with* N *runs and* $\binom{N-1}{2}$ *factors.*

PROOF. By Theorem 2.1 of Cheng (1997), $\mathbf{S}$ is $E(s^2)$-optimal if the row vectors of $[\mathbf{S} \vdots \mathbf{J}_{N,N/2-1}]$ are mutually orthogonal, where $\mathbf{J}_{N,N/2-1}$ is the $N \times (N/2-1)$ matrix of ones. For the matrix $\mathbf{D}^{\text{ext}}$ in Theorem 3.1, we have $[\mathbf{D}^{\text{ext}} \vdots \mathbf{J}_{N,N/2}] = [\mathbf{S} \vdots \mathbf{c}_1 \vdots \cdots \vdots \mathbf{c}_{N-1} \vdots \mathbf{1}_N \vdots \mathbf{J}_{N,N/2-1}]$. Since $[\mathbf{c}_1 \vdots \cdots \vdots \mathbf{c}_{N-1} \vdots \mathbf{1}_N]$ is a Hadamard matrix, its row vectors are mutually orthogonal. This and the fact that the row vectors of $[\mathbf{D}^{\text{ext}} \vdots \mathbf{J}_{N,N/2}]$ are mutually orthogonal (see the proof of Theorem 3.1) imply that the row vectors of $[\mathbf{S} \vdots \mathbf{J}_{N,N/2-1}]$ are also mutually orthogonal. $\square$

COROLLARY 3.3. *Let* $\mathbf{X}$ *be an* $OA(N, 2^{N-1}, 2)$ *obtained by deleting a column of* 1*'s from an* $N \times N$ *Hadamard matrix, and for any* i_0, $1 \le i_0 \le N-1$, *let* $\mathbf{X}^{i_0}$ *be an* $N \times (2N-3)$ *matrix consisting of the columns* $\mathbf{c}_i$ *with* $1 \le i \le N-1$ *and* $\mathbf{c}_{i_0} \odot \mathbf{c}_j$ *with* $1 \le j \le N-1$, $j \ne i_0$, *where* $\mathbf{c}_1, \dots, \mathbf{c}_{N-1}$ *are the columns of* $\mathbf{X}$. *If* $\mathbf{X}^{i_0}$ *contains no completely aliased columns, then it is an* $E(s^2)$*-optimal supersaturated design with* N *runs and* $2N-3$ *factors.*

PROOF. Let $\mathbf{T}^{i_0}$ be the matrix

$$\mathbf{T}^{i_0} = [\mathbf{c}_0 \odot \mathbf{H} \vdots \mathbf{c}_{i_0} \odot \mathbf{H}],$$

where $\mathbf{c}_0 = \mathbf{1}_N$ and $\mathbf{H} = [\mathbf{1}_N \vdots \mathbf{X}]$. Then subject to reordering of the columns, we can write $\mathbf{T}^{i_0} = [\mathbf{1}_N \vdots \mathbf{1}_N \vdots \mathbf{X}^{i_0} \vdots \mathbf{c}_{i_0}]$. Since both $\mathbf{c}_0 \odot \mathbf{H}$ and $\mathbf{c}_{i_0} \odot \mathbf{H}$ are Hadamard matrices, the row vectors of $\mathbf{T}^{i_0}$ are mutually orthogonal. Thus by Theorem 2.1 of Cheng (1997), $[\mathbf{X}^{i_0} \vdots \mathbf{c}_{i_0}]$ would be an $E(s^2)$-optimal supersaturated design with N runs and $2N-2$ factors if it has no completely aliased columns. Since $\mathbf{c}_{i_0}$ also appears in $\mathbf{X}^{i_0}$, $[\mathbf{X}^{i_0} \vdots \mathbf{c}_{i_0}]$ is not a legitimate design, but removing $\mathbf{c}_{i_0}$ from $[\mathbf{X}^{i_0} \vdots \mathbf{c}_{i_0}]$

results in a legitimate supersaturated design $\mathbf{X}^{i_0}$ with $2N-3$ factors, which by the discussion in case 3.2 on page 934 of Cheng (1997) is $E(s^2)$-optimal. $\square$

Note that the design in Theorem 3.1 consists of all the main-effect and two-factor interaction columns in a saturated main-effect plan, that in Corollary 3.2 consists of the two-factor interaction columns only and that in Corollary 3.3 consists of all the main-effect columns and the two-factor interactions involving a given factor. If we take $\mathbf{X}$ to be a Paley design $\mathbf{D}_N$ of size N, $N \geq 12$, then by Theorem 2.2, the resulting designs always have no completely aliased columns and hence are $E(s^2)$-optimal. These are the only designs constructed by Wu's (1993) method whose $E(s^2)$-optimality we are able to prove. For example, if we try to apply the same argument as in Corollary 3.3 to show the optimality of the design given by adding all pairwise Hadamard products involving two particular factors to a saturated orthogonal array, then three columns would have to be removed from a design that minimizes $E(s^2)$. There is no guarantee that the resulting design is still $E(s^2)$-optimal.

4. Some s-level designs with good projection properties. Unlike two-level designs, projection properties of designs with more than two levels have been relatively unexplored. Cheng and Wu (2001) found a nonregular $OA(27, 3^8, 2)$ such that in all projections onto four factors, the quadratic model can be estimated. Here a quadratic model refers to the second-order model that contains the intercept, all the linear and quadratic components of main effects, and all the linear$\times$linear interactions. Specifically, when there are k factors each with three quantitative levels, under the quadratic model, the mean of each response at the combination $(x_1, \dots, x_k)$ is given by $\mu + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{1 \leq i < j \leq k} \beta_{ij} x_i x_j$, where x_i is the level of factor i, and $\mu, \beta_i, \beta_{ii}, \beta_{ij}$ are unknown constants. Suppose the levels are equally spaced. Then they can be coded as 0, 1 and 2, with 0 corresponding to the lowest level and 2 corresponding to the highest level.

The four-factor projection property as described in the previous paragraph cannot be satisfied by a regular 3^{8-5} design due to the presence of some defining words of length 3. Cheng and Wu's (2001) design is constructed by taking the union of three disjoint regular 3^{8-6} fractional factorial designs of resolution two. The three 3^{8-6} fractions are chosen in such a way that there is no defining word of length three when they are pieced together. The projection property is then verified by computer. The choice of the three fractions seems to have been done by trial and error. Cheng and Wu's design almost achieves the same property for five-factor projections; among the five-factor projections, only one fails to estimate the quadratic model.

In fact, an $OA(27, 3^8, 2)$ with a slightly better projection property can be found. We have been able to find an $OA(27, 3^8, 2)$ with the property that in all projections onto five factors, the quadratic model can be estimated. This design is displayed in the following:

$$\begin{bmatrix}
0&0&0&0&2&0&0&0\\
0&1&0&1&0&2&1&2\\
0&2&0&2&1&1&2&1\\
1&0&1&0&0&2&2&1\\
1&1&1&1&1&1&0&0\\
1&2&1&2&2&0&1&2\\
2&0&2&0&1&1&1&2\\
2&1&2&1&2&0&2&1\\
2&2&2&2&0&2&0&0\\
0&0&1&1&0&0&1&0\\
0&1&1&2&2&1&0&1\\
0&2&1&0&1&2&2&2\\
1&0&2&1&2&1&2&2\\
1&1&2&2&1&2&1&0\\
1&2&2&0&0&0&0&1\\
2&0&0&1&1&2&0&1\\
2&1&0&2&0&0&2&2\\
2&2&0&0&2&1&1&0\\
0&0&2&2&0&1&0&2\\
0&1&2&0&1&0&2&0\\
0&2&2&1&2&2&1&1\\
1&0&0&2&1&0&1&1\\
1&1&0&0&2&2&0&2\\
1&2&0&1&0&1&2&0\\
2&0&1&2&2&2&2&0\\
2&1&1&0&0&1&1&1\\
2&2&1&1&1&0&0&2
\end{bmatrix}.$$

If the design is not required to be an orthogonal array, then a smaller design with the same projection property can be constructed. We describe in the following a general construction mimicking that of Paley designs.

Let s be a positive integer with $s \geq 3$. Denote by $\mathbb{Z}_s$ the set $\{0, 1, \ldots, s-1\}$ of integers modulo s. As before, for each prime power q, let $GF(q)$ be a finite field with q elements $\alpha_1, \alpha_2, \ldots, \alpha_q$. Then the nonzero elements of $GF(q)$ form a cyclic group under the field multiplication. Let g be a generator of this group, called a primitive element. Define a $q \times q$ matrix $\mathbf{A}_s^q$ such that all the diagonal elements are equal to zero, and the (i, j)th off-diagonal element is the remainder when k is divided by s, where k is the integer such that $g^k = \alpha_i - \alpha_j$ with $0 \leq k \leq q-2$. Supplementing this matrix by a row of zeros, we obtain a matrix

$$\mathbf{M}_s^q = \begin{bmatrix} \mathbf{0}_q^T \\ \mathbf{A}_s^q \end{bmatrix},$$

where $\mathbf{0}_q$ is a $q \times 1$ vector of zeros.

Now for each $1 \leq c \leq s-1$, let $\mathbf{M}_s^q + c$ be the matrix obtained by adding c to each entry of $\mathbf{M}_s^q$, where the addition is carried out modulo s. Finally let

$$\mathbf{D}_s^q = \begin{bmatrix} \mathbf{M}_s^q \\ \mathbf{M}_s^q + 1 \\ \vdots \\ \mathbf{M}_s^q + (s-1) \end{bmatrix}. \tag{4.1}$$

Then $\mathbf{D}_s^q$ is a design of size $s(q+1)$ for q s-level factors.

Design $\mathbf{D}_s^q$ has two variants $\mathbf{D}_s'^q$ and $\mathbf{D}_s''^q$ which sometimes have better projection properties. Let

$$\mathbf{M}_s'^q = \begin{bmatrix} \mathbf{0}_q^T \\ \mathbf{A}_s^q + \mathbf{I}_q \end{bmatrix}$$

and

$$\mathbf{M}_s''^q = \begin{bmatrix} \mathbf{0}_q^T \\ \mathbf{A}_s^q + \widehat{\mathbf{I}}_q \end{bmatrix},$$

where $\widehat{\mathbf{I}}_q$ is the $q \times q$ diagonal matrix with the ith diagonal entry equal to the ith entry of the $q \times 1$ vector $(0, 1, \ldots, s-1, 0, 1, \ldots, s-1, 0, 1, \ldots)$. Then $\mathbf{D}_s'^q$ and $\mathbf{D}_s''^q$ are as defined in (4.1) except that each $\mathbf{M}_s^q$ is replaced by $\mathbf{M}_s'^q$ and $\mathbf{M}_s''^q$, respectively. Finally, by adding the column $(0 \cdot \mathbf{1}_{q+1}^T \vdots 1 \cdot \mathbf{1}_{q+1}^T \vdots \cdots \vdots (s-1) \cdot \mathbf{1}_{q+1}^T)^T$ to $\mathbf{M}_s^q$, $\mathbf{M}_s'^q$ or $\mathbf{M}_s''^q$, one can accommodate one more factor. For example, let $s = 3$, $q = 7$, $g = 2$, and $\alpha_i = i - 1$, $i = 1, \ldots, 7$. Then the following 24-run design for eight 3-level factors is obtained by adding the column $(0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,1\,1\,2\,2\,2\,2\,2\,2\,2\,2)^T$ to $\mathbf{D}_3^7$:

$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&1&2&2&1&0&0\\ 0&0&0&1&2&2&1&0\\ 1&0&0&0&1&2&2&0\\ 2&1&0&0&0&1&2&0\\ 2&2&1&0&0&0&1&0\\ 1&2&2&1&0&0&0&0\\ 0&1&2&2&1&0&0&0\\ 1&1&1&1&1&1&1&1\\ 1&1&2&0&0&2&1&1\\ 1&1&1&2&0&0&2&1\\ 2&1&1&1&2&0&0&1\\ 0&2&1&1&1&2&0&1\\ 0&0&2&1&1&1&2&1\\ 2&0&0&2&1&1&1&1\\ 1&2&0&0&2&1&1&1\\ 2&2&2&2&2&2&2&2\\ 2&2&0&1&1&0&2&2\\ 2&2&2&0&1&1&0&2\\ 0&2&2&2&0&1&1&2\\ 1&0&2&2&2&0&1&2\\ 1&1&0&2&2&2&0&2\\ 0&1&1&0&2&2&2&2\\ 2&0&1&1&0&2&2&2 \end{bmatrix}.$$

We expect designs such as $\mathbf{D}_s^q$, $\mathbf{D}_s'^q$ and $\mathbf{D}_s''^q$ to have good projection properties, although they need to be verified case by case. For example, the 24-run design for eight three-level factors displayed above can estimate the quadratic model in all its projections onto five factors, even though it has three fewer runs than the two $OA(27, 3^8, 2)$'s reported in Cheng and Wu (2001) and earlier in the present paper.

For a given model, we can evaluate the overall performance of a design with respect to the D-criterion by computing the average of $[\det(\mathbf{X}^T\mathbf{X})]^{1/h}$ over all p-factor projections, where $\mathbf{X}^T\mathbf{X}$ is the information matrix of the design when it is projected onto a specific subset of p-factors, h is the number of independent parameters in the model, and $[\det(\mathbf{X}^T\mathbf{X})]^{1/h} = 0$ if the model is not estimable. Denote this averaged D-criterion by D_p. Clearly for models without three-factor and higher-order interactions, D_p is maximized by an orthogonal array of strength 4 if such an array exists. When an orthogonal array of strength 4 does not exist, the hypothetical value of D_p that would be achieved by an orthogonal array of strength 4 provides a simple upper bound for D_p. Let this upper bound be D_p^*. Then D_p/D_p^* is a lower bound for the average D-efficiency of the design over all p-factor projections. For the 24-run design with eight three-level factors displayed above, a lower bound for the average D-efficiency under the quadratic model over all the five-factor projections is 65.6%. The corresponding value for Cheng and Wu's (2001) 27-run design is 69.60%.

Under $\mathbf{D}_3^{13}$, a 42-run design for thirteen three-level factors, all the main effects and two-factor interactions can be estimated in all its three-factor projections; a lower bound on the average D-efficiency is 92.36%. Among 1,664 of the $\binom{13}{7} = 1{,}716$ seven-factor projections, the quadratic model can be estimated. On the other hand, $\mathbf{D}_3''^{13}$ can estimate the quadratic model in all the 1,716 seven-factor projections, with a lower bound of 59.78% for the average D-efficiency.

Unlike two-level designs, we do not yet have general theoretical results on the projection properties of designs with more than two levels. This deserves further study.

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