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OPTIMAL BLOCKED MAIN EFFECTS PLANS WITH NESTED ROWS AND COLUMNS AND RELATED DESIGNS

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Optimal design is studied for factorial experiments in the nested row and column setting. The approach is analogous to that of orthogonal Latin squares: main effects plans are found by the superimposition of one nested row and column design upon another. Conditions are stated for statistical orthogonality of the superimposed components, resulting in orthogonal main effects plans, and a number of constructions are given. Orthogonal collections of sets of Latin squares are introduced. All of the constructed designs are also optimal main effects plans for the row–column and the unstructured block design settings. Further applications are as optimal multidimensional incomplete block designs and as optimal designs for multistage experimentation.

1. Introduction. In the nested row and column setting, v treatments are to be compared in b blocks, each block being a cross of p rows and q columns. This setting has recently received considerable attention in the literature, including the first studies of optimality under the bottom-stratum analysis. Sufficient conditions for optimality and corresponding constructions may be found in the papers of Bagchi, Mukhopadhyay and Sinha (1990), Chang and Notz (1990, 1994), Gupta (1992) and Morgan and Uddin (1993a). The strongest of these optimal designs, in the sense of being optimal with respect to the widest class of criteria, are the *balanced nested row and column designs*, or BNRC's. A BNRC is a design for which (i) the bq columns are a balanced block design [Kiefer (1975)] with block size p and (ii) within any given block, each row is the same multiset of treatments [if r_{jil} is the number of times treatment j occurs in row l of block i, statement (ii) says exactly that r_{jil} is constant in l for fixed i and j]. The BNRC's are variance balanced for the within-rows-and-columns analysis (they have completely symmetric information matrices), hence their name. However, they are not unique in this way, for there are large classes of variance balanced nested row and column designs which are not BNRC's, such as the more aptly named balanced incomplete block designs with nested rows and columns [Singh and Dey (1979)]. We will retain the BNRC acronym, while suggesting

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that these designs be named *bottom-stratum universally optimal nested row and column designs*, a clearer identification of their properties.

The BNRC's for $b = 1$ are the well-known regular generalized Youden designs (GYD's) of Kiefer (1975), which include Latin squares and Youden squares. Thus are BNRC's a generalization of this widely studied and applied class of designs to the $b > 1$ setting. Morgan and Uddin (1993a) point out that the blocks of any BNRC may be combined into a GYD, further demonstrating the close ties between these classes. In this paper we will exploit those ties to find optimal main effects plans for the nested row and column setting.

The simplest case in terms of both analysis and combinatorics is that of Latin squares ($b = 1$, $p = q = v$). One of the many uses of a pair of orthogonal Latin squares of order v is as an optimal main effects plan for the $v \times v$ factorial in the $v \times v$ row–column setting. An immediate extension is that a set of t mutually orthogonal Latin squares is an optimal main effects plan for the v^t factorial in the same setting. For $b = 1$, and p and q each multiples of v, the relevant designs are the mutually orthogonal F-squares with row and column replication counts constant over treatments, F-squares of this type being intermediate in the generalization sequence from Latin squares to BNRC's. For the asymmetrical case (number of treatments not the same in each square), orthogonal F-squares with variable numbers of symbols have been studied [see Dénes and Keedwell (1991) for an overview of results on Latin squares and F-squares]. For each of these classes of designs, the orthogonality is a simple combinatorial property: imposition of a square on an orthogonal mate results in each symbol of the first square being imposed on each symbol of the second the same number of times.

There are of course other uses for these orthogonal sets of row–column designs. In terms of Latin squares, t pairwise orthogonal squares define a multidimensional incomplete block designs with s-way heterogeneity for the v^{t-s} factorial, $0 \leq s \leq t - 1$ [e.g., Mukhopadhyay and Mukhopadhyay (1984)]. They are also used for sequential experimentation in which distinct, noninteracting sets of v treatments each are used on the same $v \times v$ array of plots at distinct points in time, what has been called multistage experimentation [e.g., Hoblyn, Pearce and Freeman (1954), Preece (1976), Mandeli and Federer (1984), and Morgan and Uddin (1993b); the Preece paper contains many relevant older references]. Both of these applications are covered by the model, analysis and designs given here, but for simplicity of exposition we will maintain the single terminology of "main effects plans," referring to other applications only when there is a relevant point to be made. In this regard, see especially Section 3.3.

Analogous to sets of mutually orthogonal Latin squares and F-squares, we here seek sets of mutually orthogonal BNRC's to use as main effects plans in nested rows and columns. The definition of the latter (Section 2) can be made in terms of counts from the superimposition of one BNRC upon another, but as will be seen generally involves more than just the simple combinatorial orthogonality of Latin squares and F-squares, owing to symbols not possessing combinatorial orthogonality with respect to the blocking factors. The conditions fundamentally involve those of orthogonal sets of BIBD's, which are also discussed in Section 2. In Section 3, construction of the new designs using recursive, permutation, and difference techniques are given. Section 4 introduces the notion of orthogonal collections of sets of Latin squares, and such collections are constructed for prime power numbers of treatments, embedding the classical sets of mutually orthogonal Latin squares due to Bose (1938) and Stevens (1939).

When there is need to refer to specific design parameters, the notation will be as follows: BIBD(v, b, k) for a balanced incomplete block design for v treatments in b blocks of size k; YS(v, k) and GYD(v, p, q) for $k \times v$ Youden squares and $p \times q$ generalized Youden designs; and BNRC(v, b, p, q) for bottom-stratum universally optimal nested row and column designs with b blocks of size $p \times q$. A consequence of the convention, to be maintained throughout this paper, that the column (rather than the row) component design of a BNRC be a balanced block design is $p \leq q$.

2. Orthogonality, optimality and BIBD's. Again, the nested row and column setting consists of $n = bpq$ experimental units arranged in b separate $p \times q$ arrays. Placing v treatments on these units, one treatment per unit, the additive linear model is

$$Y = \mu 1 + A\tau + Z_1 \beta + Z_2 \rho + Z_3 \gamma + \varepsilon \tag{1}$$

with, for plots ordered rowwise by block, $Z_1 = I_b \otimes 1_{pq}$, $Z_2 = I_{bp} \otimes 1_q$ and $Z_3 = I_b \otimes 1_p \otimes I_q$ the plot-block, plot-row and plot-column incidence matrices; A the $n \times v$ plot-treatment incidence matrix; τ $(v \times 1)$, β $(b \times 1)$, ρ $(bp \times 1)$ and γ $(bq \times 1)$ the vectors of treatment, block, row and column effects; and ε $(n \times 1)$ a random vector of mean zero, uncorrelated, equivariable error terms.

For a factorial treatment structure the treatment set V decomposes as $V = V_1 \times V_2 \times \cdots \times V_t$, V_i being the set of levels of factor i, $|V_i| = v_i$ and $v = \Pi_{i=1}^t v_i$, each treatment being a combination of one level of each factor. A main effects model says that the factor effects are additive, and thus $A\tau = \Sigma_{i=1}^t A_i \tau_i$, where now A_i is $n \times v_i$, τ_i is $v_i \times 1$ and $A_i 1_{v_i} = 1_n$. Eliminating block, row, and column effects, the information matrix for estimation of $\tau = (\tau_1', \tau_2', \dots, \tau_t')'$ in the within-rows-and-columns stratum is the partitioned matrix $C = (C_{ii'})$, where

$$C_{ii'} = A_i' \left[I - \frac{1}{q} Z_2 Z_2' - \frac{1}{p} Z_3 Z_3' + \frac{1}{pq} Z_1 Z_1' \right] A_{i'} \tag{2}$$

[Morgan and Uddin (1993b)].

Write D_{i1} for the $(\Sigma_{i'=1}^t v_{i'} - v_i) \times (\Sigma_{i'=1}^t v_{i'} - v_i)$ matrix found by deleting the v_i rows and columns from C which contain C_{ii}. Write D_{i2} for the

$(\sum_{i'=1}^{t} v_{i'} - v_i) \times v_i$ matrix composed of the v_i columns of C containing C_{ii} but with C_{ii} deleted. Then the information matrix for the estimation of τ_i alone is

$$C_{ii}^* = C_{ii} - D_{i2}' D_{i1}^- D_{i2}. \tag{3}$$

As in Cheng (1980), a design is said to be universally optimal for estimating main effects in a $\Pi_{i=1}^{t} v_i$ factorial if it minimizes $\sum_{i=1}^{t} \Phi_i(C_{ii}^*)$ for all functions Φ_i with the following characteristics:

1. Φ_i is convex;
2. $\Phi_i(aC)$ is nondecreasing in scalar $a \geq 0$;
3. Φ_i is invariant under simultaneous row and column permutations of C_{ii}^*.

Sufficient for this is that each C_{ii}^* is completely symmetric of maximum trace. Nonnegative definiteness of C implies $\mathrm{tr}(C_{ii}^*) \leq \mathrm{tr}(C_{ii})$. So sufficient conditions for an optimal main effects plan in nested rows and columns are

$$C_{ii} \text{ is completely symmetric of maximum trace for all } i \tag{4}$$

and

$$C_{ii'} = 0 \quad \text{for all } i \neq i'. \tag{5}$$

Condition (4) says that the nested row and column design formed by just the levels of factor i is a BNRC. Condition (5), which makes $D_{i2} = 0$, is the orthogonality condition. When (5) holds, main effects contrasts for factors i and i' are orthogonally estimated, that is, $\mathrm{cov}(\widehat{l'\tau_i}, \widehat{m'\tau_{i'}}) = 0$, an immediate consequence of C^+ being a variance–covariance matrix for $\hat{\tau}$. To state condition (5) in terms of pairwise concurrence counts, define the following:

$\lambda_{jj'}^{Pii'} = (A_i' A_{i'})_{jj'}$, the number of plots on which $j \in V_i$ and $j' \in V_{i'}$ concur;

$\lambda_{jj'}^{Bii'} = (A_i' Z_1 Z_1' A_{i'})_{jj'}$, the number of pairs $j \in V_i$, $j' \in V_{i'}$, occurring in blocks;

$\lambda_{jj'}^{Rii'} = (A_i' Z_2 Z_2' A_{i'})_{jj'}$, the number of pairs $j \in V_i$, $j' \in V_{i'}$, occurring in rows;

$\lambda_{jj'}^{Gii'} = (A_i' Z_3 Z_3' A_{i'})_{jj'}$, the number of pairs $j \in V_i$, $j' \in V_{i'}$, occurring in columns.

Then $C_{ii'} = 0$ if and only if

$$\lambda_{jj'}^{Pii'} - \frac{1}{q}\lambda_{jj'}^{Rii'} - \frac{1}{p}\lambda_{jj'}^{Gii'} + \frac{1}{pq}\lambda_{jj'}^{Bii'} = 0, \tag{6}$$

for all j, j'. However, conditions (4) and (5) are not unrelated.

THEOREM 2.1. *If the designs d_i and $d_{i'}$ formed by levels of factors i and i' are each BNRC's, then $C_{ii'} = 0$ if and only if*

$$\lambda_{jj'}^{Pii'} - \frac{1}{p}\lambda_{jj'}^{Gii'} = 0, \tag{7}$$

for all j, j'.

PROOF. Let d_1 and d_2 be any two BNRC's with common b, p and q. Suppose the superimposition of d_1 on d_2 results in $j \in V_1$ and $j' \in V_2$ occurring in s common blocks. Let $r_{j1}, \dots, r_{js}$ be the constant (within blocks) row counts for j in those s blocks, and let $r_{j'1}, \dots, r_{j's}$ be the same counts for j'. Then

$$\lambda_{jj'}^{R12} = p\sum_{l=1}^{s} r_{jl}r_{j'l} \quad \text{and} \quad \lambda_{jj'}^{B12} = p^2\sum_{l=1}^{s} r_{jl}r_{j'l}.$$

Substituting these in (6) gives (7). □

This is suggested in the following definition.

DEFINITION. Let d_i be a BNRC(v_i, b, p, q), $i = 1, \dots, t$. Then $d_1, \dots, d_t$ are said to be a *set of mutually orthogonal BNRC's* if $\lambda_{jj'}^{Pii'} - \lambda_{jj'}^{Gii'}/p = 0$ for all $j \in V_i$, $j' \in V_{i'}$, $i \neq i' = 1, 2, \dots, t$.

The orthogonality condition (7) says that, for each occurrence of levels of two different factors on the same plot, there must be p occurrences of those two levels in the same column (one of which is the occurrence on a plot). It can be shown that this is exactly the condition for mutual orthogonality of the BIBD's (or BBD's) given by the columns of the BNRC's; that is, if rows and blocks are removed from each BNRC to leave columns as an unstructured block design, then (7) says that superimposition of these designs produces a blocked main effects plan. Hence the study of sets of mutually orthogonal BIBD's will be of primary importance to the current endeavor. Curiously, this is a topic which has not received much attention in the literature. What work is available mainly appears under the general headings mentioned in Section 1 as mathematically equivalent formulations of the main effects problem: "designs for two sets of treatments" or "designs for two successive experiments," and "multidimensional incomplete block designs." In the current terminology, the former would be superimpositions of two block designs for the $v_1 \times v_2$ factorial, although the papers in this area include many designs which are not BIBD's and/or do not satisfy (7) [e.g., Street (1981); Preece (1976) and many papers referenced there]. Indeed, the reader should be warned that many of these papers focus solely on the individual counts $\lambda_{jj'}^{Pii'}$, $\lambda_{jj'}^{Rii'}$, $\lambda_{jj'}^{Gii'}$ and $\lambda_{jj'}^{Bii'}$ arising from (2), although (3) makes it clear that balancing or partially balancing those counts alone does not assure a statistically good design. Here we briefly review known series of these designs, primarily by giving new results that include them as special cases in the sense of producing designs with the same parameters.

2.1. *Orthogonal BIBD's*: *the method of differences*. Several authors have used difference techniques over the finite fields to construct pairs of mutually orthogonal BIBD's. We generalize some of these constructions in the lemmas that follow. In the finite field GF_v, x will be used to denote a primitive element.

LEMMA 2.2. *Let $v = mf + 1$ be an odd prime power with $m = tg$ for some $t \geq 2$. Let $k = hf$ for $h \leq g$. Write $C_j = x^{(j-1)}(x^0, x^m, \dots, x^{(f-1)m})'$ for the column of powers of x^m multiplied by $x^{(j-1)}$, and*

$$b_l^i = x^{(i-1)g} x^{(l-1)} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_h \end{pmatrix}.$$

Then $b_1^i, b_2^i, \dots, b_m^i$ are initial blocks for a BIBD(v, mv, k) *d_i, and $d_1, \dots, d_t$ are mutually orthogonal.*

PROOF. The orthogonality will be demonstrated. It must be shown that in the superimposed initial blocks, each difference occurs k times as often in blocks as on plots. Superimposing b_l^i on $b_l^{i'}$, the plot differences are, for $j = 1, 2, \dots, h$, the list

$$\begin{aligned} x^{l-1}\left(x^{(i-1)g} C_j - x^{(i'-1)g} C_j\right) &= x^{l-1} x^{(i-1)g} x^{j-1}\left(C_1 - x^{(i'-i)g} C_1\right) \\ &= x^{l-1} x^{(i-1)g} x^{j-1}\left(1 - x^{(i'-i)g}\right) C_1, \end{aligned}$$

which for $l = 1, \dots, m$ give each nonzero in GF_v exactly once. Changing $j = 1, 2, \dots, h$, the plot differences are each nonzero h times.

To find the block differences, subtract each member of $x^{(i'-1)g} C_{j'}$ from each of $x^{(i-1)g} C_j$ for $j, j' = 1, 2, \dots, h$. With $w_{jj'} = (i' - i)g + j' - j$, for fixed j and j' this is $x^{(i-1)g+(j-1)}$ times the differences of the members of $x^{w_{jj'}} C_1$ from those of C_1, which are the list

$$\left(1 - x^{w_{jj'}}, 1 - x^{w_{jj'}+m}, \dots, 1 - x^{w_{jj'}+(f-1)m}\right) \otimes C_1$$

(note that $0 < |w_{jj'}| < m$). Hence, for fixed l, the block differences b_l^i minus $b_l^{i'}$ are

$$x^{(l-1)} x^{(i-1)g+(j-1)}\left(1 - x^{w_{jj'}}, 1 - x^{w_{jj'}+m}, \dots, 1 - x^{w_{jj'}+(f-1)m}\right) \otimes C_1.$$

Letting $l = 1, 2, \dots, m$ for fixed j, j', this is f copies of the nonzeros of GF_v, so also varying j, j' gives $fh^2 = kh$ copies of the nonzeros, and the orthogonality is established. $\square$

COROLLARY 2.3. *Let $v = mf + 1$ be an odd prime power. Then there are m mutually orthogonal* BIBD(v, mv, f)*'s.*

COROLLARY 2.4. *Let $v = mf + 1$ be a prime power with m even. Then there is a pair of orthogonal* BIBD(v, mv, hf)*'s for each $h \leq m/2$.*

Corollary 2.3 is the case $g = 1$ in Lemma 2.2. The same result for a pair of orthogonal BIBD's can be found in Causey [(1968), Section 3], who used them to construct four-dimensional incomplete block designs. Street [(1981), Theorem 5(a)] also has this result for a pair of BIBD's, but restricted to even m. Corollary 2.4 is $g = m/2$ with even m in Lemma 2.2. This can be found in Street [(1981), Theorem 4(a)] with the restriction that f be odd.

LEMMA 2.5. *Let $v = mf + 1$ be an odd prime power. Let $k = hf + 1$ for $h \leq m$. Write $C_j = x^{(j-1)}(x^0, x^m, \dots, x^{(f-1)m})'$ and*

$$b_l^i = x^{(i-1)m} x^{(l-1)} \begin{pmatrix} 0 \\ C_1 \\ C_2 \\ \vdots \\ C_h \end{pmatrix}.$$

Then $b_1^i, b_2^i, \dots, b_m^i$ are initial blocks for a BIBD(v, mv, k) *d_i, and $d_1, \dots, d_f$ are mutually orthogonal.*

PROOF. Again the orthogonality will be demonstrated. Relative to Lemma 2.2, the initial blocks have been modified by the addition of a 0 and by changing the multiplier $x^{(i-1)g}$ to $x^{(i-1)m}$. One effect of the latter is that $b_l^{i'}$ is a permutation of b_l^i with only the 0 fixed (in Lemma 2.2 these two blocks are disjoint sets). So the plot differences are m zeros and, as in the proof of Lemma 2.2,

$$x^{l-1} x^{(i-1)m} x^{j-1} (1 - x^{(i-i')m}) C_1 \quad \text{for } l = 1, \dots, m \text{ and } j = 1, \dots, h,$$

which are the nonzeros h times each.

Also as in the proof of Lemma 2.2, the block differences for b_l^i relative to $b_l^{i'}$, if 0 is excluded from each block, are

$$x^{(l-1)} x^{(i-1)m + (j-1)} (1 - x^{w_{jj'}}, 1 - x^{w_{jj'} + m}, \dots, 1 - x^{w_{jj'} + (f-1)m}) \otimes C_1,$$

for $l = 1, \dots, m$ and $j, j' = 1, \dots, h$, which are

$$(1 - x^{w_{jj'}}, 1 - x^{w_{jj'} + m}, \dots, 1 - x^{w_{jj'} + (f-1)m}) \otimes (GF_v - 0), \qquad j, j' = 1, \dots, h,$$

where now $w_{jj'} = (i' - i)m + (j' - j)$. When $j = j'$ the list

$$(1 - x^{w_{jj'}}, 1 - x^{w_{jj'} + m}, \dots, 1 - x^{w_{jj'} + (f-1)m})$$

contains exactly one zero, and when $j \neq j'$ it contains none. So these differences are $h(v - 1) = m(k - 1)$ zeros, and $h(h - 1)f + h(f - 1) = h(hf - 1)$ of each nonzero.

Placing the 0 in each block then also gives m additional zeros and, for $j = 1, \dots, h$ and $l = 1, \dots, m$, the differences $\{-x^{l-1}x^{(i'-1)m}C_j, x^{l-1}x^{(i-1)m}C_j\}$, which are $2h$ additional copies of the nonzeros. Hence the block differences are $m(k-1) + m = mk$ zeros, and $h(hf-1) + 2h = hk$ of each nonzero, these counts being k times those for the plot differences. $\square$

Lemma 2.6 will cover a subset of the v of Lemma 2.5 while for some values of f and h allowing a larger set of BIBD's to be constructed. In Lemma 2.6, a special combinatorial array will be needed. The $p \times t$ array $A(p, t, s)$ on p symbols is said to have s-pair balance if (i) each column contains each symbol exactly once and (ii) each pair of columns has exactly s like pairs in rows. The array is said to have symmetric s-pair balance, and will be written $A^{\sigma}(p, t, s)$, if it further satisfies (iii) if the pair (α, β) is formed by the rows of a given two columns, so is the pair (β, α). Morgan and Uddin (1993b) have shown that an $A^{\sigma}(p, t, 0)$ exists if and only if $t \le 2^w$, where $p = 2^w e$ for some odd e.

LEMMA 2.6. *Let $v = 2mf + 1$ be a prime power for which f is odd. Write $k = hf + 1$ for some even $h \le m$, and suppose $h = 2^w e$ for some odd e and $w \ge 1$. Let $A = ((a_{ij}))$ be an $A^{\sigma}(h, 2^w, 0)$ on the symbols $0, 1, \dots, h-1$. Let*

$$b_1^i = \begin{pmatrix} 0 \\ C_{a_{1i}} \\ C_{a_{2i}} \\ \vdots \\ C_{a_{hi}} \end{pmatrix}$$

and $b_l^i = x^{l-1} b_1^i$. Then $b_1^i, b_2^i, \dots, b_m^i$ are initial blocks for a BIBD(v, mv, k) *d_i, and $d_1, \dots, d_{2^w}$ are mutually orthogonal.*

PROOF. For each i, b_l^i contains x^{l-1} times $C_0, \dots, C_{h-1}$ in some order. That d_i is thus a BIBD and that the block differences when superimposing b_l^i on $b_l^{i'}$ for $l = 1, \dots, m$ are each nonzero element of GF_v with frequency $h(hf+1)/2$, and 0 $m(hf+1)$ times, follows from Theorem 5(b) of Street (1981). It only need be shown that the plot differences are each nonzero with frequency $h/2$, and m copies of 0. For fixed l, the plot differences $b_l^i - b_l^{i'}$ are zero and

$$x^{l-1}\left(C_{a_{ji}} - C_{a_{ji'}}\right) = x^{l-1} x^{a_{ji} - a_{ji'}} C_0, \qquad j = 1, 2, \dots, h.$$

By the symmetry property of A, these are 0 and $\pm x^{l-1} x^{a_{ji} - a_{ji'}} C_0$ for $h/2$ of the values of j. As $l = 1, \dots, m$, $\pm x^{l-1} C_0$ generates one copy of the nonzeros of GF_v, so $b_l^i - b_l^{i'}$ gives $h/2$ copies of the nonzeros and m copies of 0. $\square$

Lemma 2.6 extends Street's (1981) Theorem 3(b) to more than two orthogonal BIBD's. Other than this, we know of no general series of designs that overlaps with Lemmas 2.5 and 2.6. Street (1981) gives two additional series of pairs of orthogonal BIBD's that are not extended here. Her Theorem 2(a) is for two different numbers of symbols: $v_1 = 4s + 3 = v_2 - 1$, $b = 2v_1$, $k = (v_1 + 1)/2$ with v_1 a prime power. In her Theorem 4(c), $v = 2mf + 2$, $b = 2m(v - 1)$ and $k = mf + 1$, where f is odd and $2mf + 1$ is a prime power. Other orthogonal pairs for small v have been tabulated by Preece (1966a).

2.2. *Orthogonal BIBD's from transitive arrays.* A transitive array of strength 2, $TA(v, b, k)$, is a $k \times b$ array on v symbols with the property that the columns of any two rows give every ordered pair of distinct symbols with equal frequency. If that frequency is 1 [implying $b = v(v - 1)$], the TA is equivalent to a set of $k - 2$ mutually orthogonal Latin squares with a common transversal. Transitive arrays played an integral role in Bose, Parker and Shrikhande's (1960) proof of the falsity of Euler's conjecture. See Dénes and Keedwell [(1991), Chapter 2] for recent results on transversals and Latin squares.

LEMMA 2.7. *Existence of the $TA(v, b, tk)$ implies the existence of t mutually orthogonal* BIBD(v, b, k)'s.

PROOF. Use consecutive sets of k rows to create t nonoverlapping subarrays from the TA. The columns of these subarrays are the required design. □

For v a prime power there is always a $TA(v, v(v - 1), v)$: use the $v - 1$ initial columns $x^l(0, x^0, x^1, \dots, x^{v-2})'$, $l = 0, 1, \dots, v - 1$.

2.3. *Orthogonal BIBD's from SOLS's.* A self-orthogonal idempotent Latin square (SOLS) L is a Latin square with diagonal transversal which is orthogonal to its own transpose. Brayton, Coppersmith and Hoffman (1974) have shown that there is a SOLS for every order v except 2, 3 and 6.

LEMMA 2.8. *Existence of a SOLS of order v implies the existence of a pair of mutually orthogonal* BIBD$(v, v(v - 1)/2, 2)$'s.

PROOF. Let $L = ((L_{ij}))$ be a SOLS and assume that $L_{ii} = i$ for each i. The blocks of the design d_1 are $\binom{i}{j}$ and the corresponding blocks of d_2 are $\binom{L_{ij}}{L_{ji}}$, $1 \le i < j \le v$. Verification of (7) is routine. □

For v not covered by the lemma, orthogonal pairs with block size 2 are given in Table 1. These designs illustrate that the plot incidence structure $A_1' A_2$ need not be of the form $xI + y11'$, as was the case for all of the designs in Lemmas 2.2 and 2.5–2.8.

3. Orthogonal sets of BNRC's.

3.1. *Recursive and related constructions.* Bagchi, Mukhopadhyay and Sinha (1990), Chang and Notz (1990, 1994), Gupta (1992) and Morgan and

TABLE 1
Orthogonal BIBD's with $k = 2$

$v = 2$

1	1
2	2
1	2
2	1

$v = 3$

1	3	2	1	3	2
2	1	3	2	1	3
1	2	3	2	3	1
2	3	1	1	2	3

$v = 6$

1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	5	5
2	2	3	3	4	4	5	5	6	6	3	3	4	4	5	5	6	6	4	4	5	5	6	6	5	5	6	6	6	6
1	2	1	3	1	4	1	5	1	6	3	4	3	5	3	6	4	5	4	6	5	6	2	3	2	4	2	5	2	6
2	1	3	1	4	1	5	1	6	1	4	3	5	3	6	3	5	4	6	4	6	5	3	2	4	2	5	2	6	2

Uddin (1993a) provide a variety of methods for constructing BNRC's. Certainly a pragmatic first approach to constructing orthogonal sets of these designs is to investigate known methods for single BNRC's for their adaptability to the problem at hand. The first of these is a very useful basic compositional result.

THEOREM 3.1. *If there exists a set of t mutually orthogonal* BIBD(v, b_1, k)*'s, and a* BNRC *for k treatments in b_2 blocks of p rows and q columns, then there exists a set of t mutually orthogonal* BNRC$(v, b = b_1 b_2, p, q)$*'s.*

The proof is simple: constructing the BNRC for k treatments for each block of one of the BIBD's produces one of the BNRC$(v, b_1 b_2, p, q)$'s. Compare Bagchi, Mukhopadhyay and Sinha [(1990), Theorem 3.2.1], Chang and Notz [(1990), Theorem 3.1] and Morgan and Uddin[(1993a), Theorem 2]. Applying this to Lemma 2.8 and Table 1 using the simplest BNRC (a 2×2 Latin square) yields the following corollary.

COROLLARY 3.2. *There is a pair of orthogonal* BNRC$(v, v(v-1)/2, 2, 2)$*'s for every $v \neq 2, 3, 6$ and a pair of orthogonal* BNRC$(v, v(v-1), 2, 2)$*'s for every v.*

The same techniques can be applied for the 2×3 and 3×3 cases (use a Youden square and a Latin square for three treatments) if one can construct the orthogonal BIBD's with $k = 3$. Alternatively, a simple modification of Theorem 3.1 requires the orthogonality to hold for the starting BNRC rather than for the BIBD.

THEOREM 3.3. *If there exist a* BIBD(v, b_1, k) *and a set of t mutually orthogonal BNRC's for k treatments in b_2 blocks of p rows and q columns, then there exists a set of t mutually orthogonal* BNRC$(v, b = b_1 b_2, p, q)$*'s.*

COROLLARY 3.4. *There is a pair of orthogonal* BNRC$(v, b, 3, 3)$*'s for every* $v \geq 3$, *where*

$$b = \begin{cases} \dfrac{v(v-1)}{6}, & \text{if } v \equiv 1 \text{ or } 3 \pmod 6, \\ \dfrac{v(v-1)}{3}, & \text{if } v \equiv 0 \text{ or } 4 \pmod 6, \\ \dfrac{v(v-1)}{2}, & \text{if } v \equiv 5 \pmod 6, \\ v(v-1), & \text{if } v \equiv 2 \pmod 6. \end{cases}$$

Corollary 3.4 takes advantage of the fact that for $k = 3$ the necessary conditions for existence of a BIBD are also sufficient, and combines those designs with a pair of orthogonal Latin squares of side 3. The corresponding result for 2×3's is not so clean, however, for there is no orthogonal pair of YS$(3, 2)$'s. In fact, a fairly broad non-existence result can be established.

THEOREM 3.5. *There is no orthogonal pair of* BIBD(v, v, k)*'s, and hence no orthogonal pair of* YS(v, k)*'s, for any* v.

PROOF. Let d_1 and d_2 be BIBD(v, v, k)'s with common replication $r = k$, and suppose they are orthogonal. Imposing d_1 on d_2, one may assume that $\lambda_{11}^{P12} = 1$, for $\lambda_{jj'}^{P12} > 0$ for some j, j', and $\lambda_{11}^{P12} > 1$ requires $\lambda_{11}^{G12} = k\lambda_{11}^{P12} > k$ for orthogonality, contradicting $r = k$. Hence $\lambda_{11}^{P12} = 1$, $\lambda_{11}^{G12} = k$, and treatment 1 of d_1 occurs in every block containing treatment 1 of d_2. Also, once treatment 1 of d_1 appears in a block with any treatment of d_2, they must appear together on a plot if they are to satisfy (7). It follows from the above reasoning that treatment 1 of d_1 appears in every block containing every treatment of d_2 that appears in a block with treatment 1 of d_2, that is, treatment 1 of d_1 appears in every block, contradicting the fact that d_1 is a BIBD. □

Returning to the 2×3 problem, one can apply Theorem 3.3 using an orthogonal pair of BNRC$(3, 2, 2, 3)$'s as given in Table 1, meaning that an orthogonal pair BNRC$(v, b, 2, 3)$ can be constructed for any v with b equal to twice that given in Corollary 3.4. Again, this can be improved if Theorem 3.1 can be applied, which is possible at least for $v = 5$ and 6. For $v = 5$ the relevant pair of orthogonal BIBD's is in Preece (1966a). For $v = 6$ the two sets of initial blocks $(0, 1, 2), (0, 2, \infty) \pmod 5$ and $(3, \infty, 4), (4, 3, 1) \pmod 5$ generate orthogonal BIBD's. Improvement in this manner for $v = 4$ and 7 is ruled out by Theorem 3.5.

Furthermore, all of the series of Lemmas 2.2, 2.5 and 2.6 can be used in either Theorem 3.1 or Theorem 3.3. For Theorem 3.3, since they are all based on initial blocks over finite fields, the BIBD's may be written as orthogonal BNRC(v, b_0, k, v)'s, where b_0 is the number of initial blocks, then combined with any BIBD($v^*, b^*, k^* = v$) to give orthogonal BNRC($v^*, b_0 b^*, k, v$)'s. They can also be combined into orthogonal sets of GYD(v, b, k)'s, or into orthogonal BNRC's with intermediate numbers of blocks, as shown by the next theorem [cf. Morgan and Uddin (1993a), Theorem 8 and its proof].

THEOREM 3.6. *The existence of orthogonal* BNRC(v_i, b, p, q)*'s*, $i = 1, \dots, t$, *for which* b *is a multiple of an integer* s, *implies the existence of orthogonal* BNRC($v_i, b/s, p, sq$)*'s*, $i = 1, \dots, t$.

Applying Theorem 3.6 to Corollary 3.2 gives a simple example.

COROLLARY 3.7. *There is a pair of orthogonal* BNRC($v, v(v-1)/4, 2, 4$) *for every* $v \equiv 0$ *or* $1 \pmod 4$.

Starting with a balanced incomplete block design with nested rows and columns, Cheng [(1986), Theorem 2.1] constructed another design of the same type and with fewer rows by retaining only those rows corresponding to treatments in blocks of a BIBD. Gupta (1992) constructed BNRC's using the same technique but starting with a YS(v, k) in place of the BIBRC. In fact one may start with any BNRC, and the method will preserve orthogonality if starting with an orthogonal set. For completeness this is stated as the last theorem of this subsection.

THEOREM 3.8. *If there exists a* BIBD($v_1 = p_2, b_1, k = p$) *and a set of* t *mutually orthogonal BNRC's for* v *treatments in* b_2 *blocks of size* $p_2 \times q$, *then there exists a set of* t *mutually orthogonal* BNRC($v, b = b_1 b_2, p, q$)*'s*.

3.2. *Constructions using the method of differences*. The next few results directly construct orthogonal sets of BNRC's using initial blocks based on finite fields.

THEOREM 3.9. *Let* $v = mq + 1$ *be a prime power and let* $2 \le p \le q$. *The initial blocks*

$$A_{i,l} = x^{i+l-2}(x^0, x^m, \dots, x^{(p-1)m})' \otimes (x^0, x^m, \dots, x^{(q-1)m}),$$

$l = 1, \dots, m$, *generate a* BNRC(v, mv, p, q) d_i, *and* $d_1, \dots, d_m$ *are mutually orthogonal*.

PROOF. The BNRC d_1 is due to Bagchi, Mukhopadhyay and Sinha [(1990), Corollary 3.2.2]. The column component design of d_i is generated by mq initial columns which can be written as the $p \times mq$ array

$$\begin{aligned}(x^0, x^1, \dots, x^{m-1}) &\otimes (x^0, x^m, \dots, x^{(q-1)m}) \otimes x^{i-1}(x^0, x^m, \dots, x^{(p-1)m})' \\ &= (GF_v - \{0\}) \otimes x^{i-1}(x^0, x^m, \dots, x^{(p-1)m})'.\end{aligned}$$

It is apparent that differences for plots and for columns of d_i versus $d_{i'}$ are balanced and contain no zeros. Since the number of column differences is p times that of plot differences, orthogonality is established. □

The design d_1 of Theorem 3.9 may be orthogonalized in at least two other ways. The proofs of this and Theorem 3.11 are further straightforward exercises in the method of differences, so they are omitted. These theorems employ the arrays $A(p,t,1)$ and $A^{\sigma}(p,t,1)$ defined prior to Lemma 2.6.

THEOREM 3.10. *Let $A_{1,l}$, $l = 1,\dots,m$, be the initial blocks of d_1 of Theorem* 3.9.

(i) *Blocks $A_{i,l} = x^{(i-1)pm}A_{1,l}$ for $l = 1,\dots,m$ generate a BNRC d_i, and $d_1, d_2,\dots,d_{\mathrm{int}(q/p)}$ are mutually orthogonal.*

(ii) *Let π_i be the permutation of $(1,2,\dots,p)$ defined by the ith column of an $A(p,t,1)$. Thinking of this as a permutation of the rows of an array, let $A_{i,l} = \pi_i(A_{1,l})$. Then the $A_{i,l}$, $l = 1,\dots,m$, generate a BNRC d_i, and $d_1, d_2,\dots,d_t$ are mutually orthogonal.*

A table of arrays $A(p,t,1)$ for $3 \le p \le 7$ is given in Morgan and Uddin [(1993b), page 440].

THEOREM 3.11. *Let $v = 2mq + 1$ with odd q be a prime power, and let $2 \le p \le q$. The initial blocks $A_{i,l}$ found by permuting the rows of*

$$A_{1,l} = x^{l-1}(x^0, x^{2m},\dots,x^{(p-1)2m})' \otimes (x^0, x^{2m},\dots,x^{(q-1)2m}),$$

$l = 1,\dots,m$, according to the ith column of an $A^{\sigma}(p,t,1)$ generate a BNRC(v, mv, p, q) *d_i, and $d_1,\dots,d_t$ are mutually orthogonal.*

The BNRC d_1 of Theorem 3.11 is Theorem 7 of Morgan and Uddin (1993a)

THEOREM 3.12. *Let $v = 2mq + 1$ with odd q be a prime power.*

(i) *There exist $(q-1)/2$ mutually orthogonal* BNRC$(v, mv, 2, q)$*'s.*

(ii) *Let $t \le (q-1)/2$ be written as $t = t^* p^*$. Then there exist t^* mutually orthogonal* BNRC$(v, mv, 2p^*, q)$*'s.*

PROOF. The initial blocks for the BNRC d_i, $i = 1,\dots,(q-1)/2$, are

$$A_{i,l} = x^{(l-1)}\begin{pmatrix} x^{2m(i-1)} & x^{2m(i-1)+2m} & \cdots & x^{2m(i-1)+2(q-1)m} \\ x^{2m(q-i)} & x^{2m(q-i)+2m} & \cdots & x^{2m(q-i)+2(q-1)m} \end{pmatrix},$$

$l = 1, \ldots, m$. For part (ii), the initial blocks for d_i, $i = 1, \ldots, t^*$, are

$$B_{i,l} = \begin{pmatrix} A_{(i-1)p^*+1,l} \\ A_{(i-1)p^*+2,l} \\ \vdots \\ A_{(i-1)p^*+p^*,l} \end{pmatrix},$$

$l = 1, \ldots, m$. □

3.3. *Related designs*. Having established orthogonality conditions for, and via Theorem 3.6 given numerous examples of, superimposed Youden designs, a comparison of these results to superimposed Youden squares already in the literature is warrented. Hedayat, Seiden and Federer (1972), generalizing an ideal of Clarke (1963), define a *mutually balanced Youden design for ordered pairs* as a set of YS$(v, v, v-1)$'s for which superimposition of any pair of the designs produces each ordered pair of distinct treatments on plots exactly once. They define a *mutually balanced Youden design for unordered pairs* as a set of YS$(v, v, (v-1)/2)$'s for which superimposition of any pair of the designs produces each unordered pair of distinct treatments on plots exactly once. However the authors do not explicitly mention a model or a specific analysis, leaving somewhat vague the motivation for these properties. It is evident from (2), (3) and (7) that these requirements do not determine the information matrix for the bottom-stratum analysis, and Theorem 3.5 says that they cannot imply orthogonality. While it is possible to construct the designs for ordered pairs with the column concurrences required for variance balance in the estimation of main effects [using the method of Preece (1966b), page 5, to make $C_{ii'} = xI + y11'$ for $i \neq i'$, $v \neq 2, 6$; see also Hedayat, Seiden and Federer (1972), Theorem 3.1], designs like those of Hedayat, Seiden and Federer [(1972), Theorem 3.2] do not have this property, and it is an open question as to what statistically meaningful benefits they may offer. Similar comments apply to the designs for unordered pairs and to a related class due to Afsarinejad and Hedayat (1975): the design conditions they impose, in and of themselves, have no apparent usefulness for the standard analysis. Nonetheless, many of the designs these authors construct do have the additional properties needed for variance balance [see Singh and Singh (1984)], and some could turn out to be optimal for their settings, in which (5) cannot hold.

Using Theorem 3.6, orthogonal sets of GYD's can be constructed with more than v columns. Relative to the designs just discussed, comparable parameter values are found with Corollary 2.3 and Lemma 2.5 (with $m = 2$ and $h = 1$) results, giving orthogonal GYD$(v, 2v, (v-1)/2)$'s and orthogonal GYD$(v, 2v, (v+1)/2)$'s. To see the gain afforded by the orthogonality, consider the orthogonal GYD$(7, 14, 3)$'s and the variance balanced YS$(7, 7, 3)$'s for unordered pairs of Hedayat, Seiden and Federer [(1972), Theorem 4.4] generated by the two initial columns $(1, 2, 4)'$ and $(2, 4, 1)'$. The value tr$(C_{11} -$

$C_{12}C_{22}^{-}C_{21})^{-}$ for the former is $\frac{9}{7}$, for the latter is 6. Multiplying the former by 2 to account for the additional experimental units, the relative efficiency for the nonorthogonal design is only $\frac{3}{7}$. Better YS$(7,7,3)$'s for unordered pairs can be found: the same comparison for the two initial columns $(1,2,4)'$ and $(3,6,5)'$ gives a relative efficiency of $\frac{6}{7}$. This also demonstrates that within the bounds of the conditions for Youden designs for unordered pairs a great range of efficiency behavior is possible.

Orthogonal pairs of Youden designs with different numbers of treatments have been constructed by Preece (1982, 1991, 1992, 1993, 1994) and Christofi (1994). These *double Youden rectangles* are superimpositions of a $p \times v$ GYD for v treatments on a $p \times v$ GYD for p treatments so that orthogonality holds in the bottom-stratum analysis. Unlike in the framework of this paper, one of the GYD's has symbols orthogonal to rows, the other to columns. In that situation, (7) is not sufficient for orthogonality of the pair.

Finally, Hedayat, Parker and Federer (1970), using pairs of orthogonal Latin squares with common transversal [in the language of Section 2.2, $TA(v, v(v-1), 4)$], construct pairs of GYD$(v, v+1, v)$'s that are variance balanced for the bottom-stratum analysis.

This is also a good opportunity to explain the relationship with multidimensional incomplete block designs. If one begins with any t superimposed orthogonal GYD(v,b,k)'s, and lets $t-1$ of the treatments sets denote levels of $t-1$ additional blocking factors, the result is a $k \times b \times v^{t-1}$ $(t+1)$-dimensional incomplete block design for v treatments. That the design is universally optimal for treatment comparisons follows easily from the orthogonality. This technique has been used at least as far back as Causey (1968) and as recently as Stewart and Bradley (1991). The class IIIa designs of Stewart and Bradley (1991) are constructed in this manner starting with the GYD$(v, 2v, (v+1)/2)$'s of Lemma 2.5, although they realized this construction only for prime v of the form $4s+3$. Their result is now extended in terms of the available v as well as the number of dimensions. One can also start with t mutually orthogonal BIBD's for $v_1, \dots, v_t$ treatments which do not arrange into GYD's and get a $b \times v_1 \times v_2 \times \cdots \times v_{t-1}$ t-dimensional incomplete block design for v_t treatments. This is the construction of classes IIa, IIb and IIIb of Stewart and Bradley (1991), starting with the BIBD's from Theorems 2(a) and 4(c) of Street (1981) mentioned at the end of Section 2.1. The IIa and IIb classes can be had for any odd prime power if instead the orthogonal BIBD's of Seberry (1979) are used. Related constructions are in Agrawal and Sharma (1978).

4. Orthogonal collections of Latin squares. To motivate the topic of this section, return to the problem of constructing a pair of orthogonal 2×2 BNRC's. In Section 3 this was accomplished by applying Theorem 3.1 to the BIBD's of Lemma 2.8 using a 2×2 Latin square. A different solution would be to apply Theorem 3.3 using any BIBD$(v, v(v-1)/2, 2)$ and a pair of mutually orthogonal Latin squares of side 2, but for the fact that the Latin squares do not exist. Howevery, this does not mean that there are not orthogonal sets of t BNRC$(2, b, 2, 2)$'s for larger values of b and possibly $t > 2$. For instance, with $b = 2$ there is

$$
\begin{array}{ccc}
d_1 & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \\[1em]
d_2 & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array}
\end{array}
$$

and for $b = 4$ there is

$$
\begin{array}{ccccc}
d_1 & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \\[1em]
d_2 & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} & \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \\[1em]
d_3 & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \\[1em]
d_4 & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} & \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} & \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}
\end{array}
$$

DEFINITION. A collection of t ordered sets each containing b Latin squares of order v is said to be a (t, b)-*orthogonal collection* if, upon superimposition of the b corresponding squares of any two sets, every ordered pair of symbols occurs b times.

Each set of b squares in a (t, b)-orthogonal collection is a BNRC(v, b, v, v). A pair of these BNRC's is orthognal if and only if (7) holds, and, since $\lambda_{jj'}^{Gii'} = bv$ for all pairs and symbols, the orthogonality requirement is simply $\lambda_{jj'}^{Pii'} = b$, the requirement of the definition. Although the motivation for these designs has been as input in Theorem 3.3 for constructions with $p = q < v$, they are also legimately useful designs in and of themselves. For $v = 4$, for instance, $b = 1 \Rightarrow t \leq 3$, but $b = 2$ allows t as large as 5, this being a special case of the main result of this section. Before proceeding to that result, the $v = 2$ case will be disposed of.

THEOREM 4.1. *There is a* (t, b)-*orthogonal collection of Latin squares of order* 2 *provided a Hadamard matrix of order b exists*, *where*

$$b = 4 \times \left(1 + \text{largest integer less than } \frac{t}{4}\right).$$

PROOF. There are only two Latin squares of order 2. Represent these by 1 and -1. Then an ordered b-set of order 2 Latin squares can be thought of as a vector of 1's and -1's. Two such ordered b-sets are orthogonal if and only if their vectors are orthogonal in the usual sense. So given t orthogonal $b \times 1$

vectors of 1's and -1's, the collection can be constructed. These are any t rows of the Hadamard matrix. □

For a comprehensive introductory treatment of Hadamard matrices, see Hall (1986). For $t = 2$, Theorem 4.1 should be eschewed in favor of the design displayed in the first paragraph of this section. Any even b can be accommodated for $t = 2$.

Combining squares of a (t, b)-orthogonal collection in the manner of Theorem 3.6 produces t orthogonal BNRC$(v, 1, v, bv)$, which are t orthogonal $v \times bv$ F-squares. Typically one should expect to achieve larger t for the F-squares, however, as that setting is less restrictive than this one. The only mention we have found in the literature of superimposing a new treatment set on $b > 1$ Latin squares is Freeman [(1972), Section 6], who briefly discusses the 3×3 and 4×4 squares.

Bose (1938) and Stevens (1939) made an invaluable contribution to statistical theory when they established the existence of complete sets of mutually orthogonal Latin squares of order v, a prime power. In the BNRC terminology, these would be $(v - 1, 1)$-orthogonal collections. Their technique was to use the finite field of order v to define initial rows that would generate the orthogonal squares. The final result of this paper takes the same approach to constructing (t, b)-orthogonal collections for $t > v - 1$ and $b > 1$.

THEOREM 4.2. *Let $v = p^n$ be a power of the prime p. Then there exists a $((2p - 1)p^{n-1} - 1, p)$-orthogonal collection of Latin squares of order v.*

Especially interesting is the case $p = 2$: there is a $(3(2^{n-1}) - 1, 2)$-orthogonal collection of Latin squares of order 2^n. For $n = 2$ this is the $(5, 2)$-collection of side 4 squares mentioned above and displayed later in Table 3. The immediate question is whether or not these are the largest such sets that can be constructed; that is, what is the upper bound on t? A simple counting of degrees of freedom shows that in any (t, b)-collection, $t \le b(v - 1)$. For $v = b = 2$ this is achieved by Theorem 4.2, but for $v = 4$ and $b = 2$ the theorem falls one short. We have managed to enumerate all of the possibilities for $v = 4$ and $b = 2$ and by doing so have established that, for this case, 5 is indeed the largest achievable t. So unlike for the Latin square ($b = 1$) case, "complete sets" do not necessarily exist even for prime power numbers of treatments. Other examples of this observation are provided by the designs with $v = 2$, for which the bound is $t \le b$. If b is a multiple of 4 and the Hadamard matrix exists, Theorem 4.1 attains the bound, but if b is odd only $t = 1$ is possible, and if $b \equiv 2 \pmod 4$ only $t = 2$ can be had.

To start the proof of Theorem 4.2, define $v - 1$ vectors a_j by

$$a_j = x^j(0, x^0, x^1, \dots, x^{v-2}), \qquad j = 0, 1, \dots, v - 2, \tag{8}$$

and $v - 1$ vectors b_q by

$$b_q = x^q(0, x^0, x^p, \dots, x^{(v-2)p}), \qquad q = 0, 1, \dots, v - 2. \tag{9}$$

These will be used as initial rows of Latin squares in the orthogonal collection presented in (11) below, but first the differences between the a_j's and the b_q's will be studied. Define $(v-1)^2$ difference vectors $d_{j,q}$ by $d_{j,q} = a_j - b_q$. The elements of $d_{j,q}$ are 0 and $d^k_{j,q} = x^{k+j} - x^{q+kp}$ for $k = 0, 1, \dots, v-2$. Now $x^{k+j} - x^{q+kp} = x^q(x^{k+j-q} - x^{kp}) = x^q(x^{j-q}y - y^p)$, where $y = x^k$. Going over all values of k goes over all nonzero values of y, so $d_{j,q}$ in some order is

$$d_{jq} = \{x^q(y^p - x^{j-q}y) \colon y \in GF_v\} \tag{10}$$

Next a series of statements concerning properties of the $d_{j,q}$'s will be derived. Several of these employ the relationship $(w+u)^p = w^p + u^p$ for every $w, u \in GF_{p^n}$. Also needed is that the elements of the subfield $\mathscr{Z}_p$ [the integers (mod p)] in GF_{p^n} are $x^{a((p^n-1)/(p-1))}$, $a = 0, 1, \dots, p-2$.

PROPERTY 1. Write $j - q = l(p-1) + m$ for some $l \in \{0, 1, \dots, (p^n - 1)/(p-1) - 1\}$ and some $m \in \{0, 1, \dots, p-2\}$. Then (10) is

$$x^q x^{lp}(y^p - x^m y), \qquad y \in GF_v.$$

PROOF. The factor involving y in (10) is

$$\begin{aligned} y^p - x^{j-q}y = y^p - x^{l(p-1)+m}y &= x^{l(p-1)}\left[x^{-l(p-1)}y^p - x^m y\right] \\ &= x^{l(p-1)}\left[x^l(x^{-l}y)^p - x^{m+l}(x^{-l}y)\right] \\ &= x^{lp}\left[(y')^p - x^m(y')\right], \qquad y' = x^{-l}y. \end{aligned}$$ □

PROPERTY 2. If $j - q \not\equiv 0 \pmod{p-1}$, then (10) is an ordering of GF_v.

PROOF. This is true if and only if $y_1 \neq y_2 \Rightarrow y_1^p - x^{j-q}y_1 \neq y_2^p - x^{j-q}y_2$. By property 1, it is sufficient to prove this for $j - q = m \in \{1, 2, \dots, p-2\}$. If $y_1^p - x^{j-q}y_1 = y_2^p - x^{j-q}y_2$, then $y_1^p - y_2^p = x^m(y_1 - y_2) \Rightarrow (y_1 - y_2)^p = x^m(y_1 - y_2) \Rightarrow x^m = (y_1 - y_2)^{p-1}$. Now $y_1 - y_2 = x^a$ for some a, so $x^m = x^{a(p-1)} \Rightarrow m \equiv a(p-1)(\text{mod } p^n - 1) \Rightarrow m(\text{mod } p-1) \equiv [a(p-1)(\text{mod } p^n - 1)](\text{mod } p-1) = 0$, a contradiction. □

PROPERTY 3. If $j - q \equiv 0 \pmod{p-1}$, then (10) is p copies of each of p^{n-1} elements of GF_v, and these p^{n-1} elements form an additive subgroup.

PROOF. By Property 1, (10) is $x^q x^{lp}(y^p - y)$ for $y \in GF_v$, so consider the list $(y^p - y)$, $y \in GF_v$. Let w be any element of GF_v and let $i \in \mathscr{Z}_p$. Then

$$(w+i)^p - (w+i) = w^p + i^p - (w+i) = w^p + i - (w+i) = w^p - w,$$

showing that $y = w, w+1, \dots, w+p-1$ all give the same value to $y^p - y$. Since $y^p - y = z$ has at most p roots for any fixed z, the list is indeed p^{n-1} different values p times each. It is now simple to check that the group requirements hold. □

Properties 4 and 5 relate different d_{jq}'s of (10) for special values of j and q. They are easy consequences of Property 1.

PROPERTY 4. If $q_2 - q_1 = h(p-1)$, then $d_{j,q_2} = x^{-h} d_{j,q_1}$.

PROPERTY 5. If $j_2 - j_1 = h(p-1)$, then $d_{j_2,q} = x^{hp} d_{j_1,q}$.

PROPERTY 6. If $d^*_{0,0}$ is the set of p^{n-1} distinct elements of $d_{0,0}$, then the following hold:

(i) $d^*_{0,0}$ is an additive subgroup;
(ii) $id^*_{0,0}$ is a reordering of $d^*_{0,0}$ for any $i \in \mathscr{Z}_p$;
(iii) the cosets of $d^*_{0,0}$ in GF_v are $iw_1 + d^*_{0,0}$, $i \in \mathscr{Z}_p$, for any fixed $w_1 \notin d^*_{0,0}$.

PROOF. Property 3 gives (i). For $z = y^p - y \in d^*_{0,0}$, $iz = iy^p - iy = i^p y^p - iy = (iy)^p - (iy) \in d^*_{0,0} \Rightarrow$ (ii). If $i_1 w_1$ and $i_2 w_1$ are in the same coset, then $i_1 w_1 - i_2 w_1 = i_3 w_1 \in d^*_{0,0}$ $(i_3 = i_1 - i_2) \Rightarrow i_3 = 0$ or [by (ii)] $w_1 \in d^*_{0,0} \Rightarrow$ (iii). □

Now fix w_1 as any element of GF_v which is not of the form $y^p - y$, that is, which is not in $d_{0,0}$. Let $w_1, w_2, \dots, w_{p^{n-1}}$ be an ordering of the elements of $w_1 + d^*_{0,0}$. Let h_i be defined by $x^{h_i} = w_1^{-1} w_i$, $i = 1, 2, \dots, p^{n-1}$. Consider the following collection of $(p^n - 1) + (p-1)p^{n-1}$ ordered sets of p initial rows each. Each initial row generates a Latin square in the usual way, so this is specifying $(2p-1)p^{n-1} - 1$ BNRC(v, p, v, v)'s which, to satisfy Theorem 4.2, we claim are mutually orthogonal. The sets are:

$$\{a_j + \alpha_{j0}, a_j + \alpha_{j1}, \dots, a_j + \alpha_{j,p-1}\} \quad \text{for } j = 0, 1, \dots, v-2;$$

$$(11) \qquad \{b_q, b_q, \dots, b_q\} \quad \text{for } q = (p-1)h_i + g \text{ with } i = 1, \dots, p^{n-1} \text{ and } g = 0, 1, \dots, p-2.$$

The a_j's are from (8), and the b_q's from (9), so the p initial rows in each set are p vectors of order $1 \times v$. The α's are defined by

$$\alpha_{00} = 0, \qquad \alpha_{0k} = w_1 x^{(k-1)((p^n-1)/(p-1))} \quad \text{and} \quad \alpha_{jk} = x^{lp+c}\alpha_{0k},$$

for $k = 1, 2, \dots, p-1$, $j = l(p-1) + c$, $c = 0, 1, \dots, p-2$ and $l = 0, 1, \dots, (p^n - 1)/(p-1) - 1$. The subscripting implicit in the definition of the q's has been suppressed to ease the notation.

First it will be shown that the b_q's are all distinct, that is, that the q's are all distinct (mod $p^n - 1$). Write $q_1 = (p-1)h_i + g_1$ and $q_2 = (p-1)h_{i'} +$

g_2, where, with no loss of generality, $g_2 \geq g_1$. If $q_1 \equiv q_2 \pmod{p^n - 1}$, then $(p-1)(h_i - h_{i'}) \equiv g_1 - g_2 \pmod{p^n - 1}$

$$\Rightarrow [(p-1)(h_i - h_{i'})(\text{mod } p^n - 1)](\text{mod } p - 1) = (g_1 - g_2)(\text{mod } p - 1)$$

$$\Rightarrow g_1 = g_2 \Rightarrow (p-1)(h_i - h_{i'}) \equiv 0 \pmod{p^n - 1}$$

$$\Rightarrow h_i - h_{i'} \equiv 0 \left(\text{mod } \frac{p^n - 1}{p - 1}\right) \Rightarrow h_i = h_{i'} + a\left(\frac{p^n - 1}{p - 1}\right), \text{ some } a$$

$$\Rightarrow x^{h_i} = i'' x^{h_{i'}} \text{ for } i'' = x^{a}\left(\frac{p^n - 1}{p - 1}\right) \in \mathscr{Z}_p \Rightarrow w_i = i'' w_{i'}.$$

However, w_i and $w_{i'}$ are in the same coset of $d^*_{0,0}$, so $w_{i'} + d^*_{0,0} = i'' w_{i'} + d^*_{0,0} \Rightarrow i'' = 1$ (by Property 6) $\Rightarrow a = 0 \Rightarrow h_i = h_{i'} \Rightarrow q_1 = q_2$.

Hence the sets are all distinct. Now to show orthogonality of the sets it must be shown that the plot differences for the superimposed vectors of any two sets are p copies of GF_v. It is easy to see that a_j and $a_{j'}$ of (8) generate orthogonal Latin squares, and thus so too do $a_j + \alpha_{jk}$ and $a_{j'} + \alpha_{j'k}$ for $j \neq j'$. Likewise, for $q \neq q'$, the squares generated by b_q and $b_{q'}$ [see (9)] are also orthogonal. If $j - q \not\equiv 0 \pmod{p - 1}$, then Property 2 implies that the squares generated by $a_j + \alpha_{jk}$ and b_q are orthogonal.

Finally, consider $a_j + \alpha_{jk}$ and b_q with $j - q \equiv 0 \pmod{p-1}$. So $j = (l - 1) + c$ and $q = h_i(p-1) + c$ for some i. The differences for $k = 1, 2, \ldots, p - 1$ are

$$\begin{aligned}
a_j - b_q + \alpha_{jk} &= x^c a_{l(p-1)} - x^c b_{h_i(p-1)} + x^{lp+c}\alpha_{0k} \\
&= x^c\left[d_{l(p-1), h_i(p-1)} + x^{lp} w_1 x^{(k-1)((p^n-1)/(p-1))}\right] \\
&= x_c\left[x^{-h_i} d_{l(p-1),0} + x^{lp} w_1 x^{(k-1)((p^n-1)/(p-1))}\right] \quad \text{(Property 4)} \\
&= x^c\left[x^{-h_i} x^{lp} d_{0,0} + x^{lp} w_1 x^{(k-1)((p^n-1)/(p-1))}\right] \quad \text{(Property 5)} \\
&= x^{c-h_i+lp}\left[d_{0,0} + x^{h_i} w_1 x^{(k-1)((p^n-1)/(p-1))}\right] \\
&= x^{c-h_i+lp}\left[d_{0,0} + w_i x^{(k-1)((p^n-1)/(p-1))}\right]
\end{aligned}$$

and for $k = 0$ the differences are $a_j - b_q = x^{c-h_i+lp} d_{0,0}$. Using (iii) of Property 6, these are together all elements of GF_v p times each, completing the proof of Theorem 4.2.

As an example, the five pairs of order-4 Latin squares will be constructed. The addition table for GF_4 (also the subtraction table) is shown in Table 2. The vectors $a_0 = (0, x^0, x^1, x^2)$ and $b_0 = (0, x^0, x^2, x^1)$ give the difference vector $d_{0,0} = (0, 0, x^0, x^0)$. Taking $w_1 = x^1$ gives $w_2 = x^2$, $h_1 = 0$ and $h_2 = 1$. The alpha's are then $\alpha_{00} = \alpha_{10} = \alpha_{20} = 0$, $\alpha_{01} = x^1$, $\alpha_{11} = x^0$ and $\alpha_{21} = x^2$. The five pairs of squares are thus generated from the five pairs of vectors $(a_0, a_0 + x^1)$, $(x^1 a_0, x^1 a_0 + x^0)$, $(x^2 a_0, x^2 a_0 + x^2)$, (b_0, b_0) and $(x^1 b_0, x^1 b_0)$. Writing i in lieu of x^i, and 3 for the field zero, the design is shown in Table 3. Observe that the Bose squares are imbedded in this construction, in this example appearing as the first square in the first three pairs.

TABLE 2
Addition table for GF_4

$+$	0	x^0	x^1	x^2
0	0	x^0	x^1	x^2
x^0	x^0	0	x^2	x^1
x^1	x^1	x^2	0	x^0
x^2	x^2	x^1	x^0	0

If (t, b)-orthogonal collections are to be used as input into Theorem 3.3, one would like b as small as possible. For $b = 1$ the extensive literature on mutually orthogonal Latin squares is available. The goal with larger b is to achieve larger t than is possible with $b = 1$, that is, to accommodate more factors in the design. Most useful in this context is $b = 2$, which Theorem 4.2 addresses for v a power of 2. By other methods we have found $(t, 2)$-orthogonal collections for other values of v, including a $(4, 2)$-orthogonal collection of order-6 Latin squares. This and related results will be reported elsewhere.

5. Summary. One of the limitations to the applicability of BIBD's relative to other classes of available incomplete block designs is the number of experimental units required to be able to meet the design conditions. When there are multiple blocking factors, such as in the row–column or nested row

TABLE 3
A $(5, 2)$*-orthogonal collection of order* 4

3	0	1	2	1	2	3	0
0	3	2	1	2	1	0	3
1	2	3	0	3	0	1	2
2	1	0	3	0	3	2	1
3	1	2	0	0	2	1	3
0	2	1	3	3	1	2	0
1	3	0	2	2	0	3	1
2	0	3	1	1	3	0	2
3	2	0	1	2	3	1	0
0	1	3	2	1	0	2	3
1	0	2	3	0	1	3	2
2	3	1	0	3	2	0	1
3	0	2	1	3	0	2	1
0	3	1	2	0	3	1	2
1	2	0	3	1	2	0	3
2	1	3	0	2	1	3	0
3	1	0	2	3	1	0	2
0	2	3	1	0	2	3	1
1	3	2	0	1	3	2	0
2	0	1	3	2	0	1	3

and column settings, that number can become prohibitive if optimality and complete symmetry of the information matrix are sought, as well evidenced by most of the available series of BNRC's. The methods of this paper address that problem for factorial treatment sets in nested rows and columns in the same way that fractional fractorial plans do in unblocked factorial experiments: it may be observed that many of the designs in this paper have fewer experimental units than a single replicate of the full $(\pi_{i=1}^{t} v_i)$ factorial would require. While the expressed intent was to find these optimal main effects plans in nested row and column designs, a most welcome product of this investigation is that the same goals have been met for the row–column and the unstructured block settings. Moreover, these designs may also be used in multistage experimentation and as optimal multidimensional incomplete block designs.

Most of the designs here constructed are for the symmetrical v^t factorial, which of course can also be used for factors with numbers of levels which are factorizations of v. A simple example is a Lemma 2.8 design for $v = 12$, say, which can serve as an $\frac{11}{12}$ replicate of the $2^4 \times 3^2$ experiment in blocks of size 2, for which all of the main effects and some of the two-factor interactions are estimable. As another example, a Lemma 2.5 design for $v = 9$ $(h = 1, f = 4)$ can serve as an $\frac{10}{279}$ replicate of the 3^8 experiment in blocks of size 5.

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