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ESTIMATING INTEGRALS OF STOCHASTIC PROCESSES USING SPACE-TIME DATA[1]

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Consider a space–time stochastic process $Z_t(x) = S(x) + \xi_t(x)$ where $S(x)$ is a signal process defined on $\mathbb{R}^d$ and $\xi_t(x)$ represents measurement errors at time t. For a known measurable function $v(x)$ on $\mathbb{R}^d$ and a fixed cube $D \subset \mathbb{R}^d$, this paper proposes a linear estimator for the stochastic integral $\int_D v(x)S(x)\,dx$ based on space–time observations $\{Z_t(x_i)\colon i = 1, \dots, n;\ t = 1, \dots, T\}$. Under mild conditions, the asymptotic properties of the mean squared error of the estimator are derived as the spatial distance between spatial sampling locations tends to zero and as time T increases to infinity. Central limit theorems for the estimation error are also studied.

1. Introduction. In spatial data analyses, a common problem is to estimate integrals of the type

$$g(v, Z) = \int_D v(x)Z(x)\,dx, \tag{1.1}$$

where $D \subset \mathbb{R}^d$, $\{Z(x)\colon x \in \mathbb{R}^d\}$ is a second-order stochastic process, and $v(x)$ is a known measurable function on $\mathbb{R}^d$. Throughout this article, it is assumed that $v(x)Z(x)$ is integrable in quadratic mean such that $\mathrm{E}g^2(v, Z)$ is nonzero and finite. When $v(x) \equiv 1/\int_D dx$, the integral $\int_D Z(x)\,dx/\int_D dx$ represents the average of the process $\{Z(x)\colon x \in \mathbb{R}^d\}$ over the set D, which is often of interest in geostatistical studies.

Letting $\hat{g}(v, Z)$ be an estimator of the integral $g(v, Z)$ based on spatial observations $\{Z(x_1), \dots, Z(x_n)\colon x_i \in D\}$, it is important to study the behavior of the mean squared error $\mathrm{E}[\hat{g}(v, Z) - g(v, Z)]^2$. When the process $\{Z(x)\colon x \in \mathbb{R}^d\}$ is observed at different time periods, this article proposes new estimators for stochastic integrals related to the process and studies asymptotic properties of the mean squared error.

Currently, two types of asymptotic theories in spatial settings have been intensively investigated. The first type of asymptotics is to fix the distance between neighboring observations and let the size of D increase with the sample size n. Quenouille (1949), Iachan (1985) and Matérn (1986), among others, used this approach to study the asymptotic mean squared error for estimating the integral $g(v, Z)$. An alternative way to obtain asymptotic theory is based on fixing the size of D and letting observations within the set get increasely dense [see, e.g., Tubilla (1975), Schoenfelder and Cambanis (1982)

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and Stein (1987, 1993, 1995a, b)], which is preferable in geostatistical data analyses such as mining and hydrology where the domain D is often thought to be bounded and extra units of information come from observations taken between those already observed. Following the terminology used by Cressie (1993), the two types of asymptotics are called increasing domain asymptotics and infill asymptotics, respectively.

In many fields such as ecology, biology and meteorology, data are usually collected at different spatial locations over a long time period. For example, to monitor the water quality of a lake, ecologists may divide the lake into equal-size blocks, sample water at each block weekly or monthly for several years and analyze the samples for concentration levels of some specific chemical–physical parameters. In environmental studies, one of the main concerns about air pollution is excessive tropospheric ozone levels, which arise as a consequence of changes in precursor emissions; surface ozone data have been collected by ground-based station networks throughout the United States for many years to monitor changes and trends in ozone concentration.

Statistical techniques for analyzing space–time data have been developing rapidly in last two decades. For example, Cliff, Haggett, Ord, Bassett and Davies (1975) proposed a class of space–time autoregressive moving-average (STARMA) models. Taneja and Aroian (1980) studied the required stationarity and invertibility conditions of STARMA models, and Pfeifer and Deutsch (1980a, b) proposed a three-stage iterative procedure for building space–time models. Most recently, Niu and Tiao (1995) used space–time regression models for long-term trend assessment in the total ozone mapping spectrometer (TOMS) data. Niu, McKeague and Elsner (1997) developed a class of seasonal space–time models for general lattice systems and applied them to maps of monthly averaged 500 mb geopotential heights over a 10×10 lattice in the Northern Hemisphere for the purpose of improving climate prediction.

For the purpose of estimating integrals of spatial processes, space–time observations, if available, provide valuable information and should be used to construct new estimators. Consider a space–time process $\{Z_t(x)\colon x \in \mathbb{R}^d;\ t = 1, \dots\}$ with the decomposition

$$Z_t(x) = \mu(x) + W(x) + \xi_t(x), \tag{1.2}$$

where $\mu(x)$ is a deterministic mean structure of the process $\{Z_t(x)\}$, $\{W(x)\colon x \in \mathbb{R}^d\}$ is a zero-mean, weakly stationary spatial process and $\{\xi_t(x)\colon x \in \mathbb{R}^d\}$ is a zero-mean noise process representing measurement and other random errors at time t. In this article, the process $\{\xi_t(x)\colon x \in \mathbb{R}^d;\ t = 1, \dots\}$ is assumed to be independent of $\{W(x)\colon x \in \mathbb{R}^d\}$.

Let $S(x) = \mu(x) + W(x)$ and define

$$g(v, S) = \int_D v(x) S(x)\, dx. \tag{1.3}$$

It makes more sense to estimate the integral $g(v, S)$ instead of $g(v, Z_t)$, since one is more interested in the average value of the signal process $\{S(x)\}$ than that of the noise-distorted process $\{Z_t(x)\}$. In this study, we suppose that observations of the process $\{Z_t(x)\}$ are available at locations $\{x_1, \dots, x_n\}$ within the set D and at given time periods $t = 1, \dots, T$, that is,

$$Z_t(x_i) = \mu(x_i) + W(x_i) + \xi_t(x_i), \qquad i = 1, \dots, n;\ t = 1, \dots, T, \tag{1.4}$$

where the time intervals are assumed to be equal and fixed. Unlike spatial sampling designs, the sample size $n \times T$ may increase in two directions: n increases as the locations $\{x_1, \dots, x_n\}$ within D become dense or T increases.

Estimating integrals of spatial processes using space–time data is very useful in environmental studies. For instance, in ozone data analyses, the average levels of surface ozone concentration in a city before and after a new regulation comes into effect are often compared. Let $\{Z_t(x) = S(x) + \xi_t(x)\colon x \in D;\ t = 1, \dots, T\}$ and $\{Z_t^*(x) = S^*(x) + \xi_t^*(x)\colon x \in D;\ t = T + t_0, \dots, T^*\}$, where t_0 is a given integer, denote, respectively, the ozone concentration processes in the city before and after the regulation is enforced. Instead of estimating the value of the spatial processes at a specific location x_0 and a given time t, estimating and comparing the aggregated random variables $\int_D S(x)\,dx / \int_D dx$ and $\int_D S^*(x)\,dx / \int_D dx$ based on long-term observations of the processes will provide more reliable information about the ozone level changes in the city.

Investigating statistical properties of stochastic integral estimators based on space–time observations is a challenging topic, which involves spatial and temporal characteristics of the process $\{Z_t(x)\}$. As pointed out by Niu, McKeague and Elsner (1997), the behavior of a space–time process in the temporal direction is usually quite different from that in the spatial direction. Instead of simply viewing the time direction as one more dimension, spatial and temporal features of a space–time process should be studied separately.

The asymptotic theory of estimating stochastic integrals using space–time data will be addressed in this article, and, to some extent, it is a combination of the infill and increasing domain asymptotics. Section 2 proposes a linear estimator for $g(v, S)$ and investigates asymptotic properties of the estimator. Under conditions C1–C4 specified in the section, the mean squared error of the estimators is shown to converge to zero as both spatial and temporal sample sizes increase to infinity. The conditions are carefully discussed and examples of space–time processes satisfying the conditions are given. The distribution of the estimation error $[\hat{g}(v, S) - g(v, S)]$ is shown to be asymptotically normal when the spatial sample size is a function of T and $T \to \infty$. Some potential applications of the derived results are discussed in Section 3. Proofs of the main results are given in the Appendix.

2. Asymptotic properties of the estimators. In this section, we study the asymptotic behavior of estimators of the stochastic integral $g(v, S)$ based on observations of the space–time process $\{Z_t(x)\}$. To avoid complications at the borders, D is assumed to be a cube in the space $\mathbb{R}^d$ with the sides parallel to the axes. Without loss of any generality, let $D = [0, 1]^d$. For the three components of the space–time process $\{Z_t(x)\}$ specified in (1.2), we assume that $\int \mu(x)v(x)\,dx < \infty$, $W(x)$ is a weakly stationary process on $\mathbb{R}^d$ with mean zero and spectral density $f_1(\omega)$, $\{\xi_t(x)\colon x \in \mathbb{R}^d;\ t = 0, \pm 1, \ldots\}$ is a weakly stationary space–time process and the spectral density of $\xi_t(x)$ for any given t is denoted by $f_2(\omega)$. Following Stein (1995a), let $v(x)\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable and square-integrable function and define $V(\omega) = \int_D v(x)\exp(i\omega' x)\,dx$. Furthermore, assume that $\int |V(\omega)|^2 f(\omega)\,d\omega < \infty$ where $f(\omega) = f_1(\omega) + f_2(\omega)$. Then $\int v(x)S(x)\,dx$ is well defined as a mean squared limit of finite weighted sums of the process $\{S(x) = \mu(x) + W(x)\}$.

Let $\gamma_1(x)$ and $\gamma_2(x)$ be the covariance functions defined on $\mathbb{R}^d$ for the processes $\{W(x)\}$ and $\{\xi_t(x)\}$ at any given time t, respectively. Then

$$\gamma_1(x) = \int \exp(i\omega' x) f_1(\omega)\,d\omega, \qquad \gamma_2(x) = \int \exp(i\omega' x) f_2(\omega)\,d\omega. \tag{2.1}$$

Based on spectral theory of weakly stationary spatial processes [e.g., Matérn (1986), page 20], the first two moments of the integral $g(v, S)$ are $\mathrm{E}g(v, S) = \int_D \mu(x)v(x)\,dx$ and

$$\begin{aligned}\operatorname{Var}(g(v, S)) &= \int_D \int_D \gamma_1(x - y)v(x)v(y)\,dx\,dy \\ &= \int \left| \int_D v(x)\exp(i\omega' x)\,dx \right|^2 f_1(\omega)\,d\omega.\end{aligned}$$

For estimating the integral $g(v, S)$, measurements of the space–time process $\{Z_t(x)\}$ are needed. In this article, spatial measurement sites $\{x_1, \ldots, x_n\}$ are selected by the centered systematic sampling scheme, a common method used in meteorological studies. In the sampling design, the spatial cube D is divided into an m^d grid of smaller cubes and values of the process $\{Z_t(x)\colon x \in \mathbb{R}^d\}$ at a given time t are observed at the center of each of the m^d cubes. Therefore, for a fixed integer m, the sample size is $n = m^d$. Stein (1993, 1995a) studied the asymptotic properties of linear predictors of integrals $\int_D v(x)Z(x)\,dx$ under this sampling design, and some results of his papers will be used in this study.

For the space–time random noise process $\{\xi_t(x)\colon x \in \mathbb{R}^d;\ t = 0, \pm 1, \ldots\}$ in (1.2), define

$$\boldsymbol{\xi}_t(m) = [\xi_t(x_1), \ldots, \xi_t(x_{m^d})]' \tag{2.2}$$

and

$$\eta_t(m) = m^{-d} \sum_{i=1}^{m^d} v(x_i)\xi_t(x_i), \tag{2.3}$$

where the spatial locations $\{x_1, \ldots, x_{m^d}\}$ depend on the integer m. Since $\{\xi_t(x)\}$ is assumed to be a weakly stationary space–time process, $\{\boldsymbol{\xi}_t(m)\colon t = 0, \pm 1, \ldots\}$ for any given integer m is a temporally stationary vector process. Throughout this article, we assume that $\{\boldsymbol{\xi}_t(m)\colon t = 0, \pm 1, \ldots\}$ is in the class of infinite-order moving average vector processes with the form

$$\boldsymbol{\xi}_t(m) = \sum_{k=0}^{\infty} A(k, m)\boldsymbol{\varepsilon}_{t-k}(m), \tag{2.4}$$

where $\{\boldsymbol{\varepsilon}_t(m)\colon t = 0, \pm 1, \ldots\}$ is a sequence of uncorrelated random vectors with mean zero and $\mathrm{E}[\boldsymbol{\varepsilon}_t(m)\boldsymbol{\varepsilon}_t'(m)] = \Sigma$, and the $A(k, m)$'s are $m^d \times m^d$ real constant matrices that satisfy the condition

$$\sum_{k=0}^{\infty} |A_{ij}(k, m)| < \infty \quad \text{for } i, j = 1, \ldots, m^d. \tag{2.5}$$

For a fixed integer m, the vector process $\boldsymbol{\xi}_t(m)$ specified in (2.4) and (2.5) is stationary with autocovariance matrices

$$\Gamma(h, m) = \mathrm{E}(\boldsymbol{\xi}_t(m)\boldsymbol{\xi}_{t+h}'(m)) = \sum_{k=0}^{\infty} A(k, m)\Sigma A'(k+h, m),$$

and $\Gamma(h, m)$ satisfy the following condition:

$$\sum_{h=-\infty}^{\infty} |\Gamma_{ij}(h, m)| < \infty \quad \text{for } i, j = 1, \ldots, m^d. \tag{2.6}$$

In this case, the autocovariance matrix function $\{\Gamma(h, m)\}$ has a spectral density matrix function $\varphi(\lambda, m)$ with the form

$$\varphi(\lambda, m) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h}\Gamma(h, m), \qquad -\pi \le \lambda \le \pi \tag{2.7}$$

and the autocovariance matrix $\Gamma(h, m)$ can be expressed in terms of $\varphi(\lambda, m)$ as

$$\Gamma(h, m) = \int_{-\pi}^{\pi} e^{i\lambda h}\varphi(\lambda, m)\, d\lambda. \tag{2.8}$$

In particular, $\Gamma(0, m)$ is the variance matrix of the random vector $\boldsymbol{\xi}_t(m)$ and the elements of $\Gamma(0, m)$ are given by

$$\Gamma_{ij}(0, m) = \mathrm{E}[\xi_t(x_i)\xi_t(x_j)] = \gamma_2(x_i - x_j).$$

It is easy to see that $\{\eta_t(m)\colon t = 0, \pm 1, \ldots\}$ defined in (2.3) is weakly stationary with mean zero and autocovariance function

$$\begin{aligned} \gamma_\eta(h, m) = \mathrm{E}[\eta_t(m)\eta_{t+h}(m)] &= m^{-2d} \sum_{i=1}^{m^d}\sum_{j=1}^{m^d} v(x_i)\Gamma_{ij}(h, m)v(x_j) \\ &= m^{-2d}\mathbf{v}_m'\Gamma(h, m)\mathbf{v}_m = m^{-2d}\int_{-\pi}^{\pi} e^{i\lambda h}\left[\mathbf{v}_m'\varphi(\lambda, m)\mathbf{v}_m\right] d\lambda, \\ &= \int_{-\pi}^{\pi} e^{i\lambda h} f_\eta(\lambda, m)\, d\lambda, \end{aligned} \tag{2.9}$$

where $\mathbf{v}_m = [v(x_1), \ldots, v(x_{m^d})]'$ and $f_\eta(\lambda, m) = m^{-2d}\mathbf{v}_m'\varphi(\lambda, m)\mathbf{v}_m$ is the spectral density of the process $\{\eta_t(m)\colon t = 0, \pm 1, \ldots\}$.

Based on observations $\{Z_t(x_i)\colon i = 1, \ldots, m^d;\ t = 1, \ldots, T\}$, we propose to estimate $g(v, S)$ by

$$\hat{g}_{Tm}(v, S) = \frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{i=1}^{m^d} v(x_i) Z_t(x_i) \right\}. \tag{2.10}$$

Define

$$\hat{g}_m(v, \mu) = m^{-d} \sum_{i=1}^{m^d} v(x_i)\mu(x_i),$$

$$\hat{g}_m(v, W) = m^{-d} \sum_{i=1}^{m^d} v(x_i)W(x_i),$$

$$\hat{g}_{Tm}(v, \xi) = \frac{1}{T} \sum_{t=1}^{T} \eta_t(m).$$

Then the linear estimator $\hat{g}_{Tm}(v, S)$ can be expressed as

$$\hat{g}_{Tm}(v, S) = \hat{g}_m(v, \mu) + \hat{g}_m(v, W) + \hat{g}_{Tm}(v, \xi). \tag{2.11}$$

For spatial points $x \in D$ and $y \in D$, $\{\xi_t(x), t = 1, \ldots, T\}$ and $\{\xi_t(y), t = 1, \ldots, T\}$ are two univariate time series. When $\boldsymbol{\xi}_t = \{\xi_t(x)\colon x \in \mathbb{R}^d\}$ is a temporally weakly stationary random field, for any given $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ the cross-covariance function $\Gamma_{xy}(t, t+h) = \mathrm{E}[\xi_t(x)\xi_{t+h}(y)]$ is independent of t, that is, $\Gamma_{xy}(t, t+h) = \Gamma_{xy}(h)$.

In order to investigate the asymptotic behavior of the linear estimator, the following conditions will be imposed on the function $v(x)$, the spatial process $\{W(x)\colon x \in \mathbb{R}^d\}$ and the temporal process $\{\eta_t(m)\colon t = 0, \pm 1, \ldots\}$.

(C1) $v(x)$ has bounded partial derivatives of order $(d+1)$ on D.

(C2) The spectral density $f_1(\omega)$ of the spatial process $\{W(x)\colon x \in \mathbb{R}^d\}$ is bounded and there exists a regularly varying function $\beta(t)$ with exponent p $(d < p < 4)$ as $t \to \infty$ such that $\beta(|\omega|)f_1(\omega)$ is bounded away from 0 and ∞ as $|\omega| \to \infty$.

(C3) The limit $\gamma_\eta(h) = \lim_{m\to\infty} \gamma_\eta(h, m)$ exists and there exists a nonnegative summable sequence $\{\gamma(h)\}$ such that $|\Gamma_{xy}(h)| = |\mathrm{E}[\xi_t(x)\xi_{t+h}(y)]| \le \gamma(h)$ for any given $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$.

(C4) The product function $\mu(x)v(x)$ is of bounded variation over the hypercube $D = [0, 1]^d$ with total variation $V_D(\mu, v)$.

REMARK 1. Suppose that $\{Z(x)\colon x \in \mathbb{R}^d\}$ is a weakly stationary spatial process with mean zero and spectral density $f(\omega)$. Under conditions C1 and C2, Stein [(1995a), Proposition 3.1], showed that $\hat{Z}_m = m^{-d}\sum_{i=1}^{m^d} v(x_i)Z(x_i)$ is an asymptotically efficient estimator for the integral $\int_D v(x)Z(x)\,dx$ relative to the optimal linear estimators. The expected mean squared error satisfies

$$\mathrm{E}\left(\int_D v(x)Z(x)\,dx - \hat{Z}_m\right)^2 \sim B(m) \quad \text{as } m \to \infty,$$

where

$$B(m) = \int_{m(-\pi,\pi)^d} \Big[\sum\nolimits_* f_1(\omega + 2\pi m J)\Big] |V(\omega)|^2 \, d\omega$$

and $\sum_*$ means summing J over all elements of $\mathbf{Z}^d$ except the origin. In particular, for $v(x) \equiv 1$ and assuming that $\lim_{m\to\infty} \beta(m) f(\nu + m\omega) = \tilde{f}(\omega)$ exists, Stein (1993), Theorem 2, proved that under C2,

$$\lim_{m\to\infty} \beta(m) B(m) = (2\pi)^{d-p} \sum\nolimits_* \tilde{f}(J). \tag{2.12}$$

REMARK 2. As an example of time series with autocovariance functions satisfying C3, for any $x \in \mathbb{R}^d$ consider the following AR(1) process

$$\xi_t(x) = \phi(x)\xi_{t-1}(x) + \varepsilon_t(x), \tag{2.13}$$

where $\{\varepsilon_t(x)\colon t = 0, \pm 1, \ldots\}$ is a temporally independent process for any fixed location x. Assume $|\phi(x)| \le \phi < 1$ and define

$$\Phi(m) = \mathrm{Diag}(\phi(x_1), \ldots, \phi(x_{m^d})).$$

Then the vector process $\boldsymbol{\xi}_t(m)$ defined in (2.2) can be expressed in the form

$$\boldsymbol{\xi}_t(m) = \sum_{k=0}^{\infty} \Phi^k(m) \boldsymbol{\varepsilon}_t(m).$$

The autocovariance matrices $\{\Gamma(h, m)\}$ for this AR(1) process satisfy

$$\Gamma(h, m) = \Phi^h(m)\Gamma(0, m) \quad \text{for } h \ge 1. \tag{2.14}$$

In this case, we have

$$|\Gamma_{ij}(h, m)| = |\phi^h(x_i)\gamma_2(x_i - x_j)| \le \phi^h \gamma_2(0),$$

where $\gamma_2(0) = \mathrm{Var}(\xi_t(x))$ is finite and the sequence $\{\gamma(h) = \phi^h \gamma_2(0)\}$ is summable.

The following result characterizes the covariance structure of the process $\{\eta_t(m)\colon t = 0, \pm 1, \ldots\}$ when $\{\xi_t(x_i)\colon t = 0, \pm 1, \ldots\}$ is an AR(1) process. The proof of this lemma is given in the Appendix.

LEMMA 2.1. *For the AR*(1) *process* $\{\xi_t(x)\colon t = 0, \pm 1, \ldots\}$ *specified in* (2.13), *assume that the coefficient function* $\phi(x)$ *satisfies the condition* $|\phi(x)| \le \phi < 1$ *fox* $x \in D$. *Then*:

(i) $f_\eta(\lambda) = \lim_{m\to\infty} f_\eta(\lambda, m)$ *and* $\gamma_\eta(h) = \lim_{m\to\infty} \gamma_\eta(h, m)$ *exist*.
(ii) $\gamma_\eta(h) = \int_{-\pi}^{\pi} e^{i\lambda h} f_\eta(\lambda)\, d\lambda$.
(iii) *The sequence* $\{\gamma_\eta(h)\colon h = 0, \pm 1, \ldots\}$ *is absolutely summable*.

REMARK 3. Let $\{x_1, x_2, \dots\}$ be a sequence points in the hypercube $D = [0, 1]^d$. Let R be a subset of D of the form $R = \prod_{i=1}^d [a_i, b_i]$ and denote the volume of R by $\mathrm{Vol}(R)$. For every integer n, let $\mathscr{N}(R, n)$ be the number of the points $\{x_1, x_2, \dots, x_n\}$ that lie in R. Set

$$\mathscr{D}(n) = \sup_{R \subset D} |\mathscr{N}(R, n) - \mathrm{Vol}(R)|.$$

Then, based on the numerical integration theory [see, e.g., Davis and Rabinowitz (1975), page 268], under C4 we have

$$\left| n^{-1} \sum_{i=1}^n v(x_i)\mu(x_i) - \int_D v(x)\mu(x)\,dx \right| \le V_D(\mu, v)\mathscr{D}(n).$$

In particular, Davis and Rabinowitz [(1975), page 267], showed that

$$\begin{aligned} &\left| m^{-d} \sum_{k_1,\dots,k_d=0}^{m-1} v(k_1/m, \dots, k_d/m)\mu(k_1/m, \dots, k_d/m) \right. \\ &\qquad \left. - \int_D v(x)\mu(x)\,dx \right| \\ &\quad \le \frac{dV_D(\mu, v)}{m}. \end{aligned} \tag{2.15}$$

Similarly to (2.15), it can be shown that the following result is valid for the centered systematic sampling scheme.

LEMMA 2.2. *Suppose that* $\{x_1, x_2, \dots, x_{m^d}\}$ *are the centers of the* m^d *cubes of side* m^{-1}. *Under* C4, *we have*

$$\left| m^{-d} \sum_{i=1}^{m_d} v(x_i)\mu(x_i) - \int_D v(x)\mu(x)\,dx \right| \le \frac{dV_D(\mu, v)}{m}. \tag{2.16}$$

For the temporal process $\{\eta_t(m)\colon t = 0, \pm 1, \dots\}$ defined in (2.3), we have the following general result that will be used in the proofs of Theorem 2.1 and Lemma 2.4.

LEMMA 2.3. *Suppose that there exists a nonnegative summable sequence* $\{\gamma(h)\}$ *such that* $|\Gamma_{xy}(h)| \le \gamma(h)$ *for any* $x, y \in \mathbb{R}^d$. *Then* $\lim_{m\to\infty} \gamma_\eta(h, m) = \gamma_\eta(h)$ *exists if and only if* $f_\eta(\lambda) = \lim_{m\to\infty} f_\eta(\lambda, m)$ *exists*.

The following theorem and Corollary 2.1 give asymptotic properties of the mean squared error $\mathrm{E}(\hat{g}_{Tm}(v, S) - g(v, S))^2$.

THEOREM 2.1. *Suppose that* C1–C4 *are valid*. *Then we have*

$$\mathrm{E}\big(\hat{g}_{Tm}(v, S) - g(v, S)\big)^2 = \sigma_\mu^2(m) + \sigma_W^2(m) + \sigma_\xi^2(T, m), \tag{2.17}$$

where

$$\sigma_\mu^2(m) = \left(\hat{g}_m(v, \mu) - \int_D v(x)\mu(x)\,dx\right)^2 \le \frac{(dV_D(\mu, v))^2}{m^2},$$

$$\sigma_W^2(m) = \mathrm{E}\left(\hat{g}_m(v, W) - \int_D v(x)W(x)\,dx\right)^2 \sim B(m) \quad \text{as } m \to \infty$$

and

$$\sigma_\xi^2(T, m) = \mathrm{E}\hat{g}_{Tm}^2(v, \xi) \sim T^{-1} \sum_{h=-\infty}^{\infty} |\gamma_\eta(h)| \quad \text{as } m \to \infty \text{ and } T \to \infty.$$

COROLLARY 2.1. *Suppose that* $\lim_{m\to\infty} \beta(m) f(\nu + m\omega) = \tilde{f}(\omega)$ *exists and* $v(x) \equiv 1$. *Then under* C1–C4 *we have*

$$\mathrm{E}\big(\hat{g}_{Tm}(v, S) - g(v, S)\big)^2 \to 0 \quad \text{as } m \to \infty \text{ and } T \to \infty. \tag{2.18}$$

PROOF. Based on the results of Stein [(1993), Theorem 2], (2.12) is valid under C2, which implies that $B(m) \to 0$ as $m \to \infty$. Then (2.18) follows directly from Theorem 2.1. □

Before studying the limiting distribution theory of the linear estimator $\hat{g}_{Tm}(v, S)$, we discuss further the asymptotic behavior of the temporal process $\{\eta_t(m)\colon t = 0, \pm 1, \dots\}$. When C3 is valid, the sequence $\{\gamma_\eta(h)\colon h = 0, \pm 1, \dots\}$ is the autocovariance function of a weakly stationary process $\{\eta_t\colon t = 0, \pm 1 \dots\}$ which has the spectral density function $f_\eta(\lambda) = \sum_{h=-\infty}^{\infty} \gamma_\eta(h) e^{-i\lambda h}$. In fact, under appropriate conditions, we have

$$\mathrm{E}(\eta_t(m) - \eta_t)^2 \to 0 \quad \text{as } m \to \infty.$$

For example, if C1 is valid, and the spectral density function $f_2(\omega)$ of the process $\{\xi_t(x)\colon x \in \mathbb{R}^d\}$ for any given t is bounded and there exists a regularly varying function $\tilde{\beta}(t)$ with exponent $d < \tilde{p} < 4$ as $t \to \infty$ such that $\tilde{\beta}(|\omega|) f_2(\omega)$ is bounded away from 0 and ∞ as $|\omega| \to \infty$, applying the results of Stein [(1995a), Proposition 3.1] to the process $\{\xi_t(x)\colon x \in D\}$ for any given t we have

$$\mathrm{E}(\eta_t(m) - \eta_t)^2 \sim \tilde{B}(m) \quad \text{as } m \to \infty,$$

where $\eta_t = \int_D v(x)\xi_t(x)\,dx$ and

$$\tilde{B}(m) = \int_{m(-\pi, \pi)^d} \Big[\sum\nolimits_* f_2(\omega + 2\pi m J)\Big] |V(\omega)|^2\,d\omega.$$

Under mild conditions, we have $\tilde{B}(m) \to 0$ as $m \to \infty$.

Letting X_n and X be random variables with respective distribution functions F_n and F, X_n converging in distribution to X will be denoted by $X_n \Rightarrow X$. Let $\hat{g}_T(v, \xi) = \sum_{t=1}^T \eta_t / T$. We give the following central limit theorem for the two-dimensional array $\{\hat{g}_{Tm}(v, \xi)\}$ and the sequence $\{\hat{g}_T(v, \xi)\}$.

LEMMA 2.4. *Suppose that* C3 *is valid and* $f_\eta(0) = \sum_{h=-\infty}^{\infty} \gamma_\eta(h) > 0$. *Then for the infinite moving average vector process* $\{\boldsymbol{\xi}_t(m): t = 0, \pm 1, \dots\}$ *specified in* (2.4) *and* (2.5), *we have*

$$\sqrt{T}\,\hat{g}_{Tm}(v, \xi) \Rightarrow N(0, f_\eta(0)) \quad \text{as } T \to \infty \text{ and } m \to \infty. \tag{2.19}$$

Furthermore, if $\mathrm{E}(\eta_t(m) - \eta_t)^2 \to 0$ *as* $m \to \infty$ *where* $\eta_t = \int_D v(x)\xi_t(x)\,dx$, *then* $\sqrt{T}\,\hat{g}_T(v, \xi)$ *has the same limiting distribution* $N(0, f_\eta(0))$.

In space–time sample designs, it is ideal to choose spatial sample size m as a function of T. In this case, we will denote the sample size at each spatial dimension by $m(T)$ and assume that $m(T) \to \infty$ as $T \to \infty$. In practice, we may choose $m(T)$ such that $TB(m(T)) \to c$ as $T \to \infty$. As an example, consider the spectral density function $f_1(\omega) = (a^2 + |\omega|^2)^{-q}$ for $a \neq 0$ and $d/2 < q < 4$. When $v(x) \equiv 1$ for $x \in D$, Stein [(1993), Theorem 2] showed that $B(m) = O(m^{-2q})$. In this case, if we choose $m(T)$ such that $T[m(T)]^{-2q} \to c_1$, then $TB(m(T)) \to c_2$ as $T \to \infty$, where c_1 and c_2 are some constants. For the asymptotic distribution for the estimation error $[\hat{g}_{Tm(T)}(v, S) - g(v, S)]$, we have the following result.

THEOREM 2.2. *Let* $\{\boldsymbol{\xi}_t(m): t = 0, \pm 1, \dots\}$ *be the infinite moving average vector process specified in* (2.4) *and* (2.5) *and suppose that* $f_\eta(0) = \sum_{h=-\infty}^{\infty} \gamma_\eta(h) > 0$.

(i) *If C*3 *is valid, we have*

$$\sqrt{T}\,\hat{g}_{Tm(T)}(v, \xi) \Rightarrow N(0, f_\eta(0)) \quad \text{as } T \to \infty. \tag{2.20}$$

(ii) *Suppose that* C1–C4 *are valid and* $\mathrm{E}(\eta_t(m) - \eta_t)^2 \to 0$ *as* $m \to \infty$ *where* $\eta_t = \int_D v(x)\xi_t(x)\,dx$. *When* $Tm^{-2}(T) \to 0$ *and* $TB(m(T)) \to 0$ *as* $T \to \infty$, *we have*

$$\sqrt{T}\big[\hat{g}_{Tm(T)}(v, S) - g(v, S)\big] \Rightarrow N(0, f_\eta(0)) \quad \text{as } T \to \infty. \tag{2.21}$$

In geostatistical and environmental studies, it is quite often assumed that $\{W(x): x \in \mathbb{R}^d\}$ is a Gaussian process. In this case, the integral $\int_D v(x)W(x)\,dx$, as the limit of $\hat{g}_m(v, W)$, is also normally distributed. Notice that

$$\mathrm{E}\left[\hat{g}_{m(T)}(v, W) - \int_D v(x)W(x)\,dx\right] = 0$$

and

$$\mathrm{E}\left[\hat{g}_{m(T)}(v, W) - \int_D v(x)W(x)\,dx\right]^2 \sim B(m(T)).$$

If $TB(m(T)) \to \sigma^2$ as $T \to \infty$, we have

$$\sqrt{T}\left[\hat{g}_{m(T)}(v, W) - \int_D v(x)W(x)\,dx\right] \Rightarrow N(0, \sigma^2) \quad \text{as } T \to \infty. \tag{2.22}$$

Since $\{W(x)\colon x \in \mathbb{R}^d\}$ is independent of $\{\xi(x)\colon x \in \mathbb{R}^d\}$, from Theorem 2.2 we can derive the following result.

THEOREM 2.3. *Suppose that* C1-C4 *are valid and sample sizes* T *and* $m(T)$ *satisfy*

$$Tm^{-2}(T) \to 0, \qquad TB(m(T)) \to \sigma^2 \quad \text{as } T \to \infty.$$

If $\{W(x)\colon x \in \mathbb{R}^d\}$ *is a Gaussian process, and if* $\mathrm{E}(\eta_t(m) - \eta_t)^2 \to 0$ *as* $m \to \infty$ *where* $\eta_t = \int_D v(x)\xi_t(x)\,dx$, *then*

$$\sqrt{T}\big[\hat{g}_{Tm(T)}(v,S) - g(v,S)\big] \Rightarrow N(0, \sigma^2 + f_\eta(0)) \quad \text{as } T \to \infty.$$

REMARK 4. In Theorem 2.3, the normality assumption on the process $\{W(x)\colon x \in \mathbb{R}^d\}$ and the condition

$$TB(m(T)) \to \sigma^2 \quad \text{as } T \to \infty$$

can be replaced by (2.22), that is, for any spatial process $\{W(x)\colon x \in \mathbb{R}^d\}$ that satisfies (2.22), the result in Theorem 2.3 is also true. Equation (2.22) may be proved under different conditions rather than the normality assumption on $\{W(x)\colon x \in \mathbb{R}^d\}$, which will not be pursued in this article.

3. Discussion. Estimating stochastic integrals of the form $g(v,S) = \int_D v(x)S(x)\,dx$ is an interesting topic in geostatistical data analyses and environmental studies. In this article, we proposed a linear estimator for $g(v,S)$ using space–time observations and investigated the asymptotic properties of the mean squared error of the estimator. The limiting distribution of the estimation error was shown to be normal under mild conditions.

The results derived from this study can be used to make statistical inferences about the stochastic integral $g(v,S)$. For example, from the result in Theorem 2.2(ii), we know that the estimation error $\big[\hat{g}_{Tm(T)}(v,S) - g(v,S)\big]$ is approximately normally distributed with mean zero and variance $f_\eta(0)/T$ when the conditions are valid and T is large. Confidence intervals for $g(v,S)$ can be constructed based on this approximate distribution and the statistic $\hat{g}_{Tm(T)}(v,S)$.

The limiting distribution results may also be used to test hypotheses about the expectation of $g(v,S)$. For instance, consider the surface ozone pollution problem mentioned in Section 1. Suppose that the ozone concentration processes at the city, before and after the enforcing of the new regulation, are $\{Z_t(x) = \mu(x) + \xi_t(x)\colon x \in D;\ t = 1,\dots,T\}$ and $\{Z_t^*(x) = \mu^*(x) + \xi_t^*(x)\colon x \in D;\ t = T + t_0,\dots,T^*\}$, respectively. Furthermore, assume the two space–time processes $\{\xi_t(x)\}$ and $\{\xi_t^*(x)\}$ are independent. In the two processes, the signal processes $S(x) = \mu(x)$ and $S^*(x) = \mu^*(x)$ are both deterministic. In this case $g(v,S) = \int_D v(x)\mu(x)\,dx$ and $g(v,S^*) = \int_D v(x)\mu^*(x)\,dx$ are simply two constants. To test the hypothesis H_0: $g(v,S) = g(v,S^*)$, we may use the statistic

$$\sqrt{T}\big[\hat{g}_{Tm(T)}(v,S) - \hat{g}_{Tm(T)}(v,S^*)\big]\Big/\sqrt{\hat{f}_\eta(0) + \hat{f}_{\eta^*}(0)},$$

which has an approximate standard normal distribution under the null hypothesis.

In practice, space–time data sets are often observed on a finite spatial lattice, that is, the spatial sample size m is fixed while the time T increases. In this case, the result given in Theorem 2.1 can be used to assess the mean squared error $\mathrm{E}(\hat{g}_{Tm}(v, S) - g(v, S))^2$. In meteorological studies, space–time data sets related to the climate system are often available for spatial lattices with different resolutions. For example, the space–time data analyzed in Niu, McKeague and Elsner (1997) are monthly averaged 500 hPa geopotential heights on a 4° latitude by 6° longitude lattice for a portion of the Northern Hemisphere over the period January 1946–December 1988. The geopotential height maps on finer spatial lattices, such as 2° latitude by 3° longitude and 1° latitude by 1° longitude, can also be obtained from the U.S. National Centers for Environmental Predictions (NCEP). When observations of a space-time process are available on a high-resolution spatial lattice, that is, the spatial sample size m is large enough, the results in Lemma 2.4, Theorem 2.2 and Theorem 2.3 will be approximately valid for making inferences about the stochastic integral $g(v, S)$.

It should be pointed out that, when a space–time process is observed on a daily or monthly basis, the data set often shows a seasonal pattern. In this case, the observed space–time process has the form

$$Y_t(x) = Z_t(x) + U_t(x),$$

where the process $Z_t(x)$ is specified in (1.2) and $U_t(x)$ represents the seasonal pattern. In order to estimate the integral $g(v, S) = \int_D v(x)S(x)\,dx$, seasonal adjustments should be performed first on the space–time process $\{Y_t(x)\}$ and the linear estimator for $g(v, S)$ defined in (2.10) then can be constructed based on the adjusted process $\{\hat{Z}_t(x) = Y_t(x) - \hat{U}_t(x)\}$.

APPENDIX

Proofs of the main results.

PROOF OF LEMMA 2.1. Notice that for a fixed m, the autocovariance matrices satisfy $\Gamma(-h, m) = \Gamma(h, m)'$. The spectral density matrices of the vector AR(1) process $\{\boldsymbol{\xi}_t(m)\colon t = 0, \pm 1, \ldots\}$ are

$$\begin{aligned}
\varphi(\lambda, m) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h}\Gamma(h, m) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h}\Phi^h(m)\Gamma(0, m) \\
&= \frac{1}{2\pi}\left\{\Gamma(0, m)\left[\sum_{h=0}^{\infty} \Phi^h(m)e^{i\lambda h}\right] + \left[\sum_{h=1}^{\infty} \Phi^h(m)e^{-i\lambda h}\right]\Gamma(0, m)\right\} \\
&= \frac{1}{2\pi}\left\{\Gamma(0, m)\tilde{\Phi}(m) + \left[\tilde{\Phi}^*(m) - I_{m^d}\right]\Gamma(0, m)\right\},
\end{aligned}$$

where "$*$" denotes complex conjugate transpose, $\tilde{\Phi}(m)$ is the diagonal matrix with elements

$$\tilde{\Phi}_{jj}(m) = \frac{1}{1 - \phi(x_j)e^{i\lambda}},$$

and I_{m^d} denotes the $m^d \times m^d$ identity matrix. It is easy to see that

$$\begin{aligned}
f_\eta(\lambda, m) &= m^{-2d}\mathbf{v}'_m \varphi(\lambda, m)\mathbf{v}_m \\
&= \frac{m^{-2d}}{2\pi} \sum_{j=1}^{m^d}\sum_{k=1}^{m^d} v(x_j)\gamma_2(x_j - x_k)v(x_k)/[1 - \phi(x_k)e^{i\lambda}] \\
&\quad + \frac{m^{-2d}}{2\pi} \sum_{j=1}^{m^d}\sum_{k=1}^{m^d} v(x_j)\gamma_2(x_j - x_k)v(x_k)\phi(x_j)e^{-i\lambda}/(1 - \phi(x_j)e^{-i\lambda}) \\
&= \frac{1}{2\pi}\int\Bigg\{\Bigg[m^{-d}\sum_{j=1}^{m^d} e^{i\omega' x_j} v(x_j)\Bigg] \qquad \text{(A.1)} \\
&\qquad \times \Bigg[m^{-d}\sum_{k=1}^{m^d} e^{-i\omega' x_k} v(x_k)/(1 - \phi(x_k)e^{i\lambda})\Bigg]\Bigg\} f_2(\omega)\, d\omega \\
&\quad + \frac{1}{2\pi}\int\Bigg\{\Bigg[m^{-d}\sum_{j=1}^{m^d} e^{i\omega' x_j} v(x_j)\frac{\phi(x_j)e^{-i\lambda}}{1 - \phi(x_j)e^{-i\lambda}}\Bigg] \\
&\qquad \times \Bigg[m^{-d}\sum_{k=1}^{m^d} e^{-i\omega' x_k} v(x_k)\Bigg]\Bigg\} f_2(\omega)\, d\omega.
\end{aligned}$$

When $|\phi(x)| \le \phi < 1$ for $x \in D$, we have

$$\begin{aligned}
f_\eta(\lambda) &= \lim_{m\to\infty} f_\eta(\lambda, m) \\
&= \frac{1}{2\pi}\int\Bigg\{V(\omega)\Bigg[\int_D e^{-i\omega' x}\frac{v(x)}{1 - \phi(x)e^{i\lambda}}\Bigg] \\
&\qquad + V^*(\omega)\Bigg[\int_D e^{i\omega' x} v(x)\frac{\phi(x)e^{-i\lambda}}{1 - \phi(x)e^{-i\lambda}}\Bigg]\Bigg\} f_2(\omega)\, d\omega.
\end{aligned}$$

In order to show that $\lim_{m\to\infty}\gamma_\eta(h, m)$ exists, notice that the autocovariance function $\gamma_\eta(h, m)$ has the form

$$\begin{aligned}
\gamma_\eta(h, m) &= m^{-2d}\mathbf{v}'_m \Phi^h(m)\Gamma(0, m)\mathbf{v}_m \\
&= m^{-2d}\sum_{j=1}^{m^d}\sum_{k=1}^{m^d} \phi^h(x_j)v(x_j)\gamma_2(x_j - x_k)v(x_k)
\end{aligned}$$

$$= \int \left\{ \left[m^{-d} \sum_{j=1}^{m^d} e^{i\omega' x_j} \phi^h(x_j) v(x_j) \right] \times \left[m^{-d} \sum_{k=1}^{m^d} e^{-i\omega' x_k} v(x_k) \right] \right\} f_2(\omega)\, d\omega,$$

which has the limit

$$\gamma_\eta(h) = \lim_{m\to\infty} \gamma_\eta(h,m) = \int \left[\int_D \exp(i\omega' x)\phi^h(x) v(x)\, dx \right] V^*(\omega) f_2(\omega)\, d\omega. \tag{A.2}$$

Moreover, we have

$$|\gamma_\eta(h)| \le \phi^h \int \left[\int_D |v(x)|\, dx \right]^2 f_2(\omega)\, d\omega.$$

Therefore the limiting autocovariance function $\{\gamma_\eta(h)\}$ is also absolutely summable.

Finally, notice that $|\gamma_2(x)| \le \gamma_2(0) < \infty$ and

$$|1-\phi(x)e^{i\lambda}|^2 = 1 + \phi^2(x) - 2\phi(x)\cos(\lambda) \le (1-|\phi(x)|)^2 \le (1-\phi)^2.$$

From (A.1), we have

$$f_\eta(\lambda,m) \le \frac{\gamma_2(0)(1+\phi)}{2\pi(1-\phi)} \left[m^{-d} \sum_{j=1}^{m^d} |v(x_j)| \right]^2 \to \frac{\gamma_2(0)(1+\phi)}{2\pi(1-\phi)} \left[\int_D |v(x)|\, dx \right]^2 < \infty \quad \text{as } m \to \infty.$$

Now $\gamma_\eta(h) = \int_{-\pi}^{\pi} e^{i\lambda h} f_\eta(\lambda)\, d\lambda$ follows from Lemma 2.1(i) and the dominated convergence theorem. □

PROOF OF LEMMA 2.3. Notice that when $|\Gamma_{xy}(h)| \le \gamma(h)$ for any $x, y \in \mathbb{R}^d$, the spectral density matrices $\{\varphi(\lambda, m)\}$ given in (2.7) satisfy

$$|\varphi_{ij}(\lambda,m)| \le \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} |\Gamma_{ij}(h,m)| \le \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) = c_0 < \infty.$$

Then we have

$$0 \le f_\eta(\lambda,m) = m^{-2d} \mathbf{v}_m' \varphi(\lambda,m) \mathbf{v}_m = m^{-2d} \sum_{i=1}^{m^d} \sum_{j=1}^{m^d} v(x_i) \varphi_{ij}(\lambda,m) v(x_j)$$

$$\le c_0 \left[m^{-d} \sum_{i=1}^{m^d} |v(x_i)| \right]^2 \to c_0 \left[\int_D |v(x)|\, dx \right]^2 < \infty \quad \text{as } m \to \infty.$$

If $f_\eta(\lambda) = \lim_{m\to\infty} f_\eta(\lambda, m)$ exists, by the dominated convergence theorem we have

$$\gamma_\eta(h) = \lim_{m\to\infty} \gamma_\eta(h, m) = \lim_{m\to\infty} \int_{-\pi}^{\pi} e^{i\lambda h} f_\eta(\lambda, m)\, d\lambda = \int_{-\pi}^{\pi} e^{i\lambda h} f_\eta(\lambda)\, d\lambda.$$

Conversely, notice that

$$\begin{aligned}\gamma_\eta(h, m) &= m^{-2d}\mathbf{v}_m'\Gamma(h, m)\mathbf{v}_m \le \gamma(h)\left[m^{-d}\sum_{i=1}^{m^d}|v(x_i)|\right]^2 \\ &\to \gamma(h)\left[\int_D |v(x)|\, dx\right]^2 \quad \text{as } m\to\infty.\end{aligned}$$

Because $\{\gamma(h)\}$ is summable, the sequence $\{\gamma_\eta(h, m)\}$ is absolutely summable and the spectral density function $f_\eta(\lambda, m)$ can be expressed in the form

$$f_\eta(\lambda, m) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} \gamma_\eta(h, m)e^{-i\lambda h}.$$

if $\lim_{m\to\infty} \gamma_\eta(h, m) = \gamma_\eta(h)$ exists, then $\{\gamma_\eta(h)\}$ is also absolutely summable and the spectral density function

$$f_\eta(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} \gamma_\eta(h)e^{-i\lambda h}$$

is well defined. The convergence of $f_\eta(\lambda, m)$ as $m\to\infty$ follows from

$$|f_\eta(\lambda, m) - f_\eta(\lambda)| \le \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} |\gamma_\eta(h, m) - \gamma_\eta(h)| \to 0 \quad \text{as } m\to\infty. \qquad \square$$

PROOF OF THEOREM 2.1. Because $\mathrm{E}W(x) = 0$, $\mathrm{E}\xi(x) = 0$, and the two processes $\{W(x)\colon x \in \mathbb{R}^d\}$ and $\{\xi_t(x)\colon x \in \mathbb{R}^d;\ t = 0, \pm 1, \dots\}$ are independent, we have

$$\begin{aligned}&\mathrm{E}\big(\hat{g}_{Tm}(v, S) - g(v, S)\big)^2 \\ &\quad= \left(\hat{g}_m(v, \mu) - \int_D v(x)\mu(x)\, dx\right)^2 \\ &\qquad + \mathrm{E}\left(\hat{g}_m(v, W) - \int_D v(x)W(x)\, dx\right)^2 + \mathrm{E}\hat{g}_{Tm}^2(v, \xi) \\ &\quad= \sigma_\mu^2(m) + \sigma_W^2(m) + \sigma_\xi^2(T, m).\end{aligned} \tag{A.3}$$

By Lemma 2.2, we have

$$\sigma_\mu^2(m) = \left(\hat{g}_m(v, \mu) - \int_D v(x)\mu(x)\, dx\right)^2 \le \frac{(dV_D(\mu, v))^2}{m^2}.$$

When C1 and C2 are valid, Stein [(1995a), Proposition 3.1] proved that

$$\sigma_W^2(m) = \mathrm{E}\biggl(\hat{g}_m(v, W) - \int_D v(x)W(x)\,dx\biggr)^2 \sim B(m) \quad \text{as } m \to \infty.$$

Finally, notice that under C3 and for a fixed T,

$$\begin{aligned} \sigma_\xi^2(T, m) &= \mathrm{E}\hat{g}_{Tm}^2(v, \xi) = \mathrm{E}\biggl[\frac{1}{T}\sum_{t=1}^{T}\eta_t(m)\biggr]^2 = T^{-2}\sum_{s=1}^{T}\sum_{t=1}^{T}\gamma_\eta(s-t) \\ &= T^{-1}\sum_{|h|<T}(1-|h|/T)\gamma_\eta(h, m) \\ &\to T^{-1}\sum_{|h|<T}(1-|h|/T)\gamma_\eta(h) \quad \text{as } m \to \infty. \end{aligned}$$

By the proof of Lemma 2.3, $\{\gamma_\eta(h)\}$ is absolutely summable. Hence we have

$$\sigma_\xi^2(T, m) = \mathrm{E}\hat{g}_{Tm}^2(v, \xi) \sim T^{-1}\sum_{h=-\infty}^{\infty}|\gamma_\eta(h)| \quad \text{as } m \to \infty \text{ and } T \to \infty. \qquad \square$$

Proof of Lemma 2.4. The spectral density matrices for the infinite moving average process $\{\boldsymbol{\xi}_t(m)\colon t = 0, \pm 1, \ldots\}$ specified in (2.4) and (2.5) have the form

$$\varphi(\lambda, m) = \frac{1}{2\pi}\biggl[\sum_{k=0}^{\infty}A(k, m)e^{ik\lambda}\biggr]\Sigma\biggl[\sum_{k=0}^{\infty}A(k, m)e^{ik\lambda}\biggr]^*.$$

Let

$$\bar{\boldsymbol{\xi}}_T(m) = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\xi}_t(m).$$

Then as $T \to \infty$, $\sqrt{T}\bar{\boldsymbol{\xi}}_T(m)$ converges weakly to a normal distribution with mean zero and covariance matrix $\varphi(0, m)$ [see, e.g., Brockwell and Davis (1991), Proposition 11.2.2]. Therefore for a fixed integer m, we have

$$\begin{aligned} \sqrt{T}\hat{g}_{Tm}(v, \xi) &= \frac{\sqrt{T}}{T}\sum_{t=1}^{T}\eta_t(m) = \frac{\sqrt{T}}{m^{-d}}\mathbf{v}_m'\bar{\boldsymbol{\xi}}_T(m) \\ &\Rightarrow Y_m \quad \text{as } T \to \infty, \end{aligned}$$

where Y_m has a normal distribution with mean zero and variance

$$m^{-2d}\mathbf{v}_m'\varphi(0, m)\mathbf{v}_m.$$

Notice that the spectral density function of the process $\{\eta_t(m)\colon t = 0, \pm 1, \ldots\}$ is

$$f_\eta(\lambda, m) = m^{-2d}\mathbf{v}_m'\varphi(\lambda, m)\mathbf{v}_m.$$

By C3 and Lemma 2.3, $f_\eta(\lambda) = \lim_{m\to\infty} f_\eta(\lambda, m)$ exists. Therefore we have

$$m^{-2d}\mathbf{v}_m'\varphi(0, m)\mathbf{v}_m \to f_\eta(0)$$

and $Y_m \Rightarrow N(0, f_\eta(0))$ as $m \to \infty$.

Furthermore, let $\tilde{\gamma}_\eta(t-s, m) = \mathrm{E}[\eta_s(m)\eta_t]$. Notice that

$$|\tilde{\gamma}_\eta(t-s, m)| = m^{-d}\left|\sum_{i=1}^{m^d} v(x_i)\int_D v(y)\Gamma_{x_i y}(t-s)\right| dy$$
$$\leq \gamma(t-s)\int_D |v(y)|\,dy\left[m^{-d}\sum_{i=1}^{m^d}|v(x_i)|\right].$$

Therefore, the sequence $\{\tilde{\gamma}_\eta(t-s, m)\}$ is absolutely summable. It is easy to see that

$$\begin{aligned}
&\mathrm{E}\big[\sqrt{T}\hat{g}_{Tm}(v,\xi) - \sqrt{T}\hat{g}_T(v,\xi)\big]^2\\
&\quad = \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathrm{E}\big[(\eta_s(m)-\eta_s)(\eta_t(m)-\eta_t)\big]\\
&\quad = \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\big[\gamma_\eta(t-s,m)+\gamma_\eta(t-s)-\tilde{\gamma}_\eta(t-s,m)-\tilde{\gamma}_\eta(s-t,m)\big]\\
&\quad \to \sum_{h=-\infty}^{\infty}\big[\gamma_\eta(h,m)+\gamma_\eta(h)-2\tilde{\gamma}_\eta(h,m)\big] \quad \text{as } T\to\infty.
\end{aligned}$$

If $\mathrm{E}(\eta_t(m)-\eta_t)^2 \to 0$ as $m\to\infty$, we have

$$\begin{aligned}
|\tilde{\gamma}_\eta(h,m)-\gamma_\eta(h)|^2 &= |\mathrm{E}[(\eta_t(m)-\eta_t)\eta_{t+h}]|^2\\
&\leq \mathrm{E}(\eta_t(m)-\eta_t)^2\mathrm{E}\eta_{t+h}^2 \to 0 \quad \text{as } m\to\infty.
\end{aligned} \tag{A.4}$$

Then (A.4) and C3 imply that

$$\lim_{m\to\infty}\lim_{T\to\infty}\mathrm{E}\big[\sqrt{T}\hat{g}_{Tm}(v,\xi)-\sqrt{T}\hat{g}_T(v,\xi)\big]^2 = 0.$$

Therefore, $\sqrt{T}\hat{g}_T(v,\xi)$ has the same limiting distribution $N(0, f_\eta(0))$. □

Proof of Theorem 2.2. (i) When the spatial sample size m is a function of T, it is clear that $\{\hat{g}_{Tm(T)}(v,\xi)\colon T=1,2,\ldots\}$ is a subsequence of the two-dimensional array $\{\hat{g}_{Tm}(v,\xi)\colon T=1,2,\ldots;\ m=1,2,\ldots\}$. Therefore (2.19) implies (2.20).

(ii) Notice that

$$\begin{aligned}
\sqrt{T}\big[\hat{g}_{Tm(T)}(v,S)-g(v,S)\big] &= \sqrt{T}\left(\hat{g}_m(v,\mu)-\int_D v(x)\mu(x)\,dx\right)\\
&\quad + \sqrt{T}\left(\hat{g}_m(v,W)-\int_D v(x)W(x)\,dx\right)\\
&\quad + \sqrt{T}\hat{g}_{Tm(T)}(v,\xi).
\end{aligned}$$

Based on the results of Theorem 2.1, we have

$$\sqrt{T}\left|\hat{g}_m(v,\mu)-\int_D v(x)\mu(x)\,dx\right| = O(\sqrt{T}m^{-1}(T))$$

and

$$T\mathrm{E}\left(\hat{g}_m(v, W) - \int_D v(x)W(x)\,dx\right)^2 \sim TB(m(T)).$$

Therefore from (i) of this theorem, when $Tm^{-2}(T) \to 0$ and $TB(m(T)) \to 0$ as $T \to \infty$, we have

$$\sqrt{T}\big[\hat{g}_{Tm(T)}(v, S) - g(v, S)\big] \Rightarrow N(0, f_\eta(0)) \quad \text{as } T \to \infty. \qquad \square$$

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