The Annals of Statistics
1998, Vol. 26, No. 2, 472–493

NONPARAMETRIC COMPARISON OF MEAN DIRECTIONS OR MEAN AXES

BY RUDOLF BERAN[1] AND NICHOLAS I. FISHER

University of California, Berkeley and CSIRO

Samples of directional or axial measurements arise in geophysical, biological and econometric contexts. We represent the rotational difference between two mean directions (or two mean axes) as a direction (or axis). We then construct nonparametric simultaneous confidence sets for all pairwise rotational differences among the mean directions or mean axes of s samples. By specialization, this methodology yields nonparametric simultaneous tests for pairwise equality of directional means or axes.

1. Introduction. Geological data often contains directions or axes measured in three dimensions. Examples are directions of remanent magnetization of lava cores, axes normal to geological folding planes, or positions on the surface of the earth. Biological measurements may include directions or axes in two dimensions. Instances are the directions in which birds or insects fly after release and time-of-day viewed as a circular variable. In econometrics, season or month-of-the-year are discrete circular variables that correspond to angular portions of the earth's orbit around the sun.

Formal statistical methods for analyzing samples of directional or axial data often rely on two parametric models: the Langevin–Fisher–von Mises distribution for directional data and the Bingham distribution for axial data [cf. Mardia (1972), Watson (1983), Fisher, Lewis and Embleton (1993), Fisher (1995)]. These two models are the simplest canonical exponential families for directional and axial data, respectively, that are closed under rotations of the coordinate system [Beran (1979)]. The cost of this mathematical simplicity is rotational symmetry of the respective density about its mean direction or mean axis. Semiparametric models studied in Watson (1983) share this symmetry. However, sets of directional or axial data in the fields of application cited above may lack even approximate rotational symmetry. The only generally applicable methods available to date are based on the spherical median direction or axis [Fisher (1985); Fisher, Lunn and Davies (1993)], and then only for large samples.

Section 1.1 recalls the definitions of random direction or random axis and of mean direction or mean axis. This paper develops nonparametric methods for comparing all pairs of mean directions (or mean axes) of s independent samples of directions (or axes). No shape assumptions are imposed upon the

Received April 1996; revised April 1997.

[1]Research partially supported by Sonderforschungsbereich 373 at Humboldt-Universität zu Berlin and by NSF Grant DMS-95-30492.

AMS 1991 *subject classifications*. Primary 62H11; secondary 62G15.

Key words and phrases. Rotational difference, simultaneous confidence sets, simultaneous tests, bootstrap.

unknown distribution of the observations in each sample. In practice, a mean direction or a mean axis summarizes a distribution most effectively when that distribution is unimodal. Central to our methodology is the representation of the rotational difference between two mean directions (or two mean axes) as a direction (or axis). This representation has two important features: it enables us to apply nonparametric methods for inference about one mean direction or one mean axis, and it suggests plots for rotational differences in two or three dimensions. Details of the representation are the subject of Section 1.2. Section 2 constructs confidence sets for the rotational difference between the mean directions (or mean axes) of two samples. Simultaneous confidence sets for all pairwise rotational differences among the mean directions (or mean axes) of s samples are developed in Section 3. Section 4 illustrates our methodology on data.

1.1. *Random directions and axes.* A *direction* in q-dimensions is a $q \times 1$ unit vector d. It is naturally visualized as a point on the unit sphere $S_q = \{u \in R^q\colon |u| = 1\}$, where $|\cdot|$ is Euclidean norm. A *random direction* x is then a random element of the sphere S_q metrized by the norm $|\cdot|$.

Suppose that $x_1, \dots, x_n$ are iid random directions, each having distribution P on S_q. Let $m(P) = \mathrm{E}x_i$ and assume that $|m(P)| > 0$. This condition excludes certain highly symmetric distributions such as the uniform distribution on S_q. The *mean direction* of x_i (or of P) is defined to be the direction $d = d(P)$ that minimizes $\mathrm{E}|x_i - d|^2$. Equivalently, $d(P) = m(P)/|m(P)|$. The *sample mean direction* is then $\hat{d}_n = \hat{m}_n/|\hat{m}_n|$, where $\hat{m}_n$ is the sample mean $n^{-1}\sum_{i=1}^n x_i$. It is readily seen that $\hat{d}_n$ is the direction d minimizing $\sum_{i=1}^n |x_i - d|^2$.

An *axis* in q-dimensions is an unordered pair of directions $\{e, -e\}$, where $e \in S_q$. We will write $\pm e$ in place of the pair and, by convention, will require that the first nonzero component of e be positive. An axis is naturally visualized as a pair of diametrically opposed points on the sphere S_q or as the one of those points that lies on a selected hemisphere of S_q. The axis $\pm e$ may equivalently be represented as the orthogonal projection ee' whose rank is 1. Given the value of ee', we can recover $\pm e$ as the sign-ambiguous eigenvector of ee' associated with the eigenvalue 1. The other eigenvalues are 0. The set of all axes in q dimensions may thus be identified with the set T_q of all orthogonal projections of R^q into one-dimensional subspaces. The Euclidean norm of a matrix $B = \{b_{ij}\colon 1 \le i,\ j \le q\}$ is defined by $||B||^2 = \mathrm{tr}(B'B)$. A *random axis* $\pm x$ corresponds to a random element xx' of the set of projections T_q metrized by the norm $||\cdot||$.

Suppose that $\pm x_1, \dots, \pm x_n$ are iid random axes, each having distribution Q on T_q. Let $M(Q) = \mathrm{E}x_i x_i'$, the second moment matrix which does not depend on the sign of x_i, and assume that the largest eigenvalue of $M(Q)$ is unique. This condition again excludes highly symmetric axial distributions. The *mean axis* of $\pm x_i$ (or of Q) is defined to be the axis $\pm e = \pm e(Q)$ that minimizes $\mathrm{E}||x_i x_i' - ee'||^2$. Equivalently, $\pm e(Q)$ is the sign-ambiguous eigenvector associated with the largest eigenvalue of $M(Q)$. The *sample mean axis* is then $\pm\hat{e}_n$, the sign-ambiguous eigenvector associated with the largest eigenvalue of the sample moment matrix $\hat{M}_n = n^{-1}\sum_{i=1}^n x_i x_i'$. It is easily seen that $\pm\hat{e}_n$ is the axis $\pm e$ minimizing $\sum_{i=1}^n ||x_i x_i' - ee'||^2$.

For further discussion of mean directions or mean axes, see Kent (1992).

1.2. *Rotations between directions or axes.*

Two dimensions. Suppose first that d_1 and d_2 are two directions in R^2. Let $\theta \in [0, 2\pi]$ denote the angle of the counterclockwise rotation that takes d_1 into d_2. Equivalent descriptions of this rotation are the angle θ or the direction vector in S_2 defined by

$$\rho_2(\theta) = (\cos(\theta), \sin(\theta))'. \tag{1.1}$$

The latter coincides with the direction vector

$$r_2(d_1, d_2) = (d_1 \cdot d_2, \det(d_1, d_2))', \tag{1.2}$$

where $\cdot$ denotes the dot product or inner product of vector analysis. The orthogonal matrix that rotates d_1 counterclockwise into d_2 is

$$\begin{pmatrix} d_1 \cdot d_2 & -\det(d_1, d_2) \\ \det(d_2, d_2) & d_1 \cdot d_2 \end{pmatrix}. \tag{1.3}$$

This matrix representation of the rotation is equivalent to the vector representations (1.2) or (1.1).

Suppose next that $\pm e_1$ and $\pm e_2$ are two axes in R^2. Two counterclockwise rotations take the first axis into the second axis. A counterclockwise rotation through angle θ (say) takes e_1 into e_2 and $-e_1$ into $-e_2$. A counterclockwise rotation through angle $\theta + \pi$ [mod 2π] then takes e_1 into $-e_2$ and $-e_1$ into e_2. Since $\rho_2(\theta + \pi) = -\rho_2(\theta)$, the pair of rotations bringing $\pm e_1$ into $\pm e_2$ can be specified by the axis $\pm\rho_2(\theta)$, where $\rho_2(\theta) = r_2(e_1, e_2)$, or by the unordered pair $\{\theta, \theta + \pi\}$ [mod 2π]. We will denote this unordered pair by $\psi_2(\theta)$.

Three dimensions. Now suppose that d_1 and d_2 are two directions in R^3. Let $\nu \in S_3$ be the normal to the plane determined by d_1 and d_2 such that, with thumb pointing in the direction ν, a right-hand-rule rotation through the angle $\theta \in [0, \pi]$ takes d_1 into d_2. Equivalent descriptions of this rotation are the angle-direction pair (θ, ν) or the direction vector in S_4 defined by

$$\rho_3(\theta, \nu) = (\cos(\theta), \sin(\theta)\nu')'. \tag{1.4}$$

The latter coincides with the direction

$$r_3(d_1, d_2) = (d_1 \cdot d_2, (d_1 \times d_2)')', \tag{1.5}$$

where $\times$ denotes the cross-product of vector analysis. For the special cases $d_1 = d_2$ or $d_1 = -d_2$, the direction $r_3(d_1, d_2)$ becomes $(1, 0, 0, 0)$ or $(-1, 0, 0, 0)$, respectively. In the form (1.4), θ is then, respectively, 0 or π and ν is arbitrary. The representation of rotation (1.5) as an orthogonal matrix is

$$d_1 d_1' + (\nu \times d_1)(\nu \times d_2)' + \nu\nu' \quad \text{where } \nu = (d_1 \times d_2)/|d_1 \times d_2|. \tag{1.6}$$

The Euler angles of the rotation can be recovered by matching this matrix with its counterpart expressed in terms of those angles.

Unlike the unit vector $r_3(d_1, d_2)$, the vector product $d_1 \times d_2$ fails to give a one-to-one representation of the rotation that takes d_1 into d_2. Indeed, $d_1 \times d_2$ equals $(0, 0, 0)'$ whether $d_1 = d_2$ or $d_1 = -d_2$. Moreover, small perturbations of d_1 and d_2 around the cases $d_1 = d_2$ and $d_1 = -d_2$ can reverse the direction of the vector product. These properties make the vector product unsuitable as a basis for confidence sets when the true rotation angle θ is near either 0 or π.

Finally, suppose that $\pm e_1$ and $\pm e_2$ are two axes in R^3. In the plane determined by these axes, two right-hand-rule rotations bring the first axis onto the second: the rotation (θ, ν) (say) that takes e_1 into e_2 and $-e_1$ into $-e_2$ and the rotation $(\pi - \theta, -\nu)$ that takes e_1 into $-e_2$ and $-e_1$ into e_2. Since $\rho_3(\pi - \theta, -\nu) = -\rho_3(\theta, \nu)$, the pair of rotations bringing $\pm e_1$ to $\pm e_2$ can be specified by the axis $\pm\rho_3(\theta, \nu)$, where $\rho_3(\theta, \nu) = r_3(e_1, e_2)$, or by the unordered pair $\{(\theta, \nu), (\pi - \theta, -\nu)\}$. We will denote this unordered pair by $\psi_3(\theta, \nu)$.

2. One pairwise comparison. The first problem is to compare the mean directions of two directional samples or the mean axes of two axial samples. Using the language of Section 1, we will represent the rotation between two mean directions as a direction and the pair of rotations between two mean axes as an axis. Nonparametric confidence cones for these representations then express our solution. Closely related to this solution are nonparametric confidence cones for the mean direction or mean axis of a single sample. Because these one-sample confidence cones are implicit in our treatment of the two-sample problem, we omit further details.

2.1. *Comparing two mean directions.* We consider two directional samples in either two or three dimensions. Suppose that the directions $x_{i,1}, \ldots, x_{i,n_i}$ in sample i are iid with common unknown distribution P_i on S_q. The samples are independent, with combined size $n = n_1 + n_2$. Let $\hat{P}_{n,i}$ denote the empirical distribution of sample i and let $\hat{P}_n = (\hat{P}_{n,1}, \hat{P}_{n,2})$ be the associated estimator of $P = (P_1, P_2)$. In the asymptotic theory of Section 5, we will suppose that neither n_1 nor n_2 is tiny relative to n. Let d_i denote the unknown mean direction of P_i, assumed to exist, and let $\hat{d}_{n,i}$ denote the ith sample mean direction, as defined in Section 2.1. The sample rotation $\hat{\rho}_{n,q} = r_q(\hat{d}_{n,1}, \hat{d}_{n,2})$ estimates the rotation $\rho_q = r_q(d_1, d_2)$ that brings d_1 into d_2.

Two dimensions. Here $\hat{\rho}_{n,2} = \rho_2(\hat{\theta}_n)$, where $\hat{\theta}_n$ is the counterclockwise angle from $\hat{d}_{n,1}$ to $\hat{d}_{n,2}$. The angle $\hat{\theta}_n$ estimates the unknown counterclockwise angle θ between the mean directions d_1 and d_2. The rotational difference $\rho_2(\hat{\theta}_n)$ may be plotted as a direction in R^2. By constructing a nonparametric confidence cone for $\rho_2(\theta)$ about $\hat{\rho}_{n,2}$, we will obtain a confidence interval for θ.

Let

$$H_{n,2}(P) = \mathscr{L}\left[\hat{\rho}_{n,2} \cdot \rho_2(\theta) | P\right] \tag{2.1}$$

denote the sampling distribution of the dot product and let $\hat{h}_{n,2}(\alpha)$ be the smallest αth quantile of the estimated sampling distribution $H_{n,2}(\hat{P}_n)$. Numerical approximation of this quantile by bootstrap resampling is discussed in Section 2.3. Our nonparametric confidence set for θ is then

$$\begin{aligned} C_{n,2} &= \{\theta\colon \hat{\rho}_{n,2}\cdot\rho_2(\theta) \geq \hat{h}_{n,2}(\alpha),\ \ \theta\in[0,2\pi]\} \\ &= \{\theta\colon \hat{\theta}_n - \cos^{-1}(\hat{h}_{n,2}(\alpha)) \leq \theta \leq \hat{\theta}_n + \cos^{-1}(\hat{h}_{n,2}(\alpha))\ [\mathrm{mod}\, 2\pi]\}. \end{aligned} \tag{2.2}$$

The range of the inverse cosine is taken to be $[0, \pi]$. We will see in Theorem 2.1 that the coverage probability of this confidence interval converges asymptotically to $1-\alpha$. If the confidence cone contains $\rho_2(\theta) = (1, 0)'$, which represents counterclockwise rotation $\theta = 0$, then the two mean directions d_1 and d_2 are not significantly different. The implied test rejects the null hypothesis $d_1 = d_2$ if and only if $\cos(\hat{\theta}_n) < \hat{h}_{n,2}(\alpha)$. It has asymptotic rejection probability α under each model satisfying that null hypothesis.

Confidence interval (2.2) may be compared with a nonparametric confidence interval for the *smaller* of θ and $2\pi - \theta$ [mod 2π] that was devised by Lewis and Fisher (1995). Because this quantity confounds the rotations θ and $2\pi-\theta$ unless $\theta = 0$ or π, the asymptotic distribution of its estimator changes sharply at those end points. This discontinuity in the asymptotics complicates the control of coverage probability. Confidence interval $C_{n,2}$ avoids the difficulty, as will be seen in the proofs, by distinguishing between θ and $2\pi - \theta$.

Three dimensions. Here $\hat{\rho}_{n,3} = \rho_3(\hat{\theta}_n, \hat{\nu}_n)$, where $\hat{\nu}_n = (\hat{d}_{n,1}\times\hat{d}_{n,2})/|\hat{d}_{n,1}\times\hat{d}_{n,2}|$ and $\hat{\theta}_n \in [0, \pi]$. With thumb pointing in direction $\hat{\nu}_n$, a right-hand-rule rotation through angle $\hat{\theta}_n$ brings $\hat{d}_{n,1}$ into coincidence with $\hat{d}_{n,2}$. The rotation $(\hat{\theta}_n, \hat{\nu}_n)$ estimates the rotation (θ, ν) that brings d_1 into d_2. The angle $\hat{\theta}_n$ may be plotted as the direction $(\cos(\hat{\theta}_n), \sin(\hat{\theta}_n))'$ in the upper halfplane of R^2. The vector $\hat{\nu}_n$ may be plotted as a direction in R^3. By constructing a nonparametric confidence cone for $\rho_3(\theta, \nu)$, we will obtain a joint confidence interval for the rotation (θ, ν).

Let

$$H_{n,3}(P) = \mathscr{L}[\hat{\rho}_{n,3}\cdot\rho_3(\theta,\nu)|P] \tag{2.3}$$

and let $\hat{h}_{n,3}(\alpha)$ denote the smallest αth quantile of the estimated sampling distribution $H_{n,3}(\hat{P}_n)$. Our confidence set for (θ, ν) is then

$$C_{n,3} = \{(\theta,\nu)\colon \hat{\rho}_{n,3}\cdot\rho_3(\theta,\nu) \geq \hat{h}_{n,3}(\alpha),\ \ \theta\in[0,\pi],\ \ \nu\in S_3\}. \tag{2.4}$$

Numerical approximation of $\hat{h}_{n,3}(\alpha)$ is discussed in Section 2.3. We will also see there that the asymptotic coverage probability of $C_{n,3}$ is $1-\alpha$.

Confidence set $C_{n,3}$ may be decomposed into slices, one slice for every acceptable value of θ. By *acceptable* value, we mean any $\theta \in [0, \pi]$ such that (θ, ν) lies in $C_{n,3}$ for at least one direction ν. The set of acceptable values is just

$$A_n = \{\theta\colon \cos(\hat{\theta}_n - \theta) \geq \hat{h}_{n,3}(\alpha),\ \ \theta\in[0,\pi]\}. \tag{2.5}$$

Indeed, if $(\theta, \nu) \in C_{n,3}$, then

$$\begin{aligned} \hat{h}_{n,3}(\alpha) &\le \hat{\rho}_{n,3} \cdot \rho_3(\theta, \nu) = \cos(\hat{\theta}_n)\cos(\theta) + \sin(\hat{\theta}_n)\sin(\theta)\hat{\nu}_n \cdot \nu \\ &\le \cos(\hat{\theta}_n - \theta). \end{aligned} \tag{2.6}$$

Conversely, if $\cos(\hat{\theta}_n - \theta) \ge \hat{h}_{n,3}(\alpha)$, then $(\theta, \hat{\nu}_n) \in C_{n,3}$ because

$$\hat{\rho}_{n,3} \cdot \rho_3(\theta, \hat{\nu}_n) = \cos(\hat{\theta}_n - \theta) \ge \hat{h}_{n,3}(\alpha). \tag{2.7}$$

For every $\theta \in A_n$, let

$$C_{n,3}(\theta) = \{\nu\colon \nu \cdot \hat{\nu}_n \ge [\hat{h}_{n,3}(\alpha) - \cos(\theta)\cos(\hat{\theta}_n)]/[\sin(\theta)\sin(\hat{\theta}_n)],\ \ \nu \in S_3\}. \tag{2.8}$$

The confidence set $C_{n,3}$ consists of those rotations (θ, ν) such that $\theta \in A_n$ and then $\nu \in C_{n,3}(\theta)$. Note that A_n contains $\theta = 0$ if and only if the confidence set $C_{n,3}$ contains $\rho_3(\theta, \nu) = (1, 0, 0, 0)'$. In this event, the directional means d_1 and d_2 are not significantly different. The implied test rejects the null hypothesis $d_1 = d_2$ if and only if $\cos(\hat{\theta}_n) < \hat{h}_{n,3}(\alpha)$. It too has asymptotic rejection probability α under each model satisfying that null hypothesis.

2.2. *Comparing two mean axes.* Comparing mean axes differs from comparing mean directions because of the sign-ambiguity of an axis. Suppose that the axes $\pm x_{i,1} \cdots \pm x_{i,n_i}$ in sample i are iid, each having unknown distribution Q_i on T_q. The samples are independent, with combined size $n = n_1 + n_2$. Let $\hat{Q}_{n,i}$ denote empirical distribution of sample i and let $\hat{Q}_n = (\hat{Q}_{n,1}, \hat{Q}_{n,2})$ be the associated estimator of $Q = (Q_1, Q_2)$. Let $\pm e_i$ denote the unknown mean axis of Q_i, assumed to exist, and let $\hat{e}_{n,i}$ denote the ith sample mean axis, as defined in Section 2.1. The unordered pair of sample rotations $\pm\hat{\rho}_{n,q}$, where $\hat{\rho}_{n,q} = r_q(\hat{e}_{n,1}, \hat{e}_{n,2})$, estimates the unordered pair of rotations $\pm\rho_q$, where $\rho_q = r_q(e_1, e_2)$, that bring $\pm e_1$ into $\pm e_2$.

Two dimensions. In this case, $\hat{\rho}_{n,2} = \rho_2(\hat{\theta}_n)$, where $\hat{\theta}_n$ is the counterclockwise angle from $\hat{e}_{n,1}$ to $\hat{e}_{n,2}$. The unordered pair $\psi_2(\hat{\theta}_n)$ estimates the unknown counterclockwise angles $\psi_2(\theta)$ that rotate mean axis $\pm e_1$ into $\pm e_2$. The rotational differences $\psi_2(\hat{\theta}_n)$ may be plotted as the axis $\pm\rho_2(\hat{\theta}_n)$ in R^2 or, alternatively, as the half of the axis that lies in the right halfplane of R^2. By constructing a nonparametric confidence double-cone for $\rho_2(\theta)$, we will obtain a confidence interval for $\psi_2(\theta)$.

Let

$$K_{n,2}(Q) = \mathscr{L}[|\hat{\rho}_{n,2} \cdot \rho_2(\theta)|\,|Q], \tag{2.9}$$

noting the absolute value operator around the dot product, and let $\hat{k}_{n,2}(\alpha)$ be the smallest αth quantile of the estimated sampling distribution $K_{n,2}(\hat{Q}_n)$. Numerical approximation of this quantile by bootstrap resampling is discussed in Section 2.3. Our nonparametric confidence set for $\psi_2(\theta)$ is then

$$\begin{aligned} D_{n,2} &= \{\theta\colon |\hat{\rho}_{n,2} \cdot \rho_2(\theta)| \ge \hat{k}_{n,2}(\alpha),\ \ \theta \in [0, 2\pi]\} \\ &= \{\psi_2(\theta)\colon \hat{\theta}_n - \cos^{-1}(\hat{k}_{n,2}(\alpha)) \le \theta \le \hat{\theta}_n + \cos^{-1}(\hat{k}_{n,2}(\alpha))\}. \end{aligned} \tag{2.10}$$

We will see in Theorem 2.1 that the coverage probability of this confidence interval converges asymptotically to $1-\alpha$. If the confidence double-cone contains $\rho_2(\theta) = \pm(1, 0)'$, which represents the rotation pair $\{0, \pi\}$, then the mean axes $\pm e_1$ and $\pm e_2$ are not significantly different. The implied formal test rejects the null hypothesis $\pm e_1 = \pm e_2$ if and only if $|\cos(\hat{\theta}_n)| < \hat{k}_{n,2}(\alpha)$.

Three dimensions. Here $\hat{\rho}_{n,3} = \rho_3(\hat{\theta}_n, \hat{\nu}_n)$, where $\hat{\nu}_n = (\hat{e}_{n,1} \times \hat{e}_{n,2})/|\hat{e}_{n,1} \times \hat{e}_{n,2}|$ and $\hat{\theta}_n \in [0, \pi]$. With thumb pointing in direction $\hat{\nu}_n$, a right-hand-rule rotation through angle $\hat{\theta}_n$ brings $\hat{e}_{n,1}$ into coincidence with $\hat{e}_{n,2}$. The unordered pair $\psi_3(\hat{\theta}_n, \hat{\nu}_n)$ estimates the pair of rotations $\psi_3(\theta, \nu)$ that bring mean axis $\pm e_1$ into mean axis $\pm e_2$. We may plot the acute angle in the pair $\{\hat{\theta}_n, \pi - \hat{\theta}_n\}$ as the direction $(\cos(\hat{\theta}_n), \sin(\hat{\theta}_n))'$ in the upper right quadrant of R^2 and may plot the half of the axis $\pm\hat{\nu}_n$ that is associated with this angle as a direction in R^3. By constructing a nonparametric confidence double-cone for $\rho_3(\theta, \nu)$, we will obtain a joint confidence interval for the rotation pair $\psi_3(\theta, \nu)$.

Let

$$K_{n,3}(Q) = \mathscr{L}\left[\left|\hat{\rho}_{n,3} \cdot \rho_3(\theta, \nu)\right| \,|\, Q\right] \tag{2.11}$$

and let $\hat{k}_{n,3}(\alpha)$ denote the smallest αth quantile of the estimated sampling distribution $K_{n,3}(\hat{Q}_n)$. Our confidence set for $\psi_3(\theta, \nu)$ is then

$$\begin{aligned} D_{n,3} &= \left\{(\theta, \nu)\colon |\hat{\rho}_{n,3} \cdot \rho_3(\theta, \nu)| \geq \hat{k}_{n,3}(\alpha),\ \ \theta \in [0, \pi],\ \ \nu \in S_3\right\} \\ &= \left\{\psi_3(\theta, \nu)\colon \hat{\rho}_{n,3} \cdot \rho_3(\theta, \nu) \geq \hat{k}_{n,3}(\alpha),\ \ \theta \in [0, \pi],\ \ \nu \in S_3\right\}. \end{aligned} \tag{2.12}$$

Numerical approximation of $\hat{k}_{n,3}(\alpha)$ is discussed in Section 2.3. We will also see there that the asymptotic coverage probability of $D_{n,3}$ is $1 - \alpha$.

Let

$$B_n = \left\{\theta\colon\ \cos(\hat{\theta}_n - \theta) \geq \hat{k}_{n,3}(\alpha),\ \ \theta \in [0, \pi]\right\}. \tag{2.13}$$

For every $\theta \in B_n$, let

$$D_{n,3}(\theta) = \left\{\nu\colon \nu \cdot \hat{\nu}_n \geq [\hat{k}_{n,3}(\alpha) - \cos(\theta)\cos(\hat{\theta}_n)]/[\sin(\theta)\sin(\hat{\theta}_n)],\ \ \nu \in S_3\right\}. \tag{2.14}$$

The confidence set $D_{n,3}$ is the set of rotation pairs $\psi_3(\theta, \nu)$ such that $\theta \in B_n$ and then $\nu \in D_{n,3}(\theta)$. Note that B_n contains $\theta = 0$ if and only if $D_{n,3}$ contains $\psi_3(\theta, \nu) = \{(0, \nu), (\pi, \nu)\}$, or equivalently, $\rho_2(\theta, \nu) = \pm(1, 0, 0, 0)'$. In that event, the mean axes $\pm e_1$ and $\pm e_2$ are not significantly different. The implied test rejects the null hypothesis $\pm e_1 = \pm e_2$ if and only if $|\cos(\hat{\theta}_n)| < \hat{k}_{n,3}(\alpha)$.

2.3. *Critical values and coverage probabilities.* Consider two artificial independent samples of directions. The first artificial sample, $x^*_{1,1}, \ldots, x^*_{1,n_1}$, is drawn randomly, with replacement, from the first real sample of directions; the second artificial sample, $x^*_{2,1}, \ldots, x^*_{2,n_2}$ is drawn randomly, with replacement, from the second real sample of directions. In other words, the conditional distribution of the random variables in the ith artificial sample, given the data, is $\hat{P}_{n,i}$ and the artificial samples are conditionally independent.

For $q = 2$ or 3, let $\rho^*_{n,q}$ denote the recomputation of $\hat{\rho}_{n,q} = r_q(\hat{d}_{n,1}, \hat{d}_{n,2})$ when the two real samples are replaced by the respective artificial samples. Then $H_{n,q}(\hat{P}_n)$ is just the conditional distribution of the dot product $\rho^*_{n,q} \cdot \hat{\rho}_{n,q}$ given the data. This fact supports the following nonparametric bootstrap algorithm for approximating the confidence sets $C_{n,q}$ defined in Section 2.1.

1. Choose positive integers m and B such that $m/(B+1) = \alpha$ and B is as large as feasible.
2. Replicate the construction of the two artificial samples B times, the replicates being conditionally independent given the data.
3. For each of the B replicates, recompute the value of the dot product $\rho^*_{n,q} \cdot \hat{\rho}_{n,q}$.
4. Of the values found in step 3, take the mth smallest as the desired approximation to the critical value $\hat{h}_{n,q}(\alpha)$ for $C_{n,q}$.

In this approximation to $C_{n,q}$, the choice $m/(B+1) = \alpha$ avoids a Monte Carlo bias in the coverage probability while a large value of B lessens the randomness of the critical value obtained in step 4 [cf. Hall (1986)].

Numerical approximation of the axial confidence sets $D_{n,q}$ in Section 2.2 is very similar. We replace the estimated distribution vector $\hat{P}_n$ with $\hat{Q}_n$ and replace the directions $x_{i,j}$, $x^*_{i,j}$ with the axes $\pm x_{i,j}$, $\pm x^*_{i,j}$. Let $\rho^*_{n,q}$ now denote the recomputation of $\hat{\rho}_{n,q} = r_q(\hat{e}_{n,1}, \hat{e}_{n,2})$ when the two samples of axes are replaced by the respective artificial samples. Then $K_{n,q}(\hat{Q}_n)$ is just the conditional distribution of $|\rho^*_{n,q} \cdot \hat{\rho}_{n,q}|$ given the data. The nonparametric bootstrap algorithm for approximating the critical value $\hat{k}_{n,q}(\alpha)$ of confidence set $D_{n,q}$ is as above, the dot product in step 3 being replaced by its absolute value.

THEOREM 2.1. *For $i = 1, 2$, suppose that the mean direction of P_i and the mean axis of Q_i are well-defined and that neither distribution is a point mass. Suppose that $n_i/n \to c_i \in (0,1)$ as $n \to \infty$ and that $\alpha \in (0,1)$. Then, for $q = 2$,*

$$\lim_{n\to\infty} \Pr[C_{n,2} \ni \theta | P] = \lim_{n\to\infty} \Pr[D_{n,2} \ni \psi_2(\theta) | Q] = 1 - \alpha \tag{2.15}$$

and, for $q = 3$,

$$\lim_{n\to\infty} \Pr[C_{n,3} \ni (\theta, \nu) | P] = \lim_{n\to\infty} \Pr[D_{n,3} \ni \psi_3(\theta, \nu) | Q] = 1 - \alpha. \tag{2.16}$$

This theorem is proved in Section 5. As the proof implicitly shows, direct asymptotic approximations may be found for the critical values of the confidence sets $C_{n,q}$ and $D_{n,q}$. These asymptotic critical values are quantiles of the distribution of a central Gaussian quadratic form whose coefficients are estimated from the data. Such an analytical approach requires substantial algebra and offers no advantage in coverage accuracy over the simple bootstrap critical values just described. Moreover, the bootstrap approach extends easily to the confidence sets for multiple comparisons developed in the next section while the analytical approach does not. Second-order improvements to the critical values are discussed at the end of Section 3.3.

3. Multiple pairwise comparisons. The question now is how to compare all pairs of directional means (or directional axes) for $s \geq 3$ samples. Our solution is a collection of confidence cones, in which confidence set (i, j) estimates the rotation that takes mean direction i into mean direction j (or the pair of rotations that take mean axis i into mean axis j). Critical values for these confidence sets are devised with two goals in mind. First, the probability that these pairwise confidence sets are simultaneously true should be $1 - \alpha$. Second, the probability that confidence set (i, j) is true should be the same for every pair (i, j). The second goal ensures that confidence set (i, j) reflects fairly the information present in the data concerning that pairwise comparison. In the normal linear model, Tukey's method for pairwise comparison of means achieves both goals exactly. In our nonparametric directional and axial models, these goals will be achieved asymptotically in the sample sizes.

3.1. *Comparing several mean directions.* The notation of Section 2.1 is retained for the directions in the s independent samples and for the respective mean directions or sample mean directions. Other notation is modified as follows. The combined size of the samples is now $n = \sum_{i=1}^{s} n_i$. The plug-in estimator of $P = (P_1, P_2, \ldots, P_s)$ based on the empirical distributions of the s samples is $\hat{P}_n = (\hat{P}_{n,1}, \hat{P}_{n,2}, \ldots, \hat{P}_{n,s})$. The superscript i, j will identify statistics or parameters used in comparing samples i and j. Thus, $\hat{\rho}_{n,q}^{i,j} = r_q(\hat{d}_{n,i}, \hat{d}_{n,j})$ estimates the rotation $\rho_q^{i,j} = r_q(d_i, d_j)$ that brings mean direction d_i into mean direction d_j.

Critical values to be used in the simultaneous confidence sets are defined as follows. Let

$$H_{n,q}^{i,j}(P) = \mathscr{L}\left[\hat{\rho}_{n,q}^{i,j} \cdot \rho_q^{i,j} | P\right] \tag{3.1}$$

and denote the cdf of this distribution by $H_{n,q}^{i,j}(\cdot, P)$. Let

$$H_{n,q}(P) = \mathscr{L}\left[\min_{1 \leq i < j \leq s} H_{n,q}^{i,j}(\hat{\rho}_{n,q}^{i,j} \cdot \rho_q^{i,j}, P) | P\right]. \tag{3.2}$$

This distribution is supported on the unit interval. Let $\hat{\beta}_{n,q}$ be the smallest αth quantile of the estimated sampling distribution $H_{n,q}(\hat{P}_n)$ and let $\hat{h}_{n,q}^{i,j}$ be the smallest $\hat{\beta}_{n,q}$th quantile of the estimated distribution $H_{n,q}^{i,j}(\hat{P}_n)$. Numerical approximation of the critical values $\hat{h}_{n,q}^{i,j}$ is discussed in Section 3.3

Two dimensions. Here $\hat{\rho}_{n,2}^{i,j} = \rho_2(\hat{\theta}_n^{i,j})$, where $\hat{\theta}_n^{i,j}$ is the counterclockwise angle from $\hat{d}_{n,i}$ to $\hat{d}_{n,j}$. The angle $\hat{\theta}_n^{i,j}$ estimates the unknown counterclockwise angle $\theta^{i,j}$ between the mean directions d_i and d_j. The simultaneous nonparametric confidence set for these angles is

$$C_{n,2} = \prod_{i<j}^{s} C_{n,2}^{i,j}, \tag{3.3}$$

the direct product of the sets

$$\begin{aligned} C_{n,2}^{i,j} = \Big\{\theta^{i,j}\colon \hat{\theta}_n^{i,j} - \cos^{-1}\big(\hat{h}_{n,2}^{i,j}(\alpha)\big) \le \theta^{i,j} \\ \le \hat{\theta}_n^{i,j} + \cos^{-1}\big(\hat{h}_{n,2}^{i,j}(\alpha)\big)\ [\mathrm{mod}\, 2\pi]\Big\}. \end{aligned} \tag{3.4}$$

In other words, $C_{n,2}$ is the simultaneous assertion of the confidence sets $C_{n,2}^{i,j}$ for every pairwise comparison of mean directions.

We will see in Theorem 3.1 that the simultaneous coverage probability of $C_{n,2}$ for the rotation angles $\{\theta^{i,j}\colon 1 \le i < j \le s\}$ converges asymptotically to $1-\alpha$, while the coverage probability of $C_{n,2}^{i,j}$ for $\theta^{i,j}$ converges to a common value that does not depend on the pair i, j. In this way, $C_{n,2}$ meets the two design goals stated at the beginning of this section.

Three dimensions. Here $\hat{\rho}_{n,3}^{i,j} = \rho_3(\hat{\theta}_n^{i,j}, \hat{\nu}_n^{i,j})$, where $\hat{\nu}_n^{i,j} = (\hat{d}_{n,i} \times \hat{d}_{n,j})/|\hat{d}_{n,i} \times \hat{d}_{n,j}|$ and $\hat{\theta}_n^{i,j} \in [0, \pi]$ identify the right-hand-rule rotation that brings $\hat{d}_{n,i}$ into coincidence with $\hat{d}_{n,j}$. The simultaneous confidence set for the rotations $\{(\theta^{i,j}, \nu^{i,j})\}$ is

$$C_{n,3} = \prod_{i<j}^{s} C_{n,3}^{i,j}, \tag{3.5}$$

where

$$\begin{aligned} C_{n,3}^{i,j} = \Big\{\big(\theta^{i,j}, \nu^{i,j}\big)\colon \hat{\rho}_{n,3}^{i,j} \cdot \rho_3\big(\theta^{i,j}, \nu^{i,j}\big) \ge \hat{h}_{n,3}^{i,j}(\alpha), \\ \theta^{i,j} \in [0, \pi],\ \nu^{i,j} \in S_3\Big\}. \end{aligned} \tag{3.6}$$

As in the two-dimensional case, the coverage probabilities of $C_{n,3}$ and $C_{n,3}^{i,j}$ achieve asymptotically our two design goals. Decomposition of $C_{n,3}^{i,j}$ into slices may be done as in the last paragraph of Section 2.1.

By analogy with the two sample case, simultaneous confidence sets for all pairwise rotational differences among the mean directions of s samples yield simultaneous tests for pairwise equality of those mean directions. The test for pair (i, j) rejects equality if and only if $\cos(\hat{\theta}_n^{i,j}) < \hat{h}_{n,q}^{i,j}(\alpha)$.

3.2. *Comparing several mean axes.* We retain the notation of Section 2.2, adding the superscript i, j as needed to identify quantities used in comparing the mean axes of samples i and j. The joint distribution of the s samples is $\prod_{i=1}^{s} Q_i^{n_i}$. The plug-in estimator of $Q = (Q_1, Q_2, \ldots, Q_s)$ based on the empirical distributions of the samples is $\hat{Q}_n = (\hat{Q}_{n,1}, \hat{Q}_{n,2}, \ldots, \hat{Q}_{n,s})$.

Critical values for the simultaneous confidence sets are defined as follows. Let

$$K_{n,q}^{i,j}(Q) = \mathscr{L}\big[\big|\hat{\rho}_{n,q}^{i,j} \cdot \rho_q^{i,j}\big|\,|Q\big], \tag{3.7}$$

denoting the cdf of this distribution by $K_{n,q}^{i,j}(\cdot, Q)$. Let

$$K_{n,q}(Q) = \mathscr{L}\Big[\min_{1\le i<j\le s} K_{n,q}^{i,j}\big(\big|\hat{\rho}_{n,q}^{i,j} \cdot \rho_q^{i,j}\big|, Q\big)|Q\Big]. \tag{3.8}$$

Let $\hat{\gamma}_{n,q}$ be the smallest αth quantile of the estimated sampling distribution $K_{n,q}(\hat{Q}_n)$ and let $\hat{k}_{n,q}^{i,j}$ be the smallest $\hat{\gamma}_{n,q}$th quantile of $K_{n,q}^{i,j}(\hat{Q}_n)$. Numerical approximation of these critical values is taken up in Section 3.3.

Two dimensions. In this case, $\hat{\rho}_{n,q}^{i,j} = \rho_2(\hat{\theta}_n^{i,j})$, where $\hat{\theta}_n^{i,j}$ is the counterclockwise angle from $\hat{e}_{n,i}$ to $\hat{e}_{n,j}$. The unordered pair $\psi_2(\hat{\theta}_n^{i,j})$ estimates the unknown counterclockwise angles $\psi_2(\theta^{i,j})$ that rotate mean axis $\pm e_i$ into $\pm e_j$. The simultaneous confidence set for these pairs of rotation angles is

$$D_{n,2} = \prod_{i<j}^{s} D_{n,2}^{i,j}, \tag{3.9}$$

where

$$D_{n,2}^{i,j} = \big\{\psi_2(\theta^{i,j})\colon \hat{\theta}_n^{i,j} - \cos^{-1}(\hat{k}_{n,2}^{i,j}(\alpha)) \le \theta^{i,j} \le \hat{\theta}_n^{i,j} + \cos^{-1}(\hat{k}_{n,2}^{i,j}(\alpha))\big\}. \tag{3.10}$$

We will see in Theorem 3.1 that the simultaneous coverage probability of $D_{n,2}$ for the rotation angle pairs $\{\psi_2(\theta^{i,j})\}$ converges asymptotically to $1-\alpha$, while the coverage probability of $D_{n,2}^{i,j}$ for the pair $\psi_2(\theta^{i,j})$ converges to a common value that does not depend on i, j.

Three dimensions. In this case, $\hat{\rho}_{n,3}^{i,j} = \rho_3(\hat{\theta}_n^{i,j}, \hat{\nu}_n^{i,j})$, where $\hat{\nu}_n^{i,j} = (\hat{e}_{n,i} \times \hat{e}_{n,j})/|\hat{e}_{n,i} \times \hat{e}_{n,j}|$ and $\hat{\theta}_n^{i,j} \in [0, \pi]$ identify the right-hand-rule rotation that take $\hat{e}_{n,1}$ into coincidence with $\hat{e}_{n,2}$. The unordered pair $\psi_3(\hat{\theta}_n^{i,j}, \hat{\nu}_n^{i,j})$ estimates the pair of rotations $\psi_3(\theta^{i,j}, \nu^{i,j})$ that bring mean axis $\pm e_i$ into mean axis $\pm e_j$. The simultaneous confidence set for these pairs of rotations is

$$D_{n,3} = \prod_{i<j}^{s} D_{n,3}^{i,j}, \tag{3.11}$$

where

$$\begin{aligned} D_{n,3}^{i,j} = \Big\{\psi_3\big(\theta^{i,j}, \nu^{i,j}\big)\colon \hat{\rho}_{n,3}^{i,j} \cdot \rho_3\big(\theta^{i,j}, \nu^{i,j}\big) \ge \hat{k}_{n,3}^{i,j}(\alpha),& \\ \theta^{i,j} \in [0, \pi],\ \nu^{i,j} \in S_3\Big\}.& \end{aligned} \tag{3.12}$$

As in the two-dimensional case, the coverage probabilities of $D_{n,3}$ and $D_{n,3}^{i,j}$ achieve asymptotically our two design goals. Decomposition of $D_{n,3}^{i,j}$ into slices may be done as in the last paragraph of Section 2.2.

By analogy with the two sample case, simultaneous confidence sets for all pairwise rotational differences among the mean axes of s samples yield simultaneous tests for pairwise equality of those mean axes. The test for pair (i, j) rejects equality if and only if $|\cos(\hat{\theta}_n^{i,j})| < \hat{k}_{n,q}^{i,j}(\alpha)$.

3.3. *Simultaneous critical values and coverage probabilities.* Construct s artificial independent samples as in Section 2.5. The ith artificial sample, $x_{i,1}^*, \ldots, x_{i,n_1}^*$, is drawn randomly, with replacement, from the ith real sample of directions. The conditional distribution of the random variables in the ith artificial sample, given the data, is $\hat{P}_{n,i}$ and the artificial samples are conditionally independent. Let $\rho_{n,q}^{i,j*}$ denote the recomputation of $\hat{\rho}_{n,q}^{i,j} = r_q(\hat{d}_{n,i}, \hat{d}_{n,j})$ when the real samples i and j are replaced by the respective artificial samples. Then $H_{n,q}^{i,j}(\hat{P}_n)$ is just the conditional distribution of the dot product $\rho_{n,q}^{i,j*} \cdot \hat{\rho}_{n,q}^{i,j}$ given the data, while $H_{n,q}(\hat{P}_n)$ is the conditional distribution of $\min_{1 \le i < j \le s} H_{n,q}^{i,j}(\rho_{n,q}^{i,j*} \cdot \hat{\rho}_{n,q}^{i,j}, \hat{P}_n)$. These identifications support the following nonparametric bootstrap algorithm for approximating the confidence set $C_{n,q}$ defined in Section 3.1.

1. Choose positive integers m and B such that $m/(B+1) = \alpha$ and B is as large as feasible.
2. Replicate the construction of the s artificial samples B times, the replicates being conditionally independent given the data.
3. Fix $i < j$. For each of the B replicates, recompute the value of the dot product $\rho_{n,q}^{i,j*} \cdot \hat{\rho}_{n,q}^{i,j}$.
4. For $1 \le k \le B$, find the rank $R_{n,k}^{i,j}$ of the kth recomputation of the dot product among the B recomputations performed in step 3. Do not break ties.
5. Repeat steps 3 and 4 for every pair $1 \le i < j \le s$. Then compute $R_{n,k} = \min_{1 \le i < j \le s} R_{n,k}^{i,j}$ for $1 \le k \le B$. Let b be the mth smallest value among the $\{R_{n,k}: 1 \le k \le B\}$.
6. Fix $i < j$. Of the B recomputations of the dot product in step 3, take the bth smallest as the desired approximation to the critical value $\hat{h}_{n,q}^{i,j}(\alpha)$ for $C_{n,q}^{i,j}$.
7. Repeat step 6 for every pair $1 \le i < j \le s$.

Numerical approximation of the simultaneous axial confidence sets $D_{n,q}^{i,j}$ in Section 3.2 is very similar. We replace the estimated joint distribution $\hat{P}_n$ with $\hat{Q}_n$ and replace the directions $x_{i,j}$, $x_{i,j}^*$ with the axes $\pm x_{i,j}$, $\pm x_{i,j}^*$. Let $\rho_{n,q}^{i,j*}$ now denote the recomputation of $\hat{\rho}_{n,q}^{i,j} = r_q(\hat{e}_{n,i}, \hat{e}_{n,j})$ when the samples i and j are replaced by the respective artificial samples. The algorithm for approximating the critical value $\hat{k}_{n,q}^{i,j}(\alpha)$ of confidence set $D_{n,q}^{i,j}$ is as above, the dot product in steps 3, 4 and 6 being replaced by its absolute value.

THEOREM 3.1. *For $1 \le i \le s$ and $s \ge 3$, suppose that the mean direction of P_i and the mean axis of Q_i are well defined and that none of these distributions* *is a point mass. Suppose that* $n_i/n \to c_i \in (0, 1)$ *as* $n \to \infty$ *and that* $\alpha \in (0, 1)$. *Then there exist constants* $\beta_q(\alpha)$ *and* $\gamma_q(\alpha)$ *such that the following hold.*

When $q = 2$,

$$\lim_{n\to\infty} \Pr\big[C_{n,2}^{i,j} \ni \theta^{i,j} | P\big] = \beta_2(\alpha) \lim_{n\to\infty} \Pr\big[D_{n,2}^{i,j} \ni \psi_2(\theta^{i,j}) | Q\big] = \gamma_2(\alpha) \tag{3.13}$$

for every $1 \le i < j \le s$. *The coverage probabilities of the simultaneous confidence sets* $C_{n,2}$ *and* $D_{n,2}$ *converge to* $1 - \alpha$ *as* $n \to \infty$.

When $q = 3$,

$$\begin{aligned} &\lim_{n\to\infty} \Pr\big[C_{n,3}^{i,j} \ni (\theta^{i,j}, \nu^{i,j}) | P\big] \\ &\quad = \beta_3(\alpha) \lim_{n\to\infty} \Pr\big[D_{n,3}^{i,j} \ni \psi_3(\theta^{i,j}, \nu^{i,j}) | Q\big] = \gamma_3(\alpha) \end{aligned} \tag{3.14}$$

for every $1 \le i < j \le s$. *The coverage probabilities of the simultaneous confidence sets* $C_{n,3}$ *and* $D_{n,3}$ *converge to* $1 - \alpha$ *as* $n \to \infty$.

This theorem is proved in Section 5. The rates at which the marginal and overall coverage probabilities converge in (3.13) and (3.14), or in (2.15) and (2.16), can be increased by suitable double bootstrap refinements to the critical values of the simultaneous confidence sets. In the present problem, double-bootstrapping achieves the order of accuracy of two-term asymptotic expansions for the sampling distributions $H_{n,q}^{i,j}(P)$ and $H_{n,q}(P)$ or $K_{n,q}^{i,j}(Q)$ and $K_{n,q}(Q)$. Single bootstrapping achieves only the order of accuracy of the limit distributions. The methodology and principles are described in Beran (1990) and Hall (1992). A detailed development lies beyond the scope of this paper.

4. Data analyses. We study two examples, for each of which the assumption of rotational symmetry of the data sets is clearly not reasonable.

EXAMPLE 1. Figure 1 shows plots of two random samples of unit vectors arising from a sociological study of the attitudes of 48 individuals to 26 different occupations [see Fisher, Lewis and Embleton (1993), page 194, Example 7.1 for a full description; the data are listed in Appendix B20, Sets C and D]. It is of interest to estimate the difference (if any) between the "preferred" occupational judgements of the two groups. Rotated versions of the raw data plots [Fisher, Lewis and Embleton (1993), page 202] indicate that neither sample exhibits rotational symmetry. Sample estimates of the two mean directions are

$$\hat{d}_{n,1} = (88.7^\circ, 31.6^\circ), \qquad \hat{d}_{n,2} = (96.6^\circ, 60.7^\circ)$$

and the estimated rotational difference is given by

$$\hat{\theta}_n = 30.1^\circ, \qquad \hat{\nu}_n = (-0.1602, 0.2174, 0.9628).$$

A 95% bootstrap confidence region for (θ, ν) based on 200 bootstrap samples is shown in Figure 2. For each vertical pair, the upper figure shows the extent of the confidence interval for θ and a few points equally spaced along

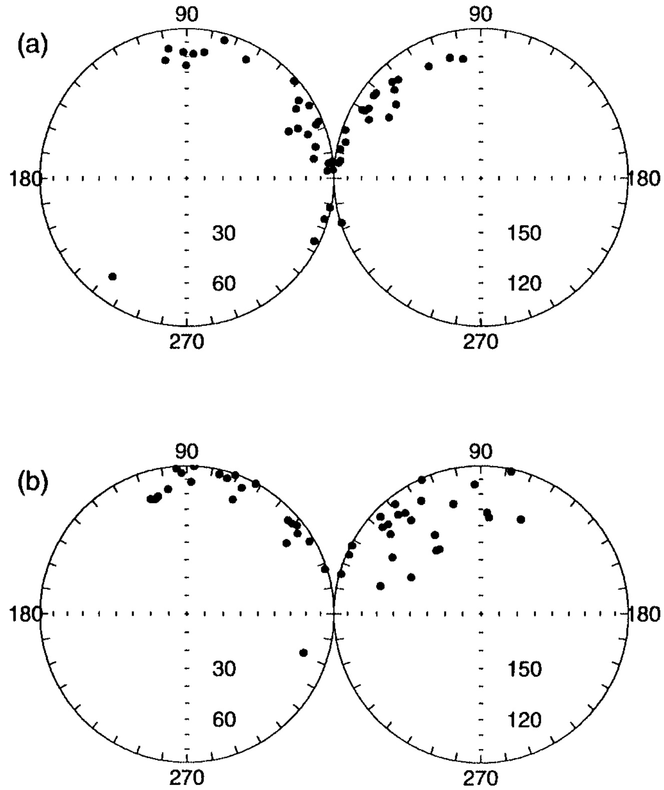

FIG. 1. *Forty-eight spherical measurements of* (a) *rewards and* (b) *social usefulness. For each set, the data are plotted in polar coordinates using an equal-area projection, with the lower hemisphere shown in reverse on the right of the upper hemisphere.*

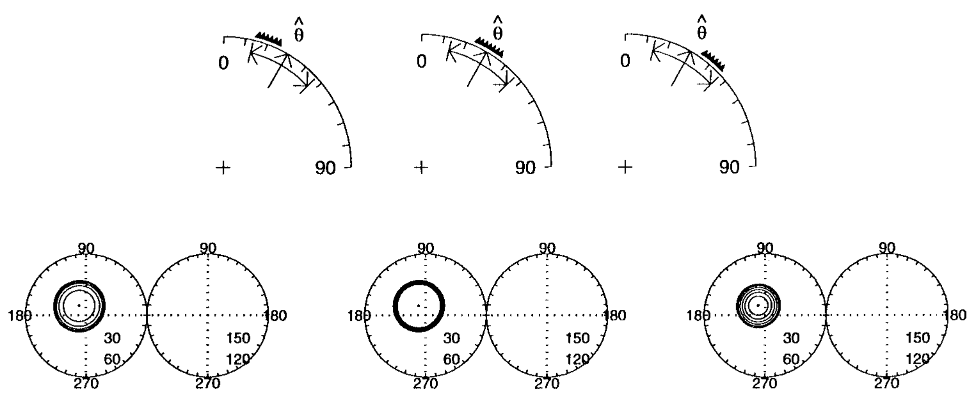

FIG. 2. *Confidence region for rotational difference between mean directions. For each vertical pair, the upper figure shows the extent of the confidence interval for θ and a few points equally spaced along a subset of the interval; and the lower figure shows, for each of these points, the associated confidence cone for ν.*

a sub–set of the interval; and the lower figure shows, for each of these points, the associated confidence cone for ν. (Ideally, one explores the joint confidence region for (θ, ν) interactively on a computer by selecting a point or small subset of the θ region and seeing the corresponding cone or subset of cones for the ν regions highlighted.) The confidence regions provide evidence of difference between the two mean directions and hence between the preferred occupational judgements of the two groups. The direction ν is near the north pole in the coordinate system.

EXAMPLE 2. Consider comparing the mean axes of the two data sets shown in Figure 3. The data are samples of L_0^1 axes (intersections between cleavage and bedding planes of F_1 folds) in Ordovician turbidites [Powell, Cole and Cudahy (1985)]. The sample mean axes are, in their original (plunge, plunge azimuth) coordinates,

$$\hat{e}_{n,1} = (1.6^\circ, 41.3^\circ), \qquad \hat{e}_{n,2} = (-15.2^\circ, 76.6^\circ),$$

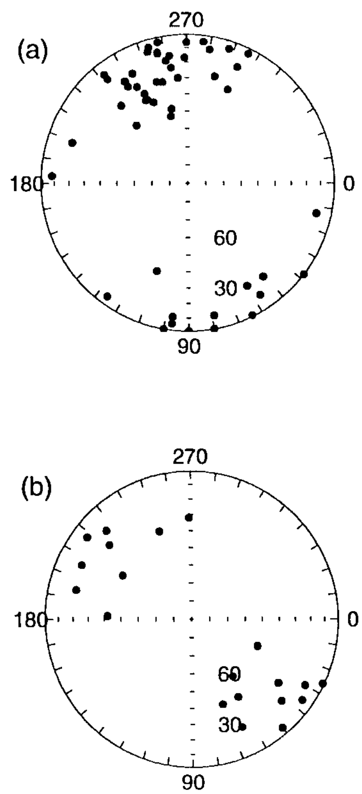

FIG. 3. *Two samples of L_0^1 axes (intersections between cleavage and bedding planes of F_1 folds) in Ordovician turbidites. The data are plotted in equal-area projection using the original coordinates (plunge, plunge azimuth).* (a) 20 *observations* (b) 50 *observations.*

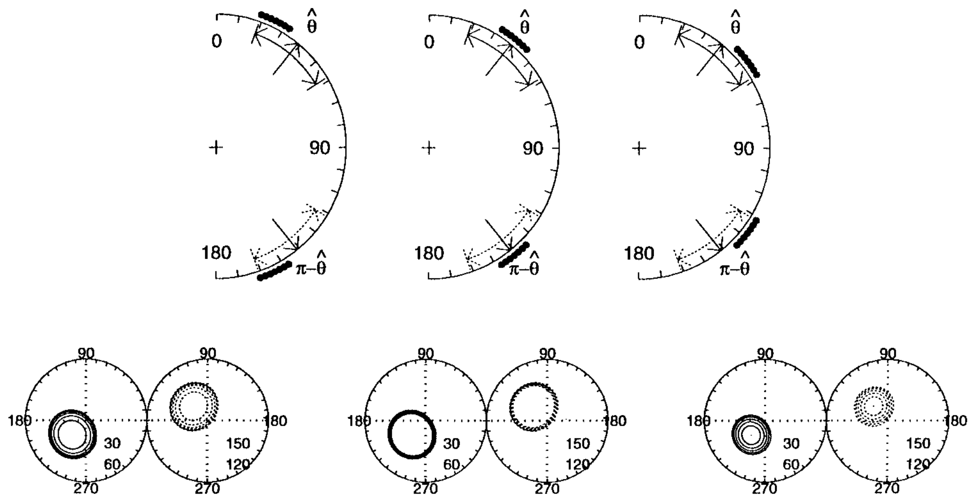

FIG. 4. *Confidence region for rotational difference between mean axes. For each vertical pair, the upper figure shows the extent of the confidence interval for θ and $\pi - \theta$ and a few points equally spaced along a subset of the interval; and the lower figure shows, for each of these points, the associated confidence cone for $\pm\nu$.*

where plunge = colatitude + 90° and plunge azimuth = 360° longitude. The estimated rotational difference is given by

$$\hat{\theta}_n = 38.8^\circ, \qquad \hat{\nu}_n = (-0.3177, -0.3244, -0.8910).$$

A 95% bootstrap confidence region for (θ, ν) based on 200 bootstrap samples is shown in Figure 4. There is clear evidence that the mean axes differ. The axis $\pm\nu$ is near the axis joining north pole and south pole in the coordinate system.

5. Proofs. In the directional case, suppose that $x_{n,1}, x_{n,2}, \ldots, x_{n,n}$ are iid random directions with distribution P_n on S_q. The distributions $\{P_n\}$ converge weakly to a distribution P. Define the sample mean vector $\hat{m}_n$ and the distribution mean vector $m(P_n)$ as in Section 2.1. Let $z(P)$ be a random vector on R^q whose distribution is normal with mean zero and covariance matrix $\Sigma(P) = \lim_{n\to\infty} \operatorname{Cov}(x_{n,1})$.

In the axial case, suppose that $\pm x_{n,1}, \pm x_{n,2} \cdots \pm x_{n,n}$ are iid random axes with distribution Q_n on T_q. The distributions $\{Q_n\}$ converge weakly to a distribution Q. Define the sample moment matrix $\hat{M}_n$ and the distribution moment matrix $M(Q_n)$ as in Section 2.1. For any $q \times q$ matrix $A = \{a_{i,j}\}$, let uvec(A) denote the $q(q+1)/2 \times 1$ column vector $\{\{a_{i,j}: 1 \le i \le j\}, 1 \le j \le q\}$ formed from the elements in the upper triangular half of A, including the diagonal elements. Let $W(Q) = \{w_{i,j}(Q)\}$ be a symmetric $q \times q$ random matrix such that $\mathscr{L}[\text{uvec}(W(Q))]$ is normal with mean zero and covariance matrix

$\Omega(Q)$. The components of $\Omega(Q)$ are determined by the condition

$$\text{Cov}[w_{i,j}(Q), w_{h,k}(Q)] = \lim_{n\to\infty} \text{Cov}\big(x_{n,1,i}x_{n,1,j}, x_{n,1,h}x_{n,1,k}\big), \tag{5.1}$$

where $x_{n,1,i}$ is the ith component of $x_{n,1}$.

LEMMA 5.1. *Suppose that $\{P_n\}$ is any sequence of distributions on S_q such that $P_n \Rightarrow P$ as $n \to \infty$. Then*

$$\mathscr{L}\big[n^{1/2}(\hat{m}_n - m(P_n))\big] \Rightarrow \mathscr{L}[z(P)]. \tag{5.2}$$

Suppose that $\{Q_n\}$ is any sequence of distributions on T_q such that $Q_n \Rightarrow Q$ and $n \to \infty$. Then

$$\mathscr{L}\big[n^{1/2}(\hat{M}_n - M(Q_n))\big] \Rightarrow \mathscr{L}[W(Q)]. \tag{5.3}$$

PROOF. To prove the axial case, let $W_{n,i} = x_{n,i}x'_{n,i} - M(Q_n)$ and observe that

$$n^{1/2}(\hat{M}_n - M(Q_n)) = n^{-1/2}\sum_{i=1}^{n} W_{n,i}. \tag{5.4}$$

Let a be any constant vector of dimension $q(q+1)/2$. By the Lindeberg central limit theorem for a triangular array,

$$\mathscr{L}\left[a'\text{uvec}\left(n^{-1/2}\sum_{i=1}^{n} W_{n,i}\right)\right] \Rightarrow N(0, a'\Omega(Q)a) = \mathscr{L}[a'\text{uvec}(W(Q))] \tag{5.5}$$

provided

$$\lim_{n\to\infty} \text{E}[a'\text{uvec}(W_{n,1})]^2 = a'\Omega(Q)a < \infty \tag{5.6}$$

and

$$\lim_{n\to\infty} \text{E}\big\{[a'\text{uvec}(W_{n,1})]^2 I[|a'\text{uvec}(W_{n,1})| > n^{1/2}\delta\big\} = 0 \tag{5.7}$$

for every positive δ.

Let G_n be the cdf of $a'\text{uvec}(W_{n,1})$ and let G be the cdf of $a'\text{uvec}(W(Q))$. By the hypotheses of the lemma, $G_n \to G$. Since the supports of the $\{G_n\}$ lie within a common compact subset,

$$\lim_{n\to\infty} \int y^2\, dG_n(y) = \int y^2\, dG(y) = a'\Omega(Q)a. \tag{5.8}$$

Properties (5.6) and (5.7) follow immediately. Hence $\mathscr{L}[n^{-1/2}\sum_{i=1}^{n} W_{n,i}] \Rightarrow \mathscr{L}[W(Q)]$, proving (5.3).

The argument for (5.2) is analogous.

PROOF OF THEOREM 2.1. We retain the notation of Sections 1 and 2, with some extensions indicated below.

Directional case. Consider the following triangular array. Each random direction in sample i has distribution $P_{n,i}$, where $P_{n,i} \Rightarrow P_i$. The joint distribution of the two samples is $P_{n,n} = P_{n,1}^{n_1} \times P_{n,2}^{n_2}$. Let $d(P_i) = m(P_i)/|m(P_i)|$. Then $\hat{d}_{n,i} = d(\hat{P}_{n,i})$ is the mean direction of sample i while $d_{n,i} = d(P_{n,i})$ is the mean direction of $P_{n,i}$. Define $P_n = (P_{n,1}, P_{n,2})$ in addition to the earlier notations $P = (P_1, P_2)$ and $\hat{P} = (\hat{P}_{n,1}, \hat{P}_{n,2})$.

Let

$$\tilde{H}_{n,q}(P_n) = \mathscr{L}\big[n|\hat{\rho}_{n,q} - \rho_{n,q}|^2 | P_n\big], \tag{5.9}$$

where $\rho_{n,q} = r_q(d_{n,1}, d_{n,2})$ and $\hat{\rho}_{n,q} = r_q(\hat{d}_{n,1}, \hat{d}_{n,2})$ for r_q defined in (1.2) and (1.5). Let $\tilde{h}_{n,q}(\alpha)$ be the largest $(1-\alpha)$th quantile of the bootstrap distribution $\tilde{H}_{n,q}(\hat{P}_n)$. Let $d_i = d(P_i)$ and $\rho_q = r_q(d_1, d_2)$. Since $|\hat{\rho}_{n,q} - \rho_q|^2 = 2 - 2\hat{\rho}_{n,q} \cdot \rho_q$, the confidence set $C_{n,q}$ may be rewritten in the form

$$C_{n,q} = \big\{\rho_q\colon n|\hat{\rho}_{n,q} - \rho_q|^2 \le \tilde{h}_{n,q}(\alpha),\ |\rho_q| = 1\big\}. \tag{5.10}$$

Under P_n,

$$\begin{aligned} \hat{d}_{n,i} - d_{n,i} &= (I - d_{n,i}d'_{n,i})\hat{d}_{n,i} - 2^{-1}|\hat{d}_{n,i} - d_{n,i}|^2 d_{n,i} \\ &= (I - d_{n,i}d'_{n,i})\hat{d}_{n,i} + O_p(n_i^{-1}) \\ &= |\hat{m}_{n,i}|^{-1}(I - d_{n,i}d'_{n,i})(\hat{m}_{n,i} - m(P_{n,i})) + O_p(n_i^{-1}). \end{aligned} \tag{5.11}$$

The first line is an algebraic identity. To verify the second line, observe that Lemma 5.1 and the differentiability of $d(P)$ as a function of $m(P)$ imply that $n^{1/2}(\hat{d}_{n,i} - d_{n,i})$ has a normal limit distribution. The third line uses the projection property $d_{n,i}d'_{n,i}m(P_{n,i}) = m(P_{n,i})$. Equation (5.11), the convergences $m(P_{n,i}) \to m(P_i)$, and Lemma 5.1 thus imply

$$\begin{aligned} &\mathscr{L}\big[n_i^{1/2}(\hat{d}_{n,i} - d_{n,i})|P_n\big] \\ &\quad \Rightarrow \mathscr{L}\big[|m(P_i)|^{-1}(I - d_i d'_i)z(P_i)\big] = \mathscr{L}[u(P_i)], \text{ say.} \end{aligned} \tag{5.12}$$

The distribution of $u(P_i)$ is Gaussian with mean zero, not a point mass, in the subspace orthogonal to d_i.

Viewing $r_q(d_1, d_2)$ as a function on $R^q \times R^q$, let $\nabla_i r_q(d_1, d_2)$ denote its partial derivative matrix with respect to d_i. In this $q \times q$ derivative matrix, row k gives the partial derivatives of the kth component of $r_q(d_1, d_2)$ with respect to the components of d_i. These partial derivatives may be computed explicitly from (1.2) and (1.5). Since $\nabla_i r_q(d_1, d_2)$ is continuous in its arguments and $d_{n,i} \to d_i = d(P_i)$, it follows from (5.12) and the assumption $n_i/n \to c_i$ that

$$n^{1/2}(\hat{\rho}_{n,q} - \rho_{n,q}) \Rightarrow \sum_{i=1}^{2} c_i^{-1/2} \nabla_i r_q(d_1, d_2) u(P_i) \tag{5.13}$$

under P_n. The limit distribution is Gaussian with mean zero and is not a point mass because the derivative matrices $\nabla_i r_q(d_1, d_2)$ are nonsingular. Consequently,

$$\tilde{H}_{n,q}(P_n) \Rightarrow \mathscr{L}\left[\left|\sum_{i=1}^{2} c_i^{-1/2}\nabla_i r_q(d_1, d_2)u(P_i)\right|^2\right] = \tilde{H}_q(P), \text{ say.} \tag{5.14}$$

Since $\hat{P}_{n,i} \Rightarrow P_i$ w.p.1 under $P_i^{n_i}$, the bootstrap distribution $\tilde{H}_{n,q}(\hat{P}_n) \Rightarrow \tilde{H}_q(P)$ w.p.1. This limit distribution is continuous, the distribution of a nondegenerate central Gaussian quadratic form. It follows that

$$\lim_{n\to\infty} \Pr\left[n|\hat{\rho}_{n,q} - \rho_q|^2 \le \tilde{h}_{n,q}(\alpha)|P\right] = 1 - \alpha \tag{5.15}$$

[cf. Theorem 1 in Beran (1984)]. In view of (5.10), this proves the directional portion of (2.15) and (2.16).

Axial case. Consider the following triangular array. Each random axis in sample i has distribution $Q_{n,i}$, where $Q_{n,i} \Rightarrow Q_i$. The joint distribution of the two samples is $Q_{n,1}^{n_1} \times Q_{n,2}^{n_2}$. Let $e(Q_i)$ denote the eigenvector associated with the unique (by assumption) largest eigenvalue of $M(Q_i)$. The sign of $e(Q_i)$ is such that the first nonzero component of $e(Q_i)$ is positive. Then $\hat{e}_{n,i} = e(\hat{Q}_{n,i})$ identifies the mean axis of sample i while $e_{n,i} = e(Q_{n,i})$ identifies the mean axis of distribution $Q_{n,i}$. Define $Q_n = (Q_{n,1}, Q_{n,2})$ in addition to the earlier notations $Q = (Q_1, Q_2)$ and $\hat{Q}_n = (\hat{Q}_{n,1}, \hat{Q}_{n,2})$.

Let

$$\tilde{K}_{n,q}(Q_n) = \mathscr{L}\left[n||\hat{\rho}_{n,q}\hat{\rho}'_{n,q} - \rho_{n,q}\rho'_{n,q}||^2|Q_n\right], \tag{5.16}$$

where now $\rho_{n,q} = r_q(e_{n,1}, e_{n,2})$ and $\hat{\rho}_{n,q} = r_q(\hat{e}_{n,1}, \hat{e}_{n,2})$ for r_q defined in (1.2) and (1.5). Let $\tilde{k}_{n,q}(\alpha)$ be the largest $(1-\alpha)$th quantile of the bootstrap distribution $\tilde{K}_{n,q}(\hat{Q}_n)$. Let $e_i = e(Q_i)$ and $\rho_q = r_q(e_1, e_2)$. Because $||\hat{\rho}_{n,q}\hat{\rho}'_{n,q} - \rho_q\rho'_q||^2 = 2 - 2(\hat{\rho}_{n,q}\rho_q)^2$, the confidence set $D_{n,q}$ may be rewritten in the form

$$D_{n,q} = \left\{\pm\rho_q\colon n||\hat{\rho}_{n,q}\hat{\rho}'_{n,q} - \rho_q\rho'_q||^2 \le \tilde{k}_{n,q}(\alpha),\ |\rho_q| = 1\right\}. \tag{5.17}$$

For the rest of the proof, we will adjust the sign of $\hat{e}_{n,i}$ so that $\hat{e}'_{n,i}e_{n,i} \ge 0$. This sign convention entails no loss of generality because the quadratic statistic defining $D_{n,q}$ is invariant under changes in the signs of either $e_{n,i}$ or $\hat{e}_{n,i}$. The eigenprojection $e(Q)e'(Q)$ is differentiable as a function of $M(Q)$ when the largest eigenvalue of $M(Q)$ is unique [cf. Kato (1982)]. This property and Lemma 5.1 imply that $n_i^{1/2}(\hat{e}_{n,i}\hat{e}'_{n,i} - e_{n,i}e'_{n,i})$ has a normal limit distribution, say $\mathscr{L}(V(Q_i))$, under Q_n.

By simple algebra,

$$\hat{e}_{n,i} - e_{n,i} = (e'_{n,i}\hat{e}_{n,i} - 1)e_{n,i} + (\hat{e}_{n,i}\hat{e}'_{n,i} - e_{n,i}e'_{n,i})\hat{e}_{n,i}. \tag{5.18}$$

By the above asymptotic normality,

$$|e'_{n,i}\hat{e}_{n,i}|^2 = 1 - 2^{-1}\mathrm{tr}[\hat{e}_{n,i}\hat{e}'_{n,i} - e_{n,i}e'_{n,i}]^2 = 1 + O_p(n_i^{-1}). \tag{5.19}$$

Since $e'_{n,i}\hat{e}_{n,i} = |e'_{n,i}\hat{e}_{n,i}|$, the first term on the right-hand side of (5.18) is $O_p(n_i^{-1})$. To analyze the second term in (5.18), note that

$$(5.20) \qquad |(\hat{e}_{n,i}\hat{e}'_{n,i} - e_{n,i}e'_{n,i})(\hat{e}_{n,i} - e_{n,i})| \le \|\hat{e}_{n,i}\hat{e}'_{n,i} - e_{n,i}e'_{n,i}\|\,|\hat{e}_{n,i} - e_{n,i}|.$$

The first factor on the right-hand side of (5.20) is $O_p(n_i^{-1/2})$ as above. The second factor is also $O_p(n_i^{-1/2})$ by applying (5.19) and the sign convention on $\hat{e}_{n,i}$ to the identity $|\hat{e}_{n,i} - e_{n,i}|^2 = 2(1 - e'_{n,i}\hat{e}_{n,i})$.

These approximations and (5.18) establish

$$(5.21) \qquad \hat{e}_{n,i} - e_{n,i} = (\hat{e}_{n,i}\hat{e}'_{n,i} - e_{n,i}e'_{n,i})e_{n,i} + O_p(n_i^{-1}).$$

Thus, since $e_{n,i} \to e_i = e(Q_i)$,

$$(5.22) \qquad \mathscr{L}\big[n_i^{1/2}(\hat{e}_{n,i} - e_{n,i})\big] \Rightarrow \mathscr{L}[V(Q_i)e_i],$$

where $V(Q_i)e_i$ has a Gaussian distribution with mean zero, not a point mass, in the subspace orthogonal to e_i.

Arguing as in the directional case shows

$$(5.23) \qquad n^{1/2}(\hat{\rho}_{n,q} - \rho_{n,q}) \Rightarrow \sum_{i=1}^{2} c_i^{-1/2}\nabla_i r_q(e_1, e_2)V(Q_i)e_i = u(Q), \text{ say.}$$

The right-hand side is normally distributed with mean zero and is not a point mass. Because $r_{n,q} \to \rho_q$, it follows now that

$$(5.24) \qquad \begin{aligned} \tilde{K}_{n,q}(Q_n) &\Rightarrow \mathscr{L}\big[\|u(Q)\rho'_q + \rho_q u'(Q)\|^2\big] \\ &= \mathscr{L}\big[2u'(Q)(I + \rho_q\rho'_q)u(Q)\big] = \tilde{K}_q(Q), \quad \text{say.} \end{aligned}$$

Since $\hat{Q}_{n,i} \Rightarrow Q_i$ w.p.1 under $Q_i^{n_i}$, the bootstrap distribution $\tilde{K}_{n,q}(\hat{Q}_n) \Rightarrow \tilde{K}_q(Q)$ w.p.1. This limit distribution is continuous, the distribution of a non-degenerate central Gaussian quadratic form. Hence,

$$(5.25) \qquad \lim_{n\to\infty} \Pr\big[n\|\hat{\rho}_{n,q}\hat{\rho}'_{n,q} - \rho_q\rho'_q\|^2 \le \tilde{k}_{n,q}(\alpha)|Q\big] = 1 - \alpha.$$

[cf. Theorem 1 in Beran (1984)]. Because of (5.17), this proves the axial portion of (2.15) and (2.16).

PROOF OF THEOREM 3.1. We augment the notation of the preceding proof with the superscript i, j to identify quantities used in comparing samples i and j.

Directional case. The triangular array now extends to the s samples, with $P_n = (P_{n,1}, P_{n,2}, \ldots, P_{n,s})$, $P = (P_1, P_2, \ldots, P_s)$, and $P_{n,i} \Rightarrow P_i$ for each i. The empirical distribution vector is $\hat{P}_n = (\hat{P}_{n,1}, \hat{P}_{n,2}, \ldots, \hat{P}_{n,s})$. Let

$$(5.26) \qquad T_n^{i,j}(\rho_q^{i,j}) = n\big|\hat{\rho}_{n,q}^{i,j} - \rho_q^{i,j}\big|^2,$$

let

$$(5.27) \qquad \tilde{H}_{n,q}^{i,j}(P_n) = \mathscr{L}\big[T_n^{i,j}(\rho_{n,q}^{i,j})|P_n\big]$$

and let

$$\tilde{H}_{n,q}(P_n) = \mathscr{L}\Big[\max_{i \le j} \tilde{H}_{n,q}^{i,j}(T_n^{i,j}(\rho_{n,q}^{i,j}), P_n)\Big], \tag{5.28}$$

where $\tilde{H}_{n,q}^{i,j}(\cdot, P_n)$ is the left continuous cdf of the distribution defined in (5.27). Similarly, let $\tilde{H}_{n,q}(\cdot, P_n)$ denote the left continuous cdf of the distribution in (5.28).

For any left continuous cdf F on the real line, the largest vth quantile is $F^{-1}(v) = \sup\{x\colon F(x) \le v\}$. Note that $F(u) \le v$ if and only if $u \le F^{-1}(v)$. By reasoning like that for (5.10), confidence set $C_{n,q}^{i,j}$ may be rewritten in the form

$$\begin{aligned} C_{n,q}^{i,j} &= \{\rho_q^{i,j}\colon T_n^{i,j}(\rho_q^{i,j}) \le \tilde{H}_{n,q}^{i,j-1}[\tilde{H}_{n,q}^{-1}(1-\alpha, \hat{P}_n), \hat{P}_n]\} \\ &= \{\rho_q^{i,j}\colon \tilde{H}_{n,q}^{i,j}(T_n^{i,j}(\rho_q^{i,j}), \hat{P}_n) \le \tilde{H}_{n,q}^{-1}(1-\alpha, \hat{P}_n)\}. \end{aligned} \tag{5.29}$$

Confidence set $C_{n,q}$ is the simultaneous assertion of the $\{C_{n,q}^{i,j}\colon 1 \le i < j \le s\}$.

By extension of the argument for Theorem 2.1, the random vector $\{T_n^{i,j}(\rho_q^{i,j})\}$ converges weakly, under P_n, to a random vector $\{T_q^{i,j}\}$. The marginal distribution of $T_q^{i,j}$ is $H_q^{i,j}(P)$, the continuous distribution of a Gaussian quadratic form. The corresponding marginal cdf's therefore converge uniformly on the real line and

$$\tilde{H}_{n,q}^{i,j}(T_n^{i,j}(\rho_q^{i,j}), P_n) \Rightarrow \tilde{H}_q^{i,j}(T_q^{i,j}, P) \quad \text{jointly in } i < j. \tag{5.30}$$

The marginal distribution of each random variable on the right-hand side of (5.30) is uniform on (0, 1). In view of (5.28) and (5.30),

$$\tilde{H}_{n,q}(P_n) \Rightarrow \mathscr{L}\Big[\max_{i<j} \tilde{H}_q^{i,j}(T_q^{i,j}, P)\Big] = H_q(P), \text{ say.} \tag{5.31}$$

This limit distribution is continuous and has full support on (0, 1).

The triangular array convergences (5.30) and (5.31) imply the following bootstrap convergences under P:

$$\mathscr{L}[\tilde{H}_{n,q}^{i,j}(T_n^{i,j}(\rho_q^{i,j}), \hat{P}_n)] \Rightarrow \text{uniform}(0,1) \quad \text{w.p.1} \tag{5.32}$$

and

$$\tilde{H}_{n,q}^{-1}(1-\alpha, \hat{P}_n) \to \tilde{H}_q^{-1}(1-\alpha, P) \quad \text{w.p.1.} \tag{5.33}$$

Combining (5.32) and (5.33) with the second line, inf (5.29) yields

$$\lim_{n\to\infty} \Pr[C_{n,q}^{i,j} \ni \rho_q^{i,j} | P] = \tilde{H}_q(1-\alpha, P). \tag{5.34}$$

This establishes the first line in (3.13) and in (3.14).

The simultaneous confidence set $C_{n,q}$ is true if and only if $\rho_q^{i,j} \in C_{n,q}^{i,j}$ for every $i < j$, which is equivalent to the inequality

$$\max_{i<j} \tilde{H}_{n,q}^{i,j}(T_n^{i,j}(\rho_q^{i,j}), \hat{P}_n) \le \tilde{H}_{n,q}^{-1}(1-\alpha, \hat{P}_n). \tag{5.35}$$

By (5.30) and the weak convergence of the $\{T_n^{i,j}(\rho_q^{i,j}\}$, the left-hand side of (5.35) converges weakly w.p.1, under P, to the random variable $\max_{i<j} \tilde{H}_q^{i,j}(T_q^{i,j}, P)$, whose distribution is $\tilde{H}_q(P)$. In view of (5.33), the coverage probability of the simultaneous confidence set $C_{n,q}$ converges to $1-\alpha$ under P.

Axial case. The argument for this is completely analogous, building on the axial part of the proof of Theorem 2.1.

REFERENCES

BERAN, R. (1979). Exponential models for directional data. *Ann. Statist.* **7** 1162–1178.

BERAN, R. (1984). Bootstrap methods in statistics. *Jahresber. Deutsch. Math. Vereini.* **86** 212–225.

BERAN, R. (1990). Refining bootstrap simultaneous confidence sets. *J. Amer. Statist. Assoc.* **85** 417–426.

FISHER, N. I. (1985). Spherical medians. *J. Roy. Statist. Soc. Ser. B* **47** 342–348.

FISHER, N. I. (1995). *Statistical Analysis of Circular Data.* Cambridge Univ. Press.

FISHER, N. I., LEWIS, T. and EMBLETON, B. J. J. (1993). *Statistical Analysis of Spherical Data.* Cambridge Univ. Press.

FISHER, N. I., LUNN, A. D. and DAVIES, S. J. (1993). Spherical median axes. *J. Roy. Statist. Soc. Ser. B* **55** 117–124.

HALL, P. (1986). On the number of bootstrap simulations required to construct a confidence interval. *Ann. Statist.* **14** 1453–1462.

HALL, P. (1992). *The Bootstrap and Edgeworth Expansion.* Springer, New York.

KATO, T. (1982). *A Short Introduction to Perturbation Theory for Linear Operators.* Springer, New York.

KENT, J. T. (1992). New directions in shape analysis. In *The Art of Statistical Science* (K. V. Mardia, ed.) 115–127. Wiley, New York.

LEWIS, T. and FISHER, N. I. (1995). Estimating the angle between the mean directions of two spherical distributions. *Austral. J. Statist.* **37** 179–191.

MARDIA, K. V. (1972). *Statistics of Directional Data.* Academic Press, London.

POWELL, C. MCA., COLE, J. P. and CUDAHY, T. J. (1985). Megakinking in the Lachlan Fold Belt. *J. Struct. Geol.* **7** 281–300.

WATSON, G. S. (1983). *Statistics on Spheres.* Wiley, New York.

UNIVERSITY OF CALIFORNIA
DEPARTMENT OF STATISTICS
367 EVANS HALL
BERKELEY, CALIFORNIA 94720-3860
E-MAIL: beran@stat.berkeley.edu

MATHEMATICAL AND INFORMATION SCIENCES
CSIRO
LOCKED BAG 17
NORTH RYDE, NSW 2113
AUSTRALIA