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A LOWER BOUND ON THE ARL TO DETECTION OF A CHANGE WITH A PROBABILITY CONSTRAINT ON FALSE ALARM[1]

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An inequality that relates the probability of false alarm of a change-point detection policy to its average run length to detection is proved. By means of this inequality, a lower bound on the rate of detection, when the change occurs after a long delay, is derived.

1. Introduction. Let $f_\theta(x) = \exp\{\theta x - \Psi(\theta)\}$ be the density, with respect to some σ-finite measure, of a one-parameter exponential family. Let (a, b) be an open interval of real numbers on which Ψ is finite, and let θ_0 and θ be known, $a < \theta_0 < \theta < b$. We will assume, as in [2, page 1268], that $\theta_0 = 0$.

Consider the following simple formulation of a change-point detection with a probability bound on false alarm: Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, and let $1 \le \nu \le \infty$, an (extended) integer, be an unknown parameter: the change-point of the sequence. With each $1 \le \nu \le \infty$ and $a < \omega \le 0$, a probability measure on the sequence of observations is associated by which the observations are independent. Under the probability measure $\mathbb{P}_{\omega,\theta}^{(\nu)}$, the marginal density of the first $\nu - 1$ observations is f_ω whereas the density of the observations that follow is f_θ. Under the probability measure $\mathbb{P}_{\omega,\theta}^{(\infty)}$ $(= \mathbb{P}_\omega)$ the observations are i.i.d. with density f_ω, and under $\mathbb{P}_{\omega,\theta}^{(1)}$ $(= \mathbb{P}_\theta)$ they are i.i.d. with density f_θ. Let $Z_i^{\omega,\theta} = \log[f_\theta(X_i)/f_\omega(X_i)]$ be the log-likelihood ratio of an observation, and set $Z_i^\theta = Z_i^{0,\theta}$. Denote by $I(\omega, \theta)$ the $\mathbb{P}_\theta$-expectation of $Z_i^{\omega,\theta}$ $[I(\theta) = I(0, \theta)]$.

A detection policy is a stopping time, defined on the sequence of observations. For the most part of this paper, the stopping times that will be considered are those that satisfy the probability constraint

$$\mathbb{P}_0(N < \infty) \le \alpha, \tag{1}$$

for a given $0 < \alpha < 1$. Among such policies one seeks the policy that minimizes the average run length (ARL) to detection:

$$\mathbb{E}_{0,\theta}^{(\nu)}(N - \nu + 1 | N \ge \nu). \tag{2}$$

However, which among the policies that satisfy (1) is best depends on the value of ν in (2). It is well known, for example, that when $\nu = 1$ the power-1

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SPRT minimizes (2) among all stopping times that satisfy (1). Nevertheless, the performance of the SPRT is poor when large values of ν are considered.

The main result in this article is an inequality that relates the probability of a false alarm in (1) to the ARL to detection, given in (2). As a corollary of this inequality (Corollary 2), we are able to confirm part of a conjecture of Pollak and Siegmund [2, Section 6], regarding the rate of divergence of (2), considered as a function of ν, as $\nu \to \infty$.

At the end of this paper we will look at stopping times that satisfy the stronger constraint

$$\mathbb{P}_{\omega}(N < \infty) \leq \alpha, \tag{3}$$

for all $a < \omega \leq 0$. The conjecture in [2] states that, for any stopping rule satisfying (3),

$$\limsup_{\nu \to \infty} \mathbb{E}^{(\nu)}_{\omega,\theta}(N - \nu + 1 | N \geq \nu)/\log \nu \geq 1/I(0, \theta). \tag{4}$$

It will be shown, via a counterexample, that the conjecture as stated is not true.

2. The main inequality. The main result is the following theorem.

THEOREM 1. *Assume that Z^{θ}_i, the log-likelihood ratio of an observation, is nonlattice under $\mathbb{P}_0$ and has a finite $\mathbb{P}_{\theta}$-expectation $I(\theta)$. Then there exists a finite constant c such that*

$$\sum_{\nu=1}^{\infty} \exp\{-I(\theta)\mathbb{E}^{(\nu)}_{0,\theta}(N - \nu + 1 | N \geq \nu)\} \leq \frac{\alpha \exp[c/(1-\alpha)]}{1-\alpha}. \tag{5}$$

The constant c does not depend on α, ν or N. However, it does depend on θ.

PROOF. Define, for each $n = 1, 2, \ldots,$

$$R^{\theta}_n = \sum_{\nu=1}^{n} \exp\left\{\sum_{i=\nu}^{n} Z^{\theta}_i\right\}.$$

These statistics are known as the Shiryayev–Roberts statistics. It is easy to check that, for any stopping time N,

$$\begin{aligned} \mathbb{P}_0(N < \infty) &= \sum_{j=1}^{\infty} \mathbb{E}_0[\mathbb{1}\{N = j\}] \\ &= \sum_{j=1}^{\infty} \sum_{\nu=1}^{j} \mathbb{E}^{(\nu)}_{0,\theta}\left[\frac{1}{R^{\theta}_j} \mathbb{1}\{N = j\}\right] \\ &= \sum_{\nu=1}^{\infty} \mathbb{E}^{(\nu)}_{0,\theta}\left[\frac{1}{R^{\theta}_N} \mathbb{1}\{N \geq \nu\}\right] \\ &\geq \sum_{\nu=1}^{\infty} \exp\{-\mathbb{E}^{(\nu)}_{0,\theta}[\log R^{\theta}_N | N \geq \nu]\}\mathbb{P}^{(\nu)}_{0,\theta}(N \geq \nu). \end{aligned}$$

In particular, if N satisfies (1), then $\mathbb{P}_{0,\theta}^{(\nu)}(N \geq \nu) \geq 1 - \alpha$, hence

$$\sum_{\nu=1}^{\infty} \exp\left\{-\mathbb{E}_{0,\theta}^{(\nu)}\left[\log R_N^{\theta} | N \geq \nu\right]\right\} \leq \frac{\alpha}{1-\alpha}. \tag{6}$$

For each $n \geq \nu$, the statistic R_n^{θ} can be represented as a product of two terms:

$$\begin{aligned} R_n^{\theta} &= \exp\left\{\sum_{i=\nu}^{n} Z_i^{\theta}\right\}\left(\sum_{j=1}^{\nu-1} \exp\left\{\sum_{i=j}^{\nu-1} Z_i^{\theta}\right\} + 1 + \sum_{j=\nu+1}^{n} \exp\left\{-\sum_{i=\nu}^{j-1} Z_i^{\theta}\right\}\right) \\ &= \exp\left\{\sum_{i=\nu}^{n} Z_i^{\theta}\right\} \times W^{\theta}(\nu, n). \end{aligned}$$

It follows, since $Z_{\nu}^{\theta}, Z_{\nu+1}^{\theta}, \dots$ are i.i.d. under the measure $\mathbb{P}_{0,\theta}^{(\nu)}(\cdot | N \geq \nu)$, that

$$\begin{aligned} &\mathbb{E}_{0,\theta}^{(\nu)}\left[\log R_N^{\theta} | N \geq \nu\right] \\ &\quad = I(\theta)\mathbb{E}_{0,\theta}^{(\nu)}(N - \nu + 1 | N \geq \nu) + \mathbb{E}_{0,\theta}^{(\nu)}(\log W^{\theta}(\nu, N) | N \geq \nu) \\ &\quad \leq I(\theta)\mathbb{E}_{0,\theta}^{(\nu)}(N - \nu + 1 | N \geq \nu) + \frac{\mathbb{E}_{0,\theta}^{(\nu)}(\log W^{\theta}(\nu, N))}{1-\alpha}. \end{aligned} \tag{7}$$

(Notice that $\log W^{\theta}$ is positive when $N \geq \nu$, and its definition can be extended to be zero when $N < \nu$.)

However, the $\mathbb{P}_{0,\theta}^{(\nu)}$-distribution of the random variable $W^{\theta}(\nu, N)$ is stochastically dominated, uniformly in ν and N, by the distribution of $W_{\infty}^{\theta} + 1 + W_1^{\theta}$, where W_{∞}^{θ} and W_1^{θ} are independent of each other, the distribution of W_1^{θ} is the $\mathbb{P}_{\theta}$-distribution of $\sum_{j=1}^{\infty} \exp\{-\sum_{i=1}^{j} Z_i^{\theta}\}$ and the distribution of W_{∞}^{θ} is the $\mathbb{P}_0$-distribution of $\sum_{j=1}^{\infty} \exp\{\sum_{i=1}^{j} Z_i^{\theta}\}$. To assert this claim, notice that the $\mathbb{P}_{0,\theta}^{(\nu)}$-distribution of $\sum_{j=1}^{\nu-1} \exp\{\sum_{i=j}^{\nu-1} Z_i^{\theta}\}$ is identical to the $\mathbb{P}_0$-distribution of $\sum_{j=1}^{\nu-1} \exp\{\sum_{k=1}^{j} Z_k^{\theta}\}$ which is $\mathbb{P}_0$-almost surely smaller than $\sum_{j=1}^{\infty} \exp\{\sum_{i=1}^{j} Z_i^{\theta}\}$. Likewise, $\sum_{j=\nu+1}^{N} \exp\{-\sum_{i=\nu}^{j-1} Z_i^{\theta}\}$ is $\mathbb{P}_{0,\theta}^{(\nu)}$-almost surely smaller than $\sum_{j=\nu}^{\infty} \exp\{-\sum_{i=\nu}^{j} Z_i^{\theta}\}$.

From Theorem 4 in [1] it follows that the second term on the right-hand side of (7) is bounded, uniformly in ν and N. Plugging (7) into (6) completes the proof. □

Corollary 1. *Assume that Z_i^{θ} is nonlattice under $\mathbb{P}_0$ and has a finite second moment under $\mathbb{P}_{\theta}$. Let $a_1, a_2, \dots$ be a sequence of positive numbers. Then the upper limit*

$$\limsup_{\nu \to \infty}\left[\mathbb{E}_{0,\theta}^{(\nu)}(N - \nu + 1 | N \geq \nu) + \frac{\log a_{\nu}}{I(\theta)}\right] \tag{8}$$

is infinite for every stopping time N that satisfies (1) *iff* $\sum_{\nu=1}^{\infty} a_{\nu} = \infty$.

PROOF. Assume first that $\sum_{\nu=1}^{\infty} a_\nu < \infty$. Let

$$\alpha_\nu = \frac{\alpha a_\nu}{\sum_{k=1}^{\infty} a_k}, \qquad 1 \le \nu < \infty. \tag{9}$$

The stopping time

$$N = \inf\left\{n\colon \sum_{\nu=1}^{n} \alpha_\nu \exp\left\{\sum_{i=\nu}^{n} Z_i^\theta\right\} + \sum_{\nu=n+1}^{\infty} \alpha_\nu \ge 1\right\},$$

which is similar to a policy suggested in [2], would produce a finite upper limit in (8). The last claim follows from the relation

$$N \le \inf\left\{n\colon \sum_{i=\nu}^{n} Z_i^\theta \ge -\log \alpha_\nu\right\}, \qquad \nu = 1, 2, \dots,$$

Wald's lemma and from the fact that the expectation of the overshoot is bounded.

Assume next that the sum of the sequence $\{a_\nu\}$ diverges. Inequality (5) can be rewritten in the form

$$\sum_{\nu=1}^{\infty} a_\nu \exp\left\{-I(\theta)\left[\mathbb{E}_{0,\theta}^{(\nu)}(N - \nu + 1 | N \ge \nu) + \frac{\log a_\nu}{I(\theta)}\right]\right\} \le \frac{\alpha \exp[c/(1-\alpha)]}{1-\alpha}.$$

It is easy to see that a finite upper limit in (8) would produce a contradiction. □

COROLLARY 2. *Under the assumptions of Corollary* 1, *relation* (59) *in* [2] *holds for* $\omega = 0$; *that is*,

$$\limsup_{\nu \to \infty} \mathbb{E}_{0,\theta}^{(\nu)}(N - \nu + 1 | N \ge \nu)/\log \nu \ge 1/I(0, \theta),$$

for any stopping rule that satisfies (3) [*and hence* (1)].

PROOF. The proof follows immediately from the proof of Corollary 1 with $a_\nu = 1/\nu$. □

3. A counterexample. Corollary 2 states that the conjecture of Pollak and Siegmund is true for $\omega = 0$. It is false, however, for $a < \omega < 0$. For any such ω one can find a stopping rule σ, satisfying (3), for which (4) is false. Indeed, fix $a < \omega < 0$. Consider, for any integer m and negative real η, $a < \eta \le 0$, the auxiliary stopping rule

$$N(m, \eta) = \inf\left\{n \ge m + 1\colon \sum_{\nu=m+1}^{n} \alpha_\nu \exp\left\{\sum_{i=\nu}^{n} Z_i^{\eta,\theta}\right\} + \sum_{\nu=n+1}^{\infty} \alpha_\nu \ge 2\right\},$$

where α_ν is defined in (9) with $a_\nu = 1/(\nu \log^2 \nu)$. Note that, for any $a < \zeta \le \eta$,

$$\mathbb{P}_\zeta(N(m, \eta) < \infty) \le \frac{\alpha}{2}$$

and that, for any $\nu > m$,

$$\mathbb{E}^{(\nu)}_{\zeta,\theta}(N(m,\eta) - \nu + 1 | N(m,\eta) \ge \nu) \le \frac{-\log(\alpha_\nu/2) + \text{const.}}{I(\eta,\theta)}.$$

The constant is a bound on the expected overshoot. It depends only on η and θ.

Given $0 < \varepsilon < -\omega$, let δ_m be a test of H_0: $\eta \ge \omega + \varepsilon$ versus H_1: $\eta < \omega + \varepsilon$, based on the first m observations, with significance level $\alpha/2$. Let $m = m(\omega, \varepsilon)$ be a sample size needed to assure that the power of the test, at $\eta = \omega$, is at least $1 - \varepsilon$.

Consider the stopping rule

$$\sigma = \sigma(\omega, \varepsilon) = N(m, (\omega + \varepsilon)\delta_m).$$

The stopping rule σ satisfies (3), since $\mathbb{P}_\eta(\sigma < \infty)$ is less than $\alpha/2$ for $\eta \le \omega + \varepsilon$ and less than α for $\omega + \varepsilon < \eta \le 0$. This stopping rule is a random mixture of the two stopping times $N(m, 0)$ and $N(m, \omega + \varepsilon)$. The mixture is based on the first m observations. It can be shown that

$$\mathbb{P}^{(\nu)}_{\omega,\theta}(\delta_m = 0 | \sigma \ge \nu) < \frac{\varepsilon}{(1 - \alpha/2)(1 - \varepsilon)}.$$

As a result it follows that the ARL to detection, for all $\nu > m$, satisfies the relation

$$\mathbb{E}^{(\nu)}_{\omega,\theta}(\sigma - \nu + 1 | \sigma \ge \nu) \le \frac{-\log a_\nu/2 + \text{const.}}{I(\omega + \varepsilon, \theta)} + \frac{\varepsilon}{(1 - \alpha/2)(1 - \varepsilon)} \frac{-\log a_\nu/2 + \text{const.}}{I(0, \theta)}.$$

A contradiction to (4) can be derived from the above inequality since $I(\omega + \varepsilon, \theta) > I(0, \theta)$ and ε can be chosen arbitrarily small.

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