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ON THE OPTIMALITY OF ORTHOGONAL ARRAY PLUS ONE RUN PLANS[1]

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Much contrary to popular belief and even a published result, it is seen that orthogonal array plus one run plans are not necessarily optimal, within the relevant class, for general $s_1 \times \cdots \times s_m$ factorials. A broad sufficient condition on $s_1, \dots, s_m$ ensuring the optimality of such plans has been worked out. This condition covers, in particular, all symmetric factorials and thus strengthens some previous results.

1. Introduction. Over the last two decades, there has been considerable work on the optimality of fractional factorial plans based on orthogonal arrays (OA's) and related structures. Cheng (1980a) pioneered research in this area by proving the universal optimality of fractional factorial plans given by OA's of strength two. Mukerjee (1982) extended this result to OA's of general strength. Subsequent work focussed on the optimality of plans obtained by adding one or more run(s) to an OA. Cheng (1980b) again initiated research in this direction and, in particular, proved that the addition of any single run to a two-symbol OA of even strength yields an optimal fraction, under a very wide range of criteria, for a 2^m factorial. This work on the optimality of OA plus one run plans was followed up by Kolyva-Machera (1989) who considered OA's of strength two for 3^m factorials under the D-criterion and Collombier (1988) who, with a complex parametrization, reported an extension of Cheng's (1980b) result to general $s_1 \times \cdots \times s_m$ factorials; see Mukerjee (1995) for a brief review of the work in this general area.

As for OA plus one run plans, one might be inclined to believe that Collombier's (1988) result, obtained with a complex parametrization, can be routinely translated to the practically more meaningful case of a real parametrization. If so, then this would completely settle the issue of the optimality of such plans. Unfortunately, there seems to be a serious gap in Colombier's proof, which is crucially based on the following claim (see page 42 of his paper); if

$$z = \sum_{j=0}^{s-1} \alpha_j \exp\{2\pi(-1)^{1/2} j/s\},$$

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where s (≥ 2) is a positive integer and $\alpha_0, \dots, \alpha_{s-1}$ are nonnegative integers such that $\alpha_0 + \cdots + \alpha_{s-1} = 1 \bmod s$, then the modulus of the complex number z is at least unity. However, taking $s = 5$, $\alpha_0 = 3$, $\alpha_1 = 1$, $\alpha_2 = 3$, $\alpha_3 = \alpha_4 = 2$, we get $\alpha_0 + \cdots + \alpha_4 = 1 \bmod s$ but at the same time the modulus of z equals 0.382. This shows the incorrectness of the above claim, thus invalidating Collombier's proof even under his complex parametrization (and even for symmetric s^m factorials where s is a prime or prime power) and rendering a translation of his approach to the case of a real parametrization impossible.

Notwithstanding the above, Collombier's (1988) final result seems to be plausible, that is, one may still believe that even for general $s_1 \times \cdots \times s_m$ factorials OA plus one run plans should be optimal and that only a rectification of his proof is needed. Quite counter-intuitively, however, we find that such optimality does not hold for arbitrary $s_1, \dots, s_m$ in general. An example to this effect is shown in Section 3 after the necessary preliminaries are presented in Section 2. This somewhat unexpected development makes the problem far more nontrivial than what was originally believed and calls for a study of conditions on $s_1, \dots, s_m$ that ensure the optimality of OA plus one run plans. This has been taken up in Section 4 where we work directly with a real parametrization and employ the Kronecker calculus for factorial arrangements to derive a broad sufficient condition. In particular, this sufficient condition covers all symmetric factorials and thus substantially strengthens the earlier results due to Cheng (1980b) and Kolyva-Machera (1989).

While concluding this section, we remark that OA plus one run plans can be of particular practical interest when the initial OA is saturated, that is, attains Rao's bound [cf. Mukerjee and Wu (1995)]. This is because if a saturated OA of strength $2t$ is augmented by one run then the resulting plan is most economical in the sense of involving the minimum number of observations needed to get a resolution $2t + 1$ plan, which also allows the use of the standard F-test for testing the significance of the relevant factorial effects.

2. Preliminaries.

2.1. *The model.* Consider a setup involving m (≥ 2) factors $F_1, \dots, F_m$ at $s_1, \dots, s_m$ (≥ 2) levels, respectively. Let $\mathscr{F}$ denote the set of the $v = \prod_{j=1}^{m} s_j$ level combinations represented by ordered m-tuples $i_1 \cdots i_m$ ($0 \leq i_j \leq s_j - 1$; $1 \leq j \leq m$). Let $\eta_{i_1 \cdots i_m}$ be the treatment effect corresponding to $i_1 \cdots i_m$ and η be a $v \times 1$ vector with elements $\eta_{i_1 \cdots i_m}$ arranged lexicographically. We intend to study resolution $2t + 1$ plans ($t \geq 1$) and thus consider a linear model that includes only the general mean and interactions involving t or less factors. As usual, a main effect is a one-factor interaction. The following notation facilitates an explicit presentation of the model.

For $1 \leq j \leq m$, let P_j be an $(s_j - 1) \times s_j$ matrix such that

$$P_j 1_{s_j} = 0, \qquad P_j P_j' = s_j I_{s_j - 1}, \tag{2.1}$$

where, for any positive integer α, 1_α is the $\alpha \times 1$ vector with all elements unity and I_α is the identity matrix of order α. Let Ω be the set of binary m-tuples. For any $x = x_1 \cdots x_m \in \Omega$, define the $a(x) \times v$ matrix

$$P^x = \bigotimes_{j=1}^{m} P_j^{x_j} = P_1^{x_1} \otimes \cdots \otimes P_m^{x_m}, \tag{2.2a}$$

where $\otimes$ denotes the Kronecker product, $a(x) = \prod_{j=1}^m (s_j - 1)^{x_j}$, and for $1 \le j \le m$,

$$P_j^{x_j} = \begin{cases} 1'_{s_j}, & \text{if } x_j = 0, \\ P_j, & \text{if } x_j = 1. \end{cases} \tag{2.2b}$$

The v columns of P^x correspond to the lexicographically ordered level combinations and for any $i_1 \cdots i_m \in \mathscr{F}$, let $p^x_{i_1 \cdots i_m}$ be the corresponding column of P^x.

Define Ω_t as a subset of Ω consisting of those binary m-tuples which have at most t components unity. For $x \in \Omega_t$, let $\theta_x = v^{-1} P^x \eta$. Then by (2.1) and (2.2), θ_x represents the general mean if $x = 00 \cdots 0$, or a complete set of orthogonal contrasts, each having squared norm v^{-1} and belonging to the interaction $F_1^{x_1} \cdots F_m^{x_m}$, if $x \neq 00 \cdots 0$; compare Gupta and Mukerjee (1989), Chapter 2. In fact, $v^{-1} P^{x\prime} P^x$ is the orthogonal projector on the space spanned by 1_v if $x = 00 \cdots 0$, or the space spanned by contrasts representing the interaction $F_1^{x_1} \cdots F_m^{x_m}$ if $x (\in \Omega_t) \neq 00 \cdots 0$. Hence if we allow only the general mean and interactions involving up to t factors in the model, then

$$\eta = \sum_{x \in \Omega_t} v^{-1} P^{x\prime} P^x \eta = \sum_{x \in \Omega_t} P^{x\prime} \theta_x. \tag{2.3}$$

Let $Y_{i_1 \cdots i_m}$ be any observation corresponding to the level combination $i_1 \cdots i_m$. Then, in view of (2.3), our linear model is given by

$$E(Y_{i_1 \cdots i_m}) = \eta_{i_1 \cdots i_m} = \sum_{x \in \Omega_t} \left(p^x_{i_1 \cdots i_m}\right)' \theta_x, \tag{2.4}$$

where the $p^x_{i_1 \cdots i_m}$ and the parameters θ_x are as explained above. As usual, we assume that the errors are uncorrelated and homoscedastic. In particular, if $s_1 = \cdots = s_m = 2$, then with $P_j = [-1, 1]$ for each j, the model (2.4) is in agreement with that in Cheng (1980b). We emphasize here that none of our findings depends on the specific choice of $P_1, \ldots, P_m$ subject to (2.1).

2.2. *More notation.* For subsequent use, we introduce some more notation and indicate a few related points. Let

$$P(t) = [\ldots, P^{x\prime}, \ldots]'_{x \in \Omega_t}, \qquad \theta(t) = (\ldots, \theta'_x, \ldots)'_{x \in \Omega_t}, \tag{2.5}$$

for example, if $m = 2$, $t = 1$, then $P(t) = [P^{00\prime}, P^{01\prime}, P^{10\prime}]'$ and $\theta(t) = (\theta'_{00}, \theta'_{01}, \theta'_{10})'$. By (2.1), (2.2) and (2.5),

$$P(t)' P(t) = \sum_{x \in \Omega_t} W^x, \tag{2.6}$$

where for each $x \in \Omega$ ($\supset \Omega_t$), the $v \times v$ matrix W^x is given by

$$W^x = P^{x\prime}P^x = \bigotimes_{j=1}^{m} W_j^{x_j}, \tag{2.7a}$$

with

$$W_j^{x_j} = P_j^{x_j'}P_j^{x_j} = \begin{cases} 1_{s_j}1'_{s_j}, & \text{if } x_j = 0, \\ s_j I_{s_j} - 1_{s_j}1'_{s_j}, & \text{if } x_j = 1. \end{cases} \tag{2.7b}$$

Following (2.5), for each $i_1 \cdots i_m \in \mathscr{F}$, let

$$p_{i_1 \cdots i_m}^{(t)} = \left[\ldots, \left(p_{i_1 \cdots i_m}^{x}\right)', \ldots\right]'_{x \in \Omega_t}$$

be the column of $P(t)$ that corresponds to $i_1 \cdots i_m$. Then by (2.6), (2.7), for any $i_1 \cdots i_m \in \mathscr{F}$ and $k_1 \cdots k_m \in \mathscr{F}$,

$$p_{i_1 \cdots i_m}^{(t)\prime} p_{k_1 \cdots k_m}^{(t)} = \sum_{x \in \Omega_t} \prod_{j=1}^{m} \left(s_j \delta_{i_j k_j} - 1\right)^{x_j}, \tag{2.8}$$

where $\delta_{i_j k_j}$ is a Kronecker delta. In particular, for any $i_1 \cdots i_m \in \mathscr{F}$,

$$p_{i_1 \cdots i_m}^{(t)\prime} p_{i_1 \cdots i_m}^{(t)} = \sum_{x \in \Omega_t} \prod_{j=1}^{m} (s_j - 1)^{x_j} = \sum_{x \in \Omega_t} a(x) = a \text{ (say)}. \tag{2.9}$$

Note that by (2.5), a is also the number of rows in $P(t)$ or the number of elements of $\theta(t)$.

2.3. *Information matrix and optimality criteria.* With reference to an $s_1 \times \cdots \times s_m$ factorial, let $\mathscr{D}_N$ be the class of all designs or plans involving N level combinations or runs which are not necessarily distinct. For any $d \in \mathscr{D}_N$ and $i_1 \cdots i_m \in \mathscr{F}$, let $r_d(i_1 \cdots i_m)$ be the number of times the level combination $i_1 \cdots i_m$ appears in d. Also, let R_d be a $v \times v$ diagonal matrix with diagonal elements $r_d(i_1 \cdots i_m)$ arranged in the lexicographic order. Then from (2.5) it follows that, under the model (2.4), the information matrix of d, with reference to the parametric vector $\theta(t)$, is proportional to

$$\mathscr{I}_d = P(t)R_d P(t)'. \tag{2.10}$$

Note that $\mathscr{I}_d$ is of order $a \times a$ and that by (2.9),

$$\operatorname{tr}(\mathscr{I}_d) = Na \quad \text{for each } d \in \mathscr{D}_N. \tag{2.11}$$

We are now in a position to introduce the optimality criteria. For any $d \in \mathscr{D}_N$, denoting the eigenvalues of $\mathscr{I}_d$ by $\mu_{d1}, \ldots, \mu_{da}$, by (2.11), $\mu_{di} \in [0, Na]$, $1 \le i \le a$. Following Cheng (1980b), an optimality criterion of type 1 is given by $\phi(\mathscr{I}_d) = \sum_{i=1}^{a} q(\mu_{di})$, where q is a real-valued function defined over $[0, Na]$ such that:

1. q is continuous, strictly convex and strictly decreasing on $[0, Na]$; we include here the possibility that $\lim_{\mu \to 0+} q(\mu) = q(0) = +\infty$;
2. q is continuously differentiable and q' is strictly concave on $(0, Na)$.

Any generalized criterion of type 1 is the pointwise limit of a sequence of type 1 criteria. The class of generalized criteria of type 1, under which we work here, is very wide and includes, in particular, the well-known D-, A- and E-criteria; compare Cheng (1980b).

Cheng (1980b) gave a sufficient condition for a plan to be optimal with respect to (the minimization of) every generalized criterion of type 1. Since $\mathrm{tr}(\mathscr{I}_d)$ is constant over $\mathscr{D}_N$ by (2.11), the condition may be stated as follows in the present context.

LEMMA 2.1. *Let there exist a plan* $d_0 \in \mathscr{D}_N$ *such that*:

(i) $\mathscr{I}_{d_0}$ *has two distinct eigenvalues, the larger of which has multiplicity unity*;
(ii) $\mathrm{tr}(\mathscr{I}_{d_0}^2) < \{\mathrm{tr}(\mathscr{I}_{d_0})\}^2/(a-1)$;
(iii) d_0 *minimizes* $\mathrm{tr}(\mathscr{I}_d^2)$ *over* $\mathscr{D}_N$.

Then d_o *is optimal in* $\mathscr{D}_N$ *with respect to every generalized criterion of type* 1.

2.4. *Some other facts.* For ease in reference, we now recall the definition of an orthogonal array (OA) and indicate some related facts. An $\mathrm{OA}(T, s_1 \times \cdots \times s_m, 2t)$ of strength $2t$ $(\le m)$ is a $T \times m$ array, with elements in the jth column from the set $\{0, 1, \ldots, s_j - 1\}$ $(1 \le j \le m)$, in which all possible combinations of symbols appear equally often in every $T \times 2t$ subarray [Rao (1973)]. The rows of an OA represent the level combinations of an $s_1 \times \cdots \times s_m$ factorial. Suppose an $\mathrm{OA}(N-1, s_1 \times \cdots \times s_m, 2t)$ is available and let $d_o \in \mathscr{D}_N$ be obtained by adding any run, say $i_1 \cdots i_m$, to the $N-1$ runs given by the rows of the array. As in Collombier (1988), then by (2.10),

$$\mathscr{I}_{d_o} = (N-1) I_a + p^{(t)}_{i_1 \cdots i_m} p^{(t)\prime}_{i_1 \cdots i_m}. \tag{2.12}$$

By (2.9) and (2.12), the eigenvalue of $\mathscr{I}_{d_o}$ are $N-1$ and $N-1+a$ with respective multiplicities $a-1$ and 1. These eigenvalues, and hence the behavior of d_o under any generalized criterion of type 1, do not depend on the particular run $i_1 \cdots i_m$ that is added to an OA.

Clearly, d_o satisfies condition (i) of Lemma 2.1. Also, using Rao's bound for OA's, $N-1 \ge a$ and hence, as in Collombier (1988), it is easy to see that d_o satisfies condition (ii) of this lemma as well. Hence d_o is optimal in $\mathscr{D}_N$ with respect to every generalized criterion of type 1 provided it satisfies condition (iii) of Lemma 2.1. However, as seen in the next section, this is not always the case. In fact, as Section 4 reveals, even when d_o satisfies condition (iii), the verification can be quite nontrivial.

3. An example. Let A be an $\mathrm{OA}(144, 2^{119} \times 3^{12}, 2)$ which can be constructed as

$$A = [A_1 * D, 0_4 * A_2],$$

where A_1 is an $\mathrm{OA}(4, 2^3, 2)$, A_2 is an $\mathrm{OA}(36, 2^{11} \times 3^{12}, 2)$, D is a difference matrix $D_{36,36,2}$ arising from a Hadamard matrix of order 36, $0_4 = (0,0,0,0)'$

and $*$ represents the Kronecker sum; see Wang and Wu (1991) for more details on this type of construction. Let d_o be a resolution three plan obtained by adding any run to the array A. Then, with $N = 145$, $a = 144$, $t = 1$, as in Section 2.4, the eigenvalues of $\mathscr{I}_{d_o}$ are seen to be 144 and 288 with respective multiplicities 143 and 1. Hence,

$$\text{(3.1)} \qquad \operatorname{tr}\left(\mathscr{I}_{d_o}^2\right) = 3048192,$$

$$\text{(3.2)} \qquad \det\left(\mathscr{I}_{d_o}\right) = 5971968(144)^{141}, \qquad \operatorname{tr}\left(\mathscr{I}_{d_o}^{-1}\right) = 0.996528.$$

Without loss of generality, suppose the first row of the array A is $00\cdots 0$ and consider a rival plan d_1 obtained from A as follows: delete the run $c_0 = 00\cdots 0$ given by the first row of A and then add the two runs $c_1 = 11\cdots 1$ and $c_2 = 11\cdots 122\cdots 2$, in c_2 the first 119 levels being one and the rest being two. Then d_1, like d_o, involves 145 runs and, analogously to (2.12),

$$\text{(3.3)} \qquad \mathscr{I}_{d_1} = 144 I_{144} - \pi_0\pi_0' + \pi_1\pi_1' + \pi_2\pi_2',$$

where π_0, π_1 and π_2 denote the columns of $P(1)$ corresponding to the level combinations c_0, c_1 and c_2, respectively. By (2.8) and (2.9), taking $t = 1$,

$$\text{(3.4)} \qquad \pi_1'\pi_2 = 108, \qquad \pi_0'\pi_i = -130\ (i = 1,2), \qquad \pi_i'\pi_i = 144\ (i = 0,1,2).$$

By (3.3) and (3.4), three of the eigenvalues of $\mathscr{I}_{d_1}$ are 180 and $198 \mp 2\sqrt{1351}$, the corresponding eigenvectors being $\pi_1 - \pi_2$ and $(99 \pm \sqrt{1351})\pi_0 + 65(\pi_1 + \pi_2)$, respectively. In addition, there is an eigenvalue 144 with multiplicity 141, the corresponding eigenspace being given by the orthocomplement of the space spanned by $\{\pi_0, \pi_1, \pi_2\}$. Hence

$$\text{(3.5)} \qquad \operatorname{tr}\left(\mathscr{I}_{d_1}^2\right) = 3045392,$$

$$\text{(3.6)} \qquad \det\left(\mathscr{I}_{d_1}\right) = 6084000(144)^{141}, \qquad \operatorname{tr}\left(\mathscr{I}_{d_1}^{-1}\right) = 0.996438.$$

By (3.1) and (3.5), d_o does not minimize $\operatorname{tr}(\mathscr{I}_d^2)$ over $\mathscr{D}_{145}$. More importantly, by (3.2) and (3.6), it is inferior to d_1 under both the D- and A-criteria. Thus this example shows that an OA plus one run plan is not necessarily optimal with respect to every generalized criterion of type 1 within the class of plans having the same number of runs. In the next section, we provide a broad sufficient condition under which such optimality holds.

4. A sufficient condition.

4.1. *Result.*

THEOREM 4.1. *Suppose there exists an* $\mathrm{OA}(N-1, s_1 \times \cdots \times s_m, 2t)$ *and let* $d_o \in \mathscr{D}_N$ *be obtained by adding any run to the* $N-1$ *runs given by the array. Then* d_o *is optimal in* $\mathscr{D}_N$ *with respect to every generalized criterion of type* 1 *if*

$$\text{(4.1)} \qquad \mathrm{HCF}(s_{i_1}, \ldots, s_{i_{2t}}) \ge 2 \text{ for each } i_1, \ldots, i_{2t}, \qquad 1 \le i_1 < \cdots < i_{2t} \le m,$$

where HCF *stands for highest common factor.*

The example in Section 3 reveals that the conclusion of Theorem 4.1 may not hold without the condition (4.1). It is satisfying to note that Theorem 4.1 has a wide applicability. It completely settles the issue of optimality of OA plus one run plans for symmetric s^m factorials, even when s is not a prime or prime power, by showing that such plans are, indeed, optimal with respect to every generalized criterion of type 1. The same conclusion holds for many asymmetric factorials of interest (e.g., $2^{m_1} \times 4^{m_2}$ factorials or $3^{m_1} \times 6^{m_2}$ factorials).

4.2. *Proof*. In the rest of this paper, we prove Theorem 4.1 using several lemmas. As noted in Section 2.4, we have to verify that, under (4.1), d_o minimizes $\operatorname{tr}(\mathscr{I}_d^2)$ over $\mathscr{D}_N$. For $d \in \mathscr{D}_N$, let r_d be a $v \times 1$ vector with elements $r_d(i_1 \cdots i_m)$ arranged lexicographically. Also, for $x \in \Omega_{2t}$, define the $v \times v$ matrix W^x as in (2.7).

The key considerations in our proof of Theorem 4.1 are as follows. In Lemma 4.1, we show that for every $d \in \mathscr{D}_N$, $\operatorname{tr}(\mathscr{I}_d^2)$ is a linear combination of the quantities $r_d' W^x r_d$, $x \in \Omega_{2t}$, where the combining coefficients are nonnegative and do not depend on d. Then Lemma 4.2(a) shows that $r_d' W^{00\cdots 0} r_d$ is the same for all $d \in \mathscr{D}_N$. Hence it suffices to show that the plan d_o minimizes $r_d' W^x r_d$ over $\mathscr{D}_N$ for each $x\ (\neq 00\cdots 0) \in \Omega_{2t}$. Next, as a consequence of Lemmas 4.2(b), 4.3 and 4.4, we note that if for any $d \in \mathscr{D}_N$ and any $x\ (\neq 00\cdots 0) \in \Omega_{2t}$, $r_d' W^x r_d < r_{d_o}' W^x r_{d_o}$ then $r_d' W^x r_d = 0$ [cf. (4.13) and (4.19) below]. Lemma 4.5 shows that this is impossible under the condition (4.1) and thus completes the proof of Theorem 4.1. In fact, for any $x\ (\neq 00\cdots 0) \in \Omega_{2t}$ and any $d \in \mathscr{D}_N$, by (2.7), the quantity $r_d' W^x r_d$ can be interpreted as v times the sum of squares due to the factorial effect $F_1^{x_1} \cdots F_m^{x_m}$ in a full $s_1 \times \cdots \times s_m$ factorial where each level combination appears once and r_d plays the role of the observational vector. In view of this interpretation, under (4.1), the arguments underlying Lemma 4.5 are intuitively anticipated for $t = 1$, noting that the elements of r_d are integers and that $N = 1 \bmod s_i s_j$ for every $1 \le i < j \le m$. Lemma 4.5 formalizes these arguments for general t.

At this stage, it is also possible to explain heuristically why the counterexample in Section 3 worked. There condition (4.1) does not hold and the plan d_1 was so constructed that $r_{d_1}' W^x r_{d_1} = 0$ for every x for which $r_{d_1}' W^x r_{d_1}$ can be interpreted as the sum of squares due to the interaction between a two-level and a three-level factor in the sense described in the last paragraph. Thus $r_{d_1}' W^x r_{d_1}$ became less than $r_{d_o}' W^x r_{d_o}$ for a large number of choices of x $(\neq 00\cdots 0) \in \Omega_{2t}$. This, in turn, yielded $\operatorname{tr}(\mathscr{I}_{d_1}^2) < \operatorname{tr}(\mathscr{I}_{d_o}^2)$ and ensured the superiority of d_1 over d_o under the D- and A-criteria.

LEMMA 4.1. *For every $d \in \mathscr{D}_N$,*

$$\operatorname{tr}\left(\mathscr{I}_d^2\right) = \sum_{x \in \Omega_{2t}} \beta(x) r_d' W^x r_d,$$

where the scalars $\beta(x)$ are nonnegative and do not depend on d.

PROOF. For $d \in \mathscr{D}_N$, recalling the definition of $p^{(t)}_{i_1 \cdots i_m}$, by (2.10),

$$\text{(4.2)} \qquad \operatorname{tr}(\mathscr{I}_d^2) = \sum\sum r_d(i_1 \cdots i_m) r_d(k_1 \cdots k_m)\left\{p^{(t)'}_{i_1 \cdots i_m} p^{(t)}_{k_1 \cdots k_m}\right\}^2,$$

where the double sum extends over $i_1 \cdots i_m \in \mathscr{F}$ and $k_1 \cdots k_m \in \mathscr{F}$. Let B be a $v \times v$ matrix with lexicographically ordered rows and columns such that the $(i_1 \cdots i_m, k_1 \cdots k_m)$th element of B is

$$\text{(4.3)} \qquad b(i_1 \cdots i_m, k_1 \cdots k_m) = \left\{p^{(t)'}_{i_1 \cdots i_m} p^{(t)}_{k_1 \cdots k_m}\right\}^2.$$

The matrix B does not depend on d and by (4.2), (4.3),

$$\text{(4.4)} \qquad \operatorname{tr}(\mathscr{I}_d^2) = r'_d B r_d.$$

Now by (2.8) and (4.3),

$$\text{(4.5)} \qquad b(i_1 \cdots i_m, k_1 \cdots k_m) = \sum_{u,\, y \in \Omega_t}\sum \prod_{j=1}^{m} \left(s_j \delta_{i_j k_j} - 1\right)^{u_j + y_j}.$$

But with any $u = u_1 \cdots u_m \in \Omega_t$ and $y = y_1 \cdots y_m \in \Omega_t$, for $1 \le j \le m$, we have

$$\text{(4.6a)} \qquad \left(s_j \delta_{i_j k_j} - 1\right)^{u_j + y_j} = 1 \quad \text{if } u_j = y_j = 0,$$

$$\text{(4.6b)} \qquad = s_j \delta_{i_j k_j} - 1 \quad \text{if } u_j = 0,\ y_j = 1 \text{ or } u_j = 1,\ y_j = 0,$$

$$\text{(4.6c)} \qquad = (s_j - 2)\left(s_j \delta_{i_j k_j} - 1\right) + (s_j - 1) \quad \text{if } u_j = y_j = 1,$$

since $\delta_{i_j k_j}$ is a Kronecker delta. Note that for any $u, y \in \Omega_t$, there are at most $2t$ choices of j for which either (4.6b) or (4.6c) can arise. Since $s_j \ge 2$ for each j, it follows that for every $u, y \in \Omega_t$, the product on j of the left-hand side of (4.6) is a linear combination of $\Pi_{j=1}^{m}(s_j \delta_{i_j k_j} - 1)^{x_j}$, over $x = x_1 \cdots x_m \in \Omega_{2t}$, the combining coefficients, possibly dependent on u and y, being all nonnegative. Comparing with (2.7), it follows from (4.5) that

$$\text{(4.7)} \qquad B = \sum_{x \in \Omega_{2t}} \beta(x) W^x,$$

where the scalars $\beta(x)$ are nonnegative and, evidently, do not depend on d. From (4.4) and (4.7), the lemma follows. □

LEMMA 4.2. (a) *For every* $d \in \mathscr{D}_N$, *if* $x = 00 \cdots 0$ *then* $r'_d W^x r_d = N^2$.

(b) *Let* d_o *be as defined in Theorem* 4.1. *Then for every* $x \in \Omega_{2t}$, $x \ne 00 \cdots 0$,

$$r'_{d_o} W^x r_{d_o} = \prod_{j=1}^{m} (s_j - 1)^{x_j}.$$

PROOF. Part (a) follows noting that $W^{00 \cdots 0} = 1_v 1'_v$ by (2.7) and that $r'_d 1_v = N$ for each $d \in \mathscr{D}_N$.

(b) Recall that d_o is obtained adding one run to an orthogonal array. Let $r^*(i_1 \cdots i_m)$ be the number of times $i_1 \cdots i_m$ appears as a row of this array and r^* be a $v \times 1$ vector with elements $r^*(i_1 \cdots i_m)$ arranged lexicographically. Then

$$\text{(4.8)} \qquad r_{d_o} = r^* + e,$$

where e is a $v \times 1$ unit vector. Since r^* arises from an OA of length $2t$, from (2.7) it is not hard to see that $W^x r^* = 0$, whenever $x \in \Omega_{2t}$ and $x \neq 00 \cdots 0$; compare Mukerjee (1982). Hence for any such x, $r'_{d_o} W^x r_{d_o}$ equals a diagonal element of W^x by (4.8) and, by (2.7), the result follows. □

LEMMA 4.3. *Let $h = (h_0, h_1, \dots, h_{s-1})'$ be an $s \times 1$ vector where $s \geq 2$. If*

$$\#\{(i,k): 0 \leq i < k \leq s-1, h_i - h_k \neq 0\} \leq s-2,$$

where # denotes the cardinality of a set, then $h_0 = h_1 = \cdots = h_{s-1}$.

PROOF. This lemma is similar to Lemma 2.1 in Chatterjee and Mukerjee (1993) on connected designs and can be proved by induction on s. □

Some more notation is needed for presenting the next lemma. Let g be a positive integer and $s_1, \dots, s_g$ (≥ 2) be integers. For $1 \leq j \leq g$, let e_{ji} ($0 \leq i \leq s_j - 1$) be the unit vectors of order s_j and Q_j be a matrix with $\frac{1}{2}s_j(s_j - 1)$ rows and s_j columns, such that the rows of Q_j are given by $(e_{ji} - e_{jk})'$, $0 \leq i < k \leq s_j - 1$. Let $w = \prod_{j=1}^{g} s_j$ and h be a $w \times 1$ vector. Then the following lemma holds.

LEMMA 4.4. *If among the elements of $(Q_1 \otimes \cdots \otimes Q_g)h$ at most $\prod_{j=1}^{g}(s_j - 1) - 1$ are nonzero, then $(Q_1 \otimes \cdots \otimes Q_g)h = 0$.*

PROOF. For $g = 1$, the result is a consequence of Lemma 4.3. To apply the method of induction, let the result hold for $g = f$ (≥ 1). Consider $g = f + 1$ and suppose among the elements of $(Q_1 \otimes \cdots \otimes Q_{f+1})h$ at most $\prod_{j=1}^{f+1}(s_j - 1) - 1$ are nonzero. Denoting the $\sigma_1 = \prod_{j=1}^{f}\{\frac{1}{2}s_j(s_j - 1)\}$ rows of $Q_1 \otimes \cdots \otimes Q_f$ by $\xi'_{1i}(0 \leq i \leq \sigma_1 - 1)$ and the $\sigma_2 = \frac{1}{2}s_{f+1}(s_{f+1} - 1)$ rows of Q_{f+1} by ξ'_{2k} ($0 \leq k \leq \sigma_2 - 1$), then among the $\sigma_1 \sigma_2$ quantities,

$$H_{ik} = (\xi'_{1i} \otimes \xi'_{2k})h, \qquad 0 \leq i \leq \sigma_1 - 1, 0 \leq k \leq \sigma_2 - 1, \tag{4.9}$$

at most $\prod_{j=1}^{f+1}(s_j - 1) - 1$ are nonzero. Hence writing $L = \{0, 1, \dots, \sigma_1 - 1\}$ and $L_0 = \{i$: $i \in L$, among the H_{ik}, $0 \leq k \leq \sigma_2 - 1$, at least $s_{f+1} - 1$ are nonzero$\}$, then

$$\#L_0 \leq \prod_{j=1}^{f}(s_j - 1) - 1. \tag{4.10}$$

Consider now any fixed $i \in L - L_0$. Among the H_{ik}, $0 \leq k \leq \sigma_2 - 1$, then at most $s_{f+1} - 2$ are nonzero, that is, by (4.9) and the definition of the vectors ξ'_{2k}, among the elements of $(\xi'_{1i} \otimes Q_{f+1})h$ at most $s_{f+1} - 2$ are nonzero. But

$$(\xi'_{1i} \otimes Q_{f+1})h = Q_{f+1}h^{(1)}, \tag{4.11}$$

where $h^{(1)} = (\xi'_{1i} \otimes I_{s_{f+1}})h$. Hence among the elements of $Q_{f+1}h^{(1)}$ at most $s_{f+1} - 2$ are nonzero. Since the result holds for $g = 1$, it follows that $Q_{f+1}h^{(1)} = 0$, that is, by (4.11), $H_{ik} = 0$ for every k. Thus

$$H_{ik} = 0, \qquad 0 \leq k \leq \sigma_2 - 1 \qquad \text{if } i \in L - L_0. \tag{4.12}$$

Consider now any fixed k $(0 \le k \le \sigma_2 - 1)$. By (4.10) and (4.12), among the H_{ik}, $0 \le i \le \sigma_1 - 1$, at most $\prod_{j=1}^{f}(s_j - 1) - 1$ are nonzero, that is, by (4.9) and the definition of the vectors ξ'_{1i}, among the elements of $(Q_1 \otimes \cdots \otimes Q_f \otimes \xi'_{2k})h$, at most $\prod_{j=1}^{f}(s_j - 1) - 1$ are nonzero. Since

$$(Q_1 \otimes \cdots \otimes Q_f \otimes \xi'_{2k})h = (Q_1 \otimes \cdots \otimes Q_f)\big(I_{s_1} \otimes \cdots \otimes I_{s_f} \otimes \xi'_{2k}\big)h,$$

by induction hypothesis, it follows that $(Q_1 \otimes \cdots \otimes Q_f \otimes \xi'_{2k})h = 0$. As k is arbitrary, this yields $(Q_1 \otimes \cdots \otimes Q_f \otimes Q_{f+1})h = 0$, and the result follows by induction. □

PROPOSITION 4.5. *Let* (4.1) *hold. Then for every* $d \in \mathscr{D}_N$,

$$r'_d W^x r_d \ge \prod_{j=1}^{m} (s_j - 1)^{x_j},$$

whenever $x \in \Omega_{2t}$ *and* $x \ne 00 \cdots 0$.

PROOF. Without loss of generality, let $x = x_1 \cdots x_m$ with $x_1 = \cdots = x_g = 1$ and $x_{g+1} = \cdots = x_m = 0$, where $1 \le g \le 2t$. Let $w = \prod_{j=1}^{g} s_j$. Then by (2.7),

$$r'_d W^x r_d = h'_d W h_d, \tag{4.13}$$

where the $w \times 1$ vector h_d and the $w \times w$ matrix W are given, respectively, by

$$h_d = \big(I_{s_1} \otimes \cdots \otimes I_{s_g} \otimes 1'_{s_{g+1}} \otimes \cdots \otimes 1'_{s_m}\big) r_d, \tag{4.14}$$

$$W = \bigotimes_{j=1}^{g} \big(s_j I_{s_j} - 1_{s_j} 1'_{s_j}\big). \tag{4.15}$$

With $Q_1, \dots, Q_g$ defined as in the context of Lemma 4.4, we have $Q'_j Q_j = s_j I_{s_j} - 1_{s_j} 1'_{s_j}$, $1 \le j \le g$. Hence by (4.13) and (4.15),

$$r'_d W^x r_d = \lambda'_d \lambda_d, \tag{4.16}$$

where

$$\lambda_d = (Q_1 \otimes \cdots \otimes Q_g) h_d. \tag{4.17}$$

If possible, suppose

$$r'_d W^x r_d < \prod_{j=1}^{m} (s_j - 1)^{x_j} = \prod_{j=1}^{g} (s_j - 1). \tag{4.18}$$

Since the elements of h_d and hence those of λ_d are integers, then by (4.16), at most $\prod_{j=1}^{g}(s_j - 1) - 1$ elements of λ_d are nonzero. By (4.17) and Lemma 4.4, this implies that $\lambda_d = 0$, that is, by (4.13) and (4.16),

$$W h_d = 0, \tag{4.19}$$

noting that W is nonnegative definite.

Now, by (4.15),

$$W = \sum_{u \in U} M_{1u_1} \otimes \cdots \otimes M_{gu_g}, \tag{4.20}$$

where U is the set of binary g-tuples, $u = u_1 \cdots u_g$ is a typical member of U and for $1 \le j \le g$,

$$M_{j0} = -1_{s_j} 1'_{s_j}, \qquad M_{j1} = s_j I_{s_j}. \tag{4.21}$$

By (4.14) and (4.21),

$$(M_{10} \otimes \cdots \otimes M_{g0}) h_d = (-1)^g 1_w 1'_v r_d = (-1)^g N 1_w,$$

as $d \in \mathscr{D}_N$. Hence by (4.19) and (4.20),

$$\sum_{\substack{u \in U \\ u \ne 00 \cdots 0}} (M_{1u_1} \otimes \cdots \otimes M_{gu_g}) h_d = -(-1)^g N 1_w. \tag{4.22}$$

Note that the elements of the vectors on both sides of (4.22) are integer-valued. Since (4.1) holds and $g \le 2t$,

$$\mathrm{HCF}(s_1, \dots, s_g) = n \text{ (say)} \ge 2. \tag{4.23}$$

As $\mathscr{D}_N$ contains d_o which is obtained by adding one run to an OA($N - 1$, $s_1 \times \cdots \times s_m, 2t$), clearly $N = 1 \bmod (s_1 \cdots s_g)$. Hence by (4.23), no element of the vector in the right-hand side of (4.22) is an integral multiple of n. On the other hand, by (4.21), every element of $M_{1u_1} \otimes \cdots \otimes M_{gu_g}$ is an integral multiple of $\prod_{j=1}^{g} s_j^{u_j}$, and hence by (4.23), every element in the left-hand side of (4.22) is an integral multiple of n. This contradiction shows the impossibility of (4.18) and proves the lemma. □

PROOF OF THEOREM 4.1. By Lemmas 4.2 and 4.5, for each $d \in \mathscr{D}_N$ and $x \in \Omega_{2t}$, we have $r'_d W^x r_d \ge r'_{d_o} W^x r_{d_o}$. Hence by Lemma 4.1, the plan d_o minimizes $\mathrm{tr}(\mathscr{I}_d^2)$ over $\mathscr{D}_N$ and the result follows using Lemma 2.1. □

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