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ADAPTIVE OPTIMIZATION AND D-OPTIMUM EXPERIMENTAL DESIGN

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We consider the situation where one has to maximize a function $\eta(\theta, \mathbf{x})$ with respect to $\mathbf{x} \in \mathbb{R}^q$, when θ is unknown and estimated by least squares through observations $y_k = \mathbf{f}^\top(\mathbf{x}_k)\theta + \varepsilon_k$, with ε_k some random error. Classical applications are regulation and extremum control problems. The approach we adopt corresponds to maximizing the sum of the current estimated objective and a penalization for poor estimation: $\mathbf{x}_{k+1}$ maximizes $\eta(\hat{\theta}^k, \mathbf{x}) + (\alpha_k/k), d_k(\mathbf{x})$, with $\hat{\theta}^k$ the estimated value of θ at step k and d_k the penalization function. Sufficient conditions for strong consistency of $\hat{\theta}^k$ and for almost sure convergence of $(1/k)\sum_{i=1}^k \eta(\theta, \mathbf{x}_i)$ to the maximum value of $\eta(\theta, \mathbf{x})$ are derived in the case where $d_k(\cdot)$ is the variance function used in the sequential construction of D-optimum designs. A classical sequential scheme from adaptive control is shown not to satisfy these conditions, and numerical simulations confirm that it indeed has convergence problems.

1. Introduction. We consider an optimization problem, where one wants to maximize a worth $\eta(\bar{\theta}, \mathbf{x})$ with respect to $\mathbf{x} \in \mathscr{X}$, with $\bar{\theta}$ the unknown true value of some parameter vector $\theta \in \mathbb{R}^p$. The worth function $\eta(\bar{\theta}, \cdot)$ may be possibly multimodal on $\mathscr{X}$, a compact subset of $\mathbb{R}^q$. The value of $\bar{\theta}$ is estimated by Least-Squares (LS) from observations

$$y_k = \mathbf{f}^\top(\mathbf{x}_k)\bar{\theta} + \varepsilon_k, \tag{1.1}$$

with $\mathbf{f}(\cdot)$ a continuous function of $\mathbf{x}$ and ε_k a random error. We assume that $\{\varepsilon_k\}$ is a martingale difference sequence with respect to an increasing sequence of σ-fields $\{\mathscr{F}_k\}$: ε_k is $\mathscr{F}_k$-measurable and $E\{\varepsilon_k|\mathscr{F}_{k-1}\} = 0$ for all k. An important example is when the ε_k's are independent random variables with zero means.

A well-known example corresponds to the so-called "self-tuning optimizer" or "self-tuning extremum control" problem; see Wellstead and Zarrop (1991), where the worth $\eta(\theta, \mathbf{x}) = \mathbf{f}^\top(\mathbf{x})\theta$ is quadratic in $\mathbf{x}$ and noisy observations of $\eta(\bar{\theta}, \mathbf{x})$ itself are available. In this case, the value $\mathbf{x}_{k+1}$ maximizing $\eta(\hat{\theta}^k, \mathbf{x})$ is obtained analytically. However, using this value at the next step [which corresponds to "certainty equivalence control"; see Bar-Shalom and Tse (1974)] does not guarantee convergence of $\mathbf{x}_k$ to $\mathbf{x}^*$ which maximizes $\eta(\bar{\theta}, \mathbf{x})$. For instance, using the ODE method of Ljung (1977), Bozin and Zarrop (1991) give the set of values of θ and x to which $\hat{\theta}^k$ and x_k may converge when $\eta(\theta, x) = \theta_1 x + \theta_2 x^2$: the set of limiting values for x_k contains $x^* = -\bar{\theta}_1/(2\bar{\theta}_2)$ but is not restricted to

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it. It has thus been suggested to randomly perturb the certainty equivalence control law in order to obtain convergence, see Bozin and Zarrop (1991). Another class of example corresponds to regulation problems, where one wishes to minimize the deviation of the response $\mathbf{f}^\top(\mathbf{x})\bar{\theta}$ from a given target. Again, the addition of random disturbances to the certainty equivalence control law can be used to obtain convergence; see, for example, Lai and Wei (1987), where the problem of how often probing inputs (disturbances) should be introduced is considered. One can refer, for example, to Aström and Wittenmark (1989) and Wellstead and Zarrop (1991) for examples of application of self-tuning systems to various control problems (distillation column, chemical reactor, ship steering, spark ignition engine, temperature regulation, etc.).

In fact, the sequence $\{\mathbf{x}_k\}$ should be chosen so as to fulfill two objectives simultaneously: (i) estimate $\bar{\theta}$, or a function of it; (ii) maximize $\eta(\bar{\theta}, \mathbf{x})$. The problem thus corresponds to dual control [see the pioneer papers by Fel'dbaum (1960, 1961)], that is to a stochastic optimal control problem [Bar-Shalom (1981)]. This is more easily exposed for a finite time-horizon N. The classical approach is then Bayesian, with a prior distribution $\pi(\theta) = \pi(\theta|\mathscr{F}_0)$ assumed for θ. A standard objective [see, e.g., Ginebra and Clayton (1995)], is then to maximize

$$E\left\{\sum_{i=1}^{N} \eta(\theta, \mathbf{x}_i)|\mathscr{F}_0\right\},$$

where the expectation is with respect to all random variables. Since the x_i's are chosen sequentially, the problem can be decomposed into

$$\max_{\mathbf{x}_1}\Bigg[E\Bigg\{\eta(\theta, \mathbf{x}_1) + \max_{\mathbf{x}_2}\left[E\left\{\eta(\theta, \mathbf{x}_2) + \cdots + \max_{\mathbf{x}_N}\left[E\left\{\eta(\theta, \mathbf{x}_N)|\mathscr{F}_{N-1}\right\}\right]\cdots\Big|\mathscr{F}_1\right\}\right]\Bigg|\mathscr{F}_0\Bigg\}\Bigg].$$

The presence of imbedded expectations and maximizations makes it extremely difficult to solve, except in very particular situations, so that simple suboptimal solutions have been proposed; see, for example, Aström and Wittenmark (1989). The already mentioned addition of random disturbances to the certainty equivalence control law is one of them. Certainty equivalence control corresponds to using at step k the optimal strategy for a deterministic system with parameters $\hat{\theta}^k$. This strategy is generally not satisfactory due to its *passive* character: $\mathbf{x}_k$ does not help to estimate θ. A suboptimal *active* strategy is proposed for instance in Kulcsár et al. (1996), and a comparison between different strategies is presented in Allison et al. (1995).

The problem is even more complicated when the horizon is infinite, which is the case considered in this paper, and the restriction is then to even simpler strategies. A possible criterion of asymptotic performance for a Bayesian strategy is the long run average $\lim_{N\to\infty}(1/N)E\{\sum_{i=1}^{N} \eta(\theta, \mathbf{x}_i)|\mathscr{F}_0\}$ and, for a frequentist approach, a natural objective is to reach

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \eta(\bar{\theta}, \mathbf{x}_i) = \max_{\mathbf{x}\in\mathscr{X}} \eta(\bar{\theta}, \mathbf{x}) \quad \text{almost surely.} \tag{1.2}$$

Note that all terms in the sequence receive the same weight in the evaluation of the performance, that is all $\mathbf{x}_i$'s are considered equally important in terms of worth $\eta(\bar{\theta}, \mathbf{x}_i)$. Opposite to this is the case where experiments are performed in two stages: first, the worth of the $\mathbf{x}_i$'s is not considered, and they are chosen according to an experimental design criterion for the estimation of $\mathbf{x}^*$ which maximizes $\eta(\bar{\theta}, \mathbf{x})$; next, the $\mathbf{x}_i$'s are fixed at the estimated value $\hat{\mathbf{x}}^*$. One can refer, for instance, to Pronzato and Walter (1993) for a survey of experimental design approaches for estimating the extremum of a response surface.

A popular strategy in the infinite horizon case consists in adding to the estimate of the function to be maximized, $\eta(\hat{\theta}^k, \mathbf{x})$ at step k, a penalty for poor estimation of θ at the next step. For instance, the penalty can be proportional to the decrease of the determinant [Aström and Wittenmark (1989)], or the trace [Wittenmark (1975)] of the covariance matrix for θ, or to the decrease of the variance of a particular component of θ [Wittenmark and Elevitch (1985)]. The upper-bound designs used in Ginebra and Clayton (1995) also belong to this family, with a penalty proportional to the standard deviation of the prediction at $\mathbf{x}$. The reason for such a deterministic choice of $\mathbf{x}_k$ is that it can be expected to give better performances than the introduction of random disturbances, as in Bozin and Zarrop (1991).

Note that when dynamical systems are concerned, the vector $\mathbf{x}_k$ is usually formed of lagged values of a scalar input variable u_k, that is $\mathbf{x}_k = (u_k, u_{k-1}, \dots, u_{k-q+1})$, which induces peculiar constraints on successive design variables $\mathbf{x}_k$ (e.g., when u_k can take two values only, $\mathbf{x}_k$ lives on a DeBruijn graph B_q). This case is not considered here, and will be the subject of further studies. We thus assume that the design set $\mathscr{X}$ is the same for all vectors $\mathbf{x}_k$, and all components of $\mathbf{x}_k$ can be chosen at step k.

We shall consider the case where at step k, that is, after the observation of $y_1, \dots, y_k$, with k larger than some K_0, the vector $\mathbf{x}_{k+1}$ is taken as

$$\mathbf{x}_{k+1} = \arg\max_{\mathbf{x}\in\mathscr{X}} \eta(\hat{\theta}^k, \mathbf{x}) + \alpha_k \mathbf{f}^\top(\mathbf{x})\mathbf{M}_k^{-1}\mathbf{f}(\mathbf{x}), \tag{1.3}$$

with $\mathbf{M}_k$ the design matrix

$$\mathbf{M}_k = \sum_{i=1}^{k} \mathbf{f}(\mathbf{x}_i)\mathbf{f}^\top(\mathbf{x}_i). \tag{1.4}$$

We consider LS estimation,

$$\hat{\theta}^k = \arg\min_{\theta\in\Theta} \sum_{i=1}^{k} \left[y_i - \mathbf{f}^\top(\mathbf{x}_i)\theta\right]^2,$$

with $\Theta \subset \mathbb{R}^p$ a compact set, but extension to Bayesian estimation could be considered as well. If the optimization problem in (1.3) has several solutions, then $\mathbf{x}_{k+1}$ is simply taken as one of them.

If the ε_k's were i.i.d. with zero mean and variance σ^2, *and if the* $\mathbf{x}_i$'s *were nonrandom constants*, then $\mathbf{M}_k/\sigma^2$ would be the usual information matrix and $\mathbf{f}^\top(\mathbf{x})\mathbf{M}_k^{-1}\mathbf{f}(\mathbf{x})$ would be proportional to the variance of the prediction of y at $\mathbf{x}$. Since $\mathbf{x}_k$ depends on previous observations (it is $\mathscr{F}_{k-1}$ measurable), $\mathbf{M}_k$ is not (proportional to) the usual information matrix. Also, conditions for strong consistency of $\hat{\theta}^k$ are more stringent than in the case where the $\mathbf{x}_i$'s are nonrandom constants. One result of the paper is that the usual choice [Aström and Wittenmark (1989)] of keeping α_k constant in (1.3) does not fulfill the sufficient conditions of Lai and Wei (1982) for strong consistency of $\hat{\theta}^k$. Numerical simulations indeed confirm that there are convergence problems when α_k is kept constant.

In Section 2 we consider the design problem only, where for $k \geq K_0$,

$$\mathbf{x}_{k+1} = \arg\max_{\mathbf{x}\in\mathscr{X}} \eta\left(\bar{\theta}, \mathbf{x}\right) + \alpha_k \mathbf{f}^\top(\mathbf{x})\mathbf{M}_k^{-1}\mathbf{f}(\mathbf{x}); \tag{1.5}$$

that is, $\bar{\theta}$ is assumed to be known and we are interested in the convergence properties of the sequence $\{\mathbf{x}_k\}$. In particular, we show that under rather general conditions (see the hypotheses below) $(1/k)\sum_{i=1}^k \eta(\bar{\theta}, \mathbf{x}_i)$ converges to $\max_{\mathbf{x}\in\mathscr{X}} \eta(\bar{\theta}, \mathbf{x})$ when α_k/k tends to zero. Convergence for the original situation (1.3) is considered in Section 3: α_k/k should tend to zero but not too fast, and we derive sufficient conditions on α_k for strong consistency of $\hat{\theta}^k$ and for (1.2) to be satisfied. An illustrative example is presented in Section 4. Finally, Section 5 concludes and points out some possible developments. We shall denote by ξ a normalized measure on $\mathscr{X}$ (that is, such that $\int_{\mathscr{X}} \xi(d\mathbf{x}) = 1$), and $\Xi(\mathscr{X})$ the set of such measures. We define $\mathbf{I}(\xi)$ as $\mathbf{I}(\xi) = \int_{\mathscr{X}} \mathbf{f}(\mathbf{x})\mathbf{f}^\top(\mathbf{x})\xi(d\mathbf{x})$. At step k, we define ξ_k as the discrete measure having support points $\mathbf{x}_i$, $i = 1, \ldots, k$, with uniform mass $1/k$, so that $\mathbf{I}(\xi_k) = \mathbf{I}_k = \mathbf{M}_k/k$. We write $z_k \nearrow$ (resp., $z_k \searrow$) for a nondecreasing (resp. nonincreasing) sequence, and $z_k \nearrow l$ (resp., $z_k \searrow l$) when the sequence converges to $l \leq \infty$ (resp., $\geq -\infty$). We indicate hereafter the assumptions we shall refer to, some consequences of which are given in the Appendix. We assume throughout the paper that A1 and A2 are satisfied.

A1. $\mathbf{f}(\mathbf{x})$ and $\eta(\bar{\theta}, \mathbf{x})$ are continuous in $\mathbf{x}$ on $\mathscr{X}$ compact.
A2. $\mathbf{x}_1, \ldots, \mathbf{x}_{K_0}$ are such that $\mathbf{M}_{K_0}$ is positive definite.
A3. $\eta(\theta, \mathbf{x})$ is continuous in $(\theta, \mathbf{x})$ on $\Theta \times \mathscr{X}$, with Θ compact.
A4. $\bar{\theta} \in \Theta$ and $\hat{\theta}^k \in \Theta$ for all k (a projection of the estimate is used if necessary).
A5. In the linear regression model (1.1), $\{\varepsilon_k\}$ is a martingale-difference sequence which satisfies

$$\sup_k E\left\{\varepsilon_k^2 | \mathscr{F}_{k-1}\right\} < \infty \quad \text{almost surely.}$$

A6. $\eta(\bar{\theta}, \mathbf{x})$ has a unique global maximizer $\mathbf{x}^*$:

$$\forall\beta > 0,\ \exists\varepsilon > 0 \quad \text{such that } \eta\left(\bar{\theta}, \mathbf{x}\right) + \varepsilon > \eta\left(\bar{\theta}, \mathbf{x}^*\right) \Rightarrow \|\mathbf{x} - \mathbf{x}^*\| < \beta.$$

2. Optimum design: the parameters are known. In this section, we consider the properties of the design generated when $\bar{\theta}$ is known (local design) and, at step $k \geq K_0$, $\mathbf{x}_{k+1}$ is given by (1.5), which we rewrite as

$$\mathbf{x}_{k+1} = \arg\max_{\mathbf{x}\in\mathscr{X}} \eta\left(\bar{\theta}, \mathbf{x}\right) + \frac{\alpha_k}{k} d_k(\mathbf{x}), \qquad k \geq K_0, \tag{2.6}$$

where

$$d_k(\mathbf{x}) = \mathbf{f}^\top(\mathbf{x})\mathbf{I}_k^{-1}\mathbf{f}(\mathbf{x}), \tag{2.7}$$

with $\mathbf{I}_k = \mathbf{M}_k/k$ the average design matrix *per* sample. If there are several solutions, $\mathbf{x}_{k+1}$ is taken as one of them. Assuming that $\bar{\theta}$ is known is of little practical interest, but the results obtained in this case form an important step for proving those of Section 3.

We assume that $\alpha_k \geq 0$, and, for reasons that will become clear in Section 3, we are mainly interested in the case $\alpha_k \to \infty$. We shall distinguish between two situations: $\alpha_k/k = \alpha$ constant and $\alpha_k/k \to 0$. The case $\alpha_k/k \to \infty$ is not considered, since A1 implies that for any ε, $\exists K_1$ such that $k\eta(\bar{\theta}, \mathbf{x})/\alpha_k < \varepsilon$ for any $k > K_1$ and any $x \in \mathscr{X}$; see (A.4). Under A1–A2, the measure ξ_k therefore converges to a D-optimal measure on $\mathscr{X}$ for the linear regression model (1.1); see Wynn (1970) for a proof of convergence when $\mathbf{x}_{k+1} = \arg\max_{\mathbf{x}\in\mathscr{X}} d_k(\mathbf{x})$.

Compromise designs: $\alpha_k = k\alpha$. When $\alpha_k = k\alpha$, the sequence generated by (2.6) makes a compromise between D-optimal design and maximization of $\eta(\bar{\theta}, \mathbf{x})$, as indicates the following theorem, the proof of which is given in Section 6.

THEOREM 2.1. *Assume that* A1–A2 *are satisfied and that* $\alpha_k = k\alpha$. *The sequence* $\{\mathbf{x}_k\}$ *generated by* (2.6) *is then such that* $H(\xi_k)$ *tends to* $H(\xi^*) = \max_{\xi\in\Xi(\mathscr{X})} H(\xi)$ *when* $k \to \infty$, *where*

$$H(\xi) = F(\xi) + \alpha \log\det \mathbf{I}(\xi), \tag{2.8}$$

with

$$F(\xi) = \int_{\mathscr{X}} \eta(\bar{\theta}, \mathbf{x})\xi(d\mathbf{x}). \tag{2.9}$$

Worth-maximizing designs: $\alpha_k/k \to 0$. When $\alpha_k/k \to 0$, the study of the convergence properties of (2.6) is more difficult because of the following circular argument:

(i) the penalizing term $(\alpha_k/k)d_k(\mathbf{x})$ can be expected to decrease since the weighing factor α_k/k tends to zero, and $\mathbf{x}_k$ can thus be expected to converge to a global maximizer of $\eta(\bar{\theta}, \mathbf{x})$; (ii) at the same time, if the set of these global maximizers $\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*$ is such that the associated regressors $\mathbf{f}(\mathbf{x}_1^*), \ldots, \mathbf{f}(\mathbf{x}_m^*)$ form a singular design, the matrix $\mathbf{I}_k$ will tend to become singular, which will increase the value of $d_k(\mathbf{x})$; see (2.7).

On the other hand, it is precisely this dual feature that will permit in Section 3 to the simultaneous accumulation of the sequence $\{\mathbf{x}_k\}$ on the global maximizer(s) of $\eta(\bar{\theta}, \mathbf{x})$ and strong consistency of the parameter estimates.

Bounds on the speed of decrease of $\det \mathbf{I}_k$ and speed of increase of $\max_{\mathbf{x}\in\mathscr{X}} d_k(\mathbf{x})$ are given in Lemmas in Section 6. The following theorems state some convergence properties for ξ_k as generated by (2.6). The first one corresponds to the easiest case where α_k is bounded. The proof of Theorem 2.3 is given in Section 6.

THEOREM 2.2. *Assume that* A1–A2 *are satisfied and that* $\alpha_k < \alpha$. *Then, the sequence* $\{\mathbf{x}_k\}$ *generated by* (2.6) *is such that* $\eta(\bar{\theta}, \mathbf{x}_k) \to \eta(\bar{\theta}, \mathbf{x}^*)$ *when* $k \to \infty$. *If, moreover,* A6 *is satisfied, then* $\mathbf{x}_k \to \mathbf{x}^*$.

PROOF. We have $\alpha_k d_k(\mathbf{x}_{k+1})/k < \alpha d_k(\mathbf{x}_{k+1})/k$ which tends to zero; see Pázman (1974). Therefore, $\forall \varepsilon > 0$, $\exists K_1(\varepsilon)$ such that $\forall k > K_1$, $\alpha_k d_k(\mathbf{x}_{k+1})/k < \varepsilon$, which gives $\eta(\bar{\theta}, \mathbf{x}_{k+1}) > \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon$. When A6 is satisfied, this implies $\|\mathbf{x}_{k+1} - \mathbf{x}^*\| < \beta$ for any β and any k larger than some $K_2(\beta)$. □

THEOREM 2.3. *Assume that* A1–A2 *are satisfied and that* $\alpha_k/k \searrow 0$, $\alpha_k \nearrow \infty$. *Then, the sequence* $\{\mathbf{x}_k\}$ *generated by* (2.6) *is such that* $F(\xi_k) \to \eta(\bar{\theta}, \mathbf{x}^*)$ *when* $k \to \infty$. *If, moreover,* A6 *is satisfied, then* $\{\xi_k\} \xrightarrow{w} \xi_{\mathbf{x}^*}$ (*in the sense of weak convergence of measures*), *with* $\xi_{\mathbf{x}}$ *the discrete measure that puts weight* 1 *at the point* $\mathbf{x}$.

3. Adaptive optimization and design. We consider now the case where $\bar{\theta}$ is unknown, and estimated by LS. At step k, $\mathbf{x}_{k+1}$ is given by

$$\mathbf{x}_{k+1} = \arg\max_{\mathbf{x} \in \mathscr{X}} \eta(\hat{\theta}^k, \mathbf{x}) + \frac{\alpha_k}{k} d_k(\mathbf{x}), \qquad k \geq K_0, \tag{3.10}$$

with $d_k(\mathbf{x})$ given by (2.7). From Corollary 3 of Lai and Wei (1982), a sufficient condition for strong consistency of $\hat{\theta}^k$, under A5, is

$$\begin{aligned} &\lambda_{\min}(\mathbf{M}_k) \to \infty, \\ &[\log \lambda_{\max}(\mathbf{M}_k)]^{1+\delta} = o[\lambda_{\min}(\mathbf{M}_k)] \quad \text{for some } \delta > 0. \end{aligned} \tag{3.11}$$

From Lemma 3 in Section 6 and (A.1), the condition $\lambda_{\min}(\mathbf{M}_k) \to \infty$ is satisfied when $\alpha_k \to \infty$. Note that when A5 is strengthened into

$$\sup_k E\{|\varepsilon_k|^\gamma | \mathscr{F}_{k-1}\} < \infty \quad \text{almost surely, for some } \gamma > 2,$$

it is enough to take $\delta = 0$ in (3.11); see Theorem 1 in Lai and Wei (1982). This property is used in Section 6 to prove the following theorems.

THEOREM 3.1. *Assume that* A1–A5 *are satisfied and that* $\alpha_k = k\alpha$. *The sequence* $\{\mathbf{x}_k\}$ *generated by* (3.10) *is then such that* $\hat{\theta}^k \to \bar{\theta}$ *and* $H(\xi_k) \to H(\xi^*) = \max_{\xi \in \Xi(\mathscr{X})} H(\xi)$ *almost surely when* $k \to \infty$, *with* $H(\xi)$ *defined by* (2.8).

THEOREM 3.2. *Assume that* $(\alpha_k/k)\log \alpha_k \searrow$ *and* $\alpha_k/(\log k)^{1+\delta} \nearrow \infty$ *for some* $\delta > 0$. *Then, under* A1–A5, *the sequence* $\{\mathbf{x}_k\}$ *generated by* (3.10) *is such that* $\hat{\theta}^k \to \bar{\theta}$ *and* $F(\xi_k) \to \eta(\bar{\theta}, \mathbf{x}^*)$ *almost surely as* $k \to \infty$, *with* $F(\xi)$ *defined by* (2.9). *If, moreover,* A6 *is satisfied, then* $\{\xi_k\} \xrightarrow{w} \xi_{\mathbf{x}^*}$ *almost surely* (*in the sense of weak convergence of measures*).

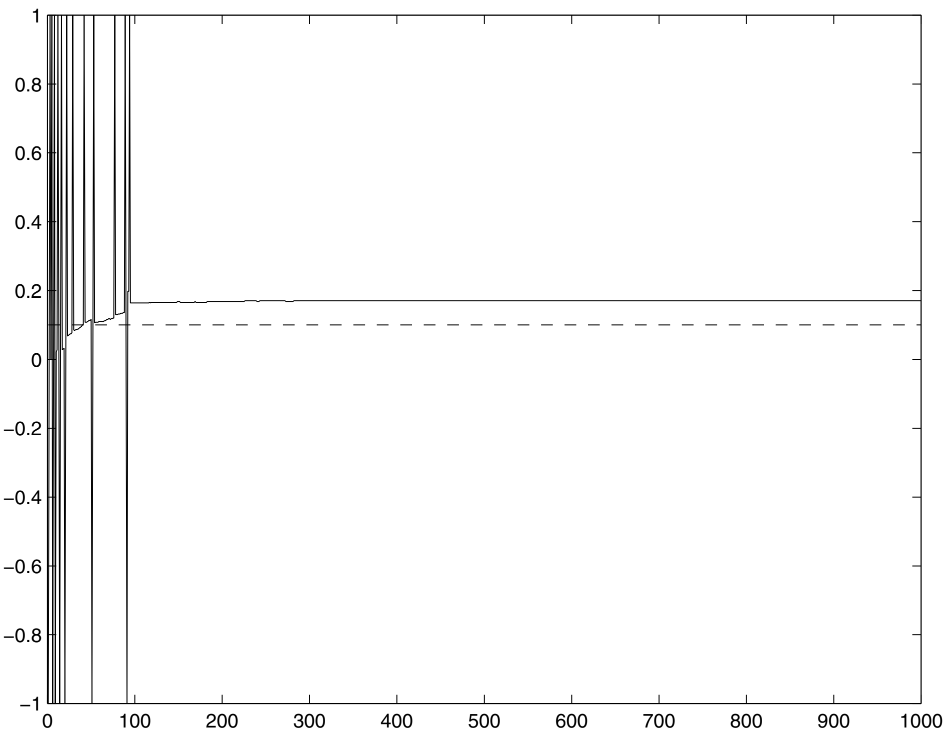

FIG. 1. *Evolution of x_k as a function of k when $\alpha_k = 1.5$.*

Note that taking a penalty function of the form $d_k(\mathbf{x}) = \lambda \det \mathbf{M}_{k+1} / \det \mathbf{M}_k$ [see Aström and Wittenmark (1989)] corresponds to taking $\alpha_k = \alpha$ constant in (3.10), which does not guarantee that $\lambda_{\min}(\mathbf{M}_k) \to \infty$. The example in the next section illustrates the convergence problems caused by this insufficient penalization for poor estimation. Also note that for the approach suggested in Alster and Blanger (1974), which corresponds to modifying the certainty-equivalence control law only when trace, $\mathbf{M}_k$ is smaller than some predefined threshold, $\lambda_{\min}(\mathbf{M}_k) \to \infty$ is not necessarily satisfied.

4. Example. We consider a self-tuning extremum control problem, with a scalar input $x_k \in \mathscr{X} = [-1, 1]$ and a quadratic response $\eta(\theta, x) = \mathbf{f}^\top(x)\theta = \theta_0 + \theta_1 x + \theta_2 x^2$. We take $\bar{\theta} = (0, 0.04, -0.2)^\top$, so that the maximum response for $\bar{\theta}$ is reached at $x^* = 0.1$. The measurement errors ε_k are i.i.d. $\mathscr{N}(0, \sigma^2)$ with $\sigma = 0.1$. The first three experiments are fixed: $x_1 = -1, x_2 = 0, x_3 = 1$, so that $K_0 = 3$ in A2.

We consider three different choices for $\{\alpha_k\}$, namely:

(i) $\alpha_k^{(i)} = 1.5$;

(ii) $\alpha_k^{(ii)} = 0.05(\log k)^2$;

(iii): $\alpha_k^{(iii)} = 0.007k$;

so that $\alpha_k^{(i)} > \alpha_k^{(ii)} > \alpha_k^{(iii)}$ for $k < 200$.

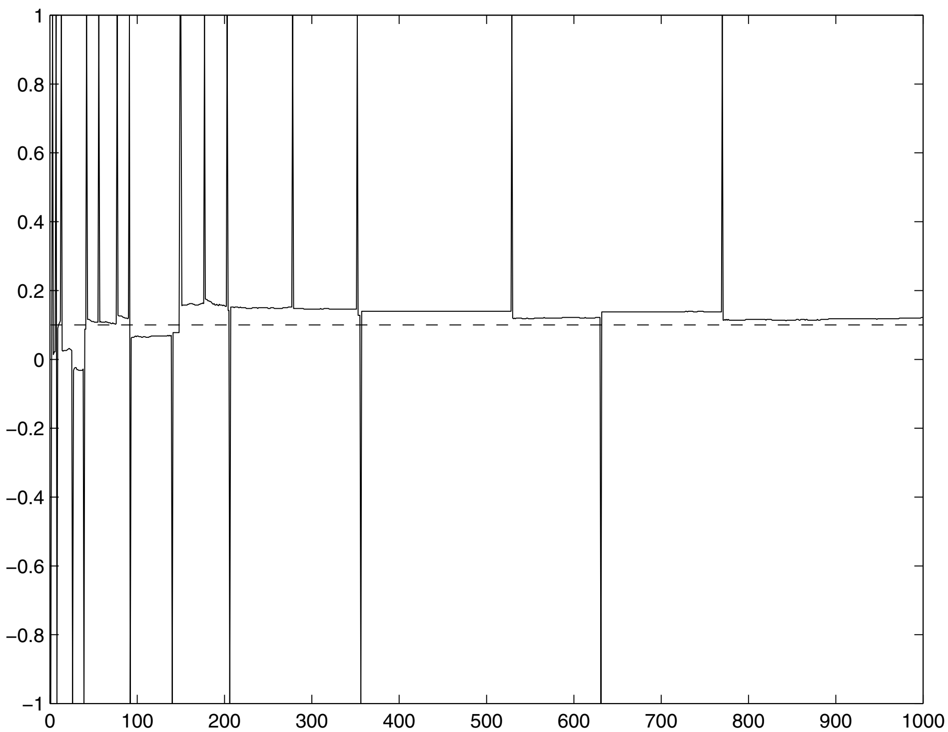

FIG. 2. *Evolution of x_k as a function of k when* $\alpha_k = 0.05(\log k)^2$.

Figures 1 and 2 respectively present the evolution of x_k as a function of k in cases (i) and (ii). The value of x^* is indicated by a dashed line. In case (iii), x_k fluctuates a lot between -1 and 1, and a histogram is presented in Figure 3.

We see in Figure 1 that ξ_k does not converge to ξ_{x^*}, although Theorem 2.2 applies: the excitation is not sufficient to estimate the model parameters and the corresponding value of x^* correctly. Figure 3 shows that the design measure ξ_k makes a compromise between a D-optimal measure, with support points $-1, 0, 1$, and the discrete measure ξ_{x^*} with a unique support point at x^* [see Theorems 2.1 and 3.1]: the excitation is too important for ξ_k to converge to ξ_{x^*}. In Figure 2, the excitation is sufficient to estimate the parameters correctly and ξ_k converges to ξ_{x^*}; see Theorem 3.2.

Figures 4 to 6 respectively present $\alpha_k d_k(x_{k+1})/k$ (A, left) and $\det \mathbf{I}_k$ (B, right) as functions of k, in cases (i), (ii) and (iii). Note that the penalization for poor estimation decreases faster in Figure 4 than in Figures 5 and 6. Also note that $\det \mathbf{I}_k$ converges to a positive value on Figure 6; see Theorem 2.1. The value of $\det \mathbf{I}(\xi_D)$, with ξ_D the D-optimal measure with support points $-1, 0, 1$ receiving weights 1/3 each, is $\det \mathbf{I}(\xi_D) = 4/9 \simeq 0.1481$.

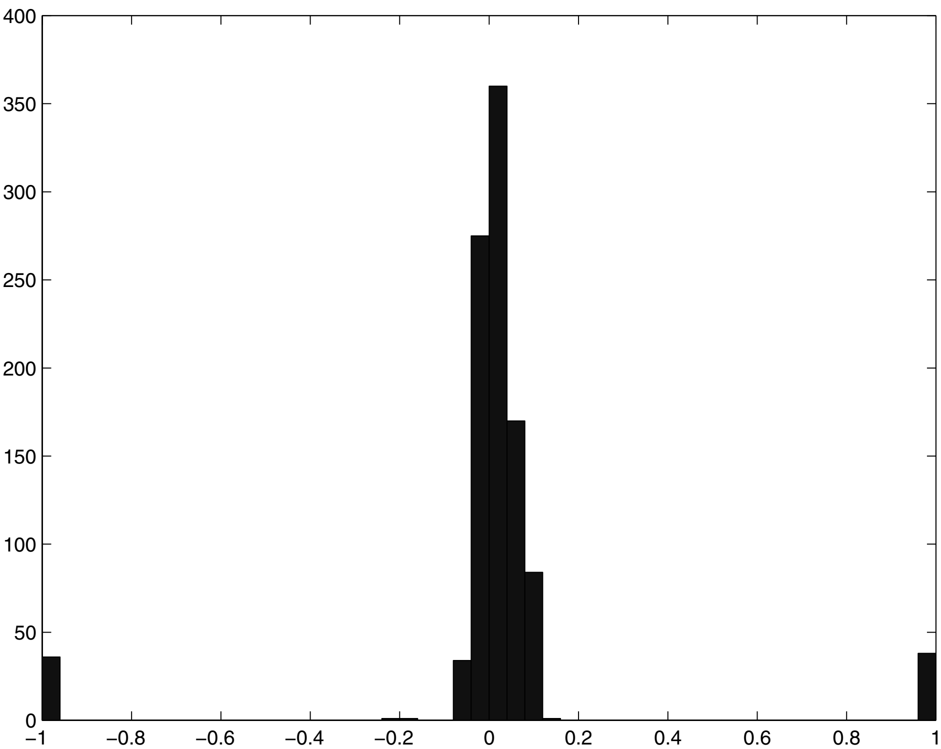

FIG. 3. *Histogram of* $\{x_k\}$ *when* $\alpha_k = 0.007k$ $(1 \leq k \leq 1000)$.

5. Conclusions and further developments. The joint problem of optimization and parameter estimation considered in this paper finds application in many different areas. Using a penalization for poor estimation, we have given conditions on the sequence of weights put on the penalizing term that simultaneously guarantee strong consistency of the parameter estimates and almost sure convergence of the empirical mean of the response to its maximum value; see (1.2). Further work might concern the characterization of the speed of convergence, in order to choose a suitable sequence of weights for a particular problem and a given time horizon. Also, the design space $\mathscr{X}$ was assumed here to be fixed. In practice, it might be useful to take $\mathbf{x}_{k+1}$ in a compact set $\mathscr{X}_k$ centered on $\mathbf{x}_k$ in order to avoid large excursions for successive values of $\mathbf{x}_k$. Finally, other penalizing functions than the one considered here, see (2.7), might be of interest. For instance, a possible choice might be $d_k(\mathbf{x}) = (\mathbf{f}^\top(\mathbf{x})\mathbf{q}_k)^2/\lambda_{\min}(\mathbf{I}_k)$, with $\|\mathbf{q}_k\| = 1$ and $\mathbf{q}_k$ an eigenvector of $\mathbf{I}_k$ associated with its minimum eigenvalue, which is related to E-optimum design [see, e.g., Silvey (1980), Pukelsheim (1993)]. Also, since the interest is in estimating $\mathbf{x}^*$, which is a function of $\bar{\theta}$, and not $\bar{\theta}$ itself, a penalty function related to c-optimality might be useful. A related idea is sometimes used in adap-

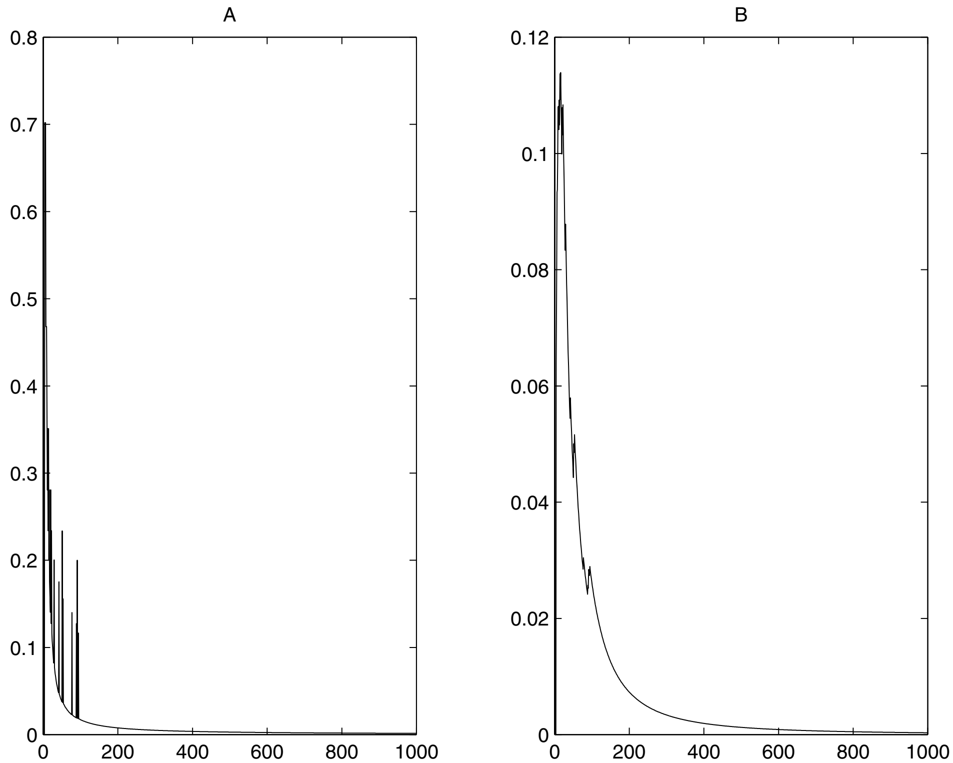

FIG. 4. *Evolution of* $\alpha_k d_k(x_{k+1})/k$ (A) *and* $\det \mathbf{I}_k$ (B) *when* $\alpha_k = 1.5$.

tive control [see, e.g., Wittenmark and Elevitch (1985)] where the penalty is related to the decrease of the variance of a particular component of θ.

6. Proofs.

PROOF OF THEOREM 2.1. Consider the first and second order directional derivatives

$$\nabla H(\xi, \xi') = \frac{\partial H}{\partial \gamma}[(1-\gamma)\xi + \gamma\xi']_{|\gamma=0^+}, \quad \nabla^2 H(\xi, \xi') = \frac{\partial^2 H}{\partial \gamma^2}[(1-\gamma)\xi + \gamma\xi']_{|\gamma=0^+}.$$

For ξ such that $\mathbf{I}(\xi)$ is nonsingular, one has

$$\nabla H(\xi, \xi') = \alpha \operatorname{trace}\{\mathbf{I}^{-1}(\xi)[\mathbf{I}(\xi') - \mathbf{I}(\xi)]\} + F(\xi') - F(\xi), \tag{6.1}$$

$$\nabla^2 H(\xi, \xi') = -\alpha \operatorname{trace}\{\mathbf{I}^{-1}(\xi)[\mathbf{I}(\xi') - \mathbf{I}(\xi)]\mathbf{I}^{-1}(\xi)[\mathbf{I}(\xi') - \mathbf{I}(\xi)]\}, \tag{6.2}$$

and $\nabla H(\xi_k, \xi')$ is maximum for $\xi' = \xi_{\mathbf{x}_{k+1}}$, the design measure putting mass 1 at the support point $\mathbf{x}_{k+1}$ given by (2.6); that is, $\mathbf{x}_{k+1} = \arg\max_{\mathbf{x}\in\mathscr{X}} \nabla H(\xi_k, \xi_{\mathbf{x}})$, with $\nabla H(\xi, \xi_{\mathbf{x}}) = \alpha[\mathbf{f}^\top(\mathbf{x})\mathbf{I}^{-1}(\xi)\mathbf{f}(\mathbf{x}) - p] + \eta(\bar{\theta}, \mathbf{x}) - F(\xi)$. One can then apply the dichotomous Theorem of Wu and Wynn (1978), which gives (i) $H(\xi_k) \to$

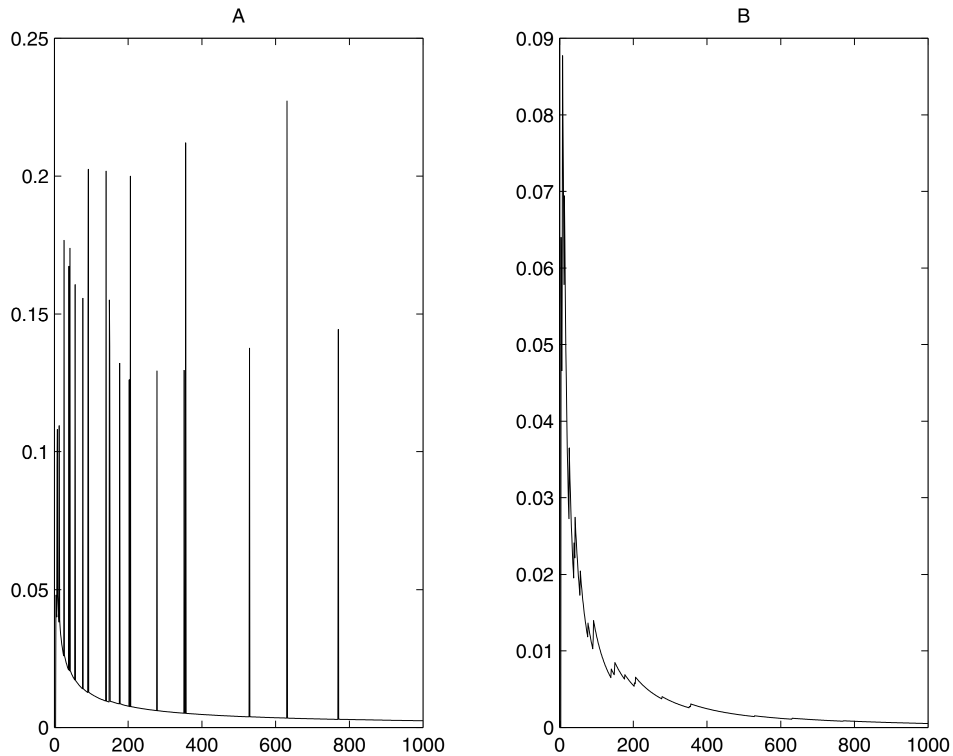

FIG. 5. *Evolution of* $\alpha_k d_k(x_{k+1})/k$ (A) *and* $\det \mathbf{I}_k$ (B) *when* $\alpha_k = 0.05(\log k)^2$.

$H(\xi^*)$ or (ii) there exists a subsequence ξ_{k_s} with $H(\xi_{k_s}) \to -\infty$. Condition C6 of the same paper can be used to eliminate the unboundedness situation (ii). All that needs to be proved is that there exist M and D such that

$$H(\xi) < M \Rightarrow D[\nabla H(\xi, \xi_{\bar{\mathbf{x}}})]^2 > -\nabla^2 H(\xi, \xi_{\bar{\mathbf{x}}}), \tag{6.3}$$

where, from (6.2), $\nabla^2 H(\xi, \xi_{\mathbf{x}}) = \alpha(1 - [\mathbf{f}^\top(\mathbf{x})\mathbf{I}^{-1}(\xi)\mathbf{f}(\mathbf{x}) - 1]^2 - p)$, and $\bar{\mathbf{x}} = \arg\max_{\mathbf{x}\in\mathscr{X}} \nabla H(\xi, \xi_{\mathbf{x}})$. From A1, $H(\xi) < M$ implies $\log\det \mathbf{I}(\xi) < (M + B)/\alpha$, see (A.4), and thus $\lambda_{\min}[\mathbf{I}(\xi)] < \exp[(M + B)/(p\alpha)]$. The inequality (A.1) then gives $\max_{\mathbf{x}\in\mathscr{X}} \mathbf{f}^\top(\mathbf{x})\mathbf{I}^{-1}(\xi)\mathbf{f}(\mathbf{x}) > \rho^2 \exp[-(M + B)/(p\alpha)]$. Using A1 again, we get $\mathbf{f}^\top(\bar{\mathbf{x}})\mathbf{I}^{-1}(\xi)\mathbf{f}(\bar{\mathbf{x}}) + \eta(\bar{\theta}, \bar{\mathbf{x}})/\alpha > \rho^2 \exp[-(M + B)/(p\alpha)] - B/\alpha$, and thus $\mathbf{f}^\top(\bar{\mathbf{x}})\mathbf{I}^{-1}(\xi)\mathbf{f}(\bar{\mathbf{x}}) > \{\rho^2 \exp[-(M+B)/(p\alpha)] - 2B/\alpha\} \to \infty$ when $M \to -\infty$. The condition (6.3) can thus always be fulfilled by choosing D and M respectively large and small enough. □

LEMMA 1. *Assume that* A1–A2 *are satisfied and that* $\{\alpha_k/k\}$ *is a nonincreasing sequence. Then, the sequence generated by* (2.6) *is such that*

$$\exists\, \Gamma > 0 \quad \text{such that } \forall k \geq K_0,\ \rho_k = \left(\frac{k}{\alpha_k}\right)^p \det \mathbf{I}_k > \Gamma, \tag{6.4}$$

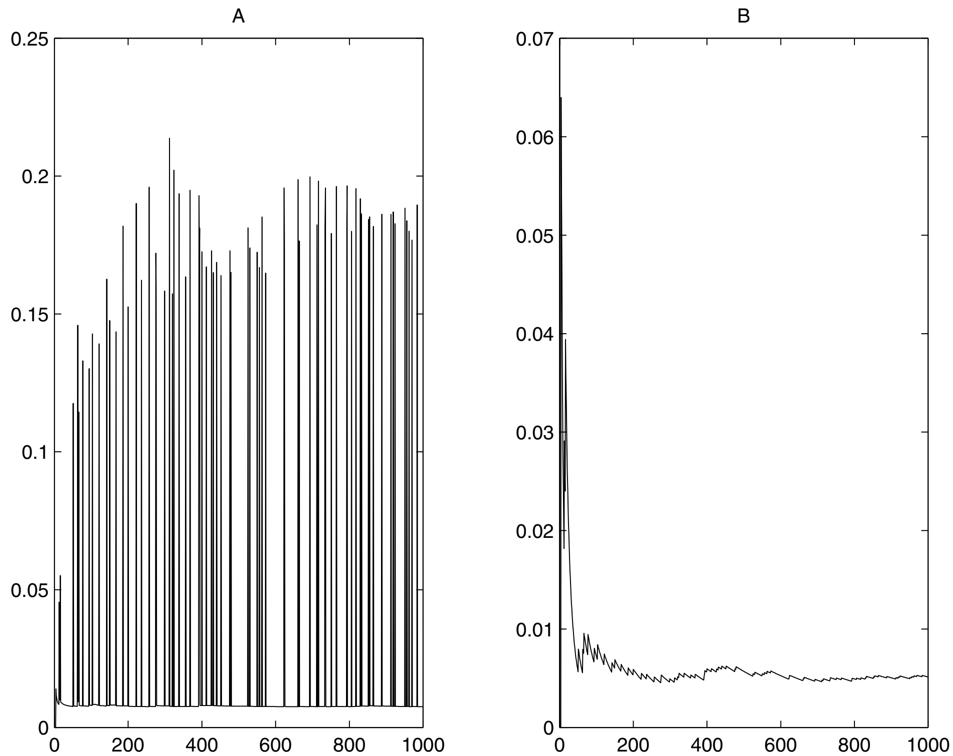

FIG. 6. *Evolution of* $\alpha_k d_k(x_{k+1})/k$ (A) *and* $\det \mathbf{I}_k$ (B) *when* $\alpha_k = 0.007k$.

where K_0 *is defined in* A2. *If, moreover,* A3 *and* A4 *are satisfied, the same is true for the sequence generated by* (3.10).

PROOF. First, we use the identity

$$\det \mathbf{I}_{k+1} = \left(\frac{k}{k+1}\right)^p \left[1 + \frac{d_k(\mathbf{x}_{k+1})}{k}\right] \det \mathbf{I}_k \tag{6.5}$$

which implies $\forall k$, $\rho_{k+1} > \rho_k \alpha_k^p / \alpha_{k+1}^p \geq \rho_k k^p/(k+1)^p$, and thus, $\forall \varepsilon$, $\exists K_1$ such that $\forall k > K_1$, $\rho_{k+1} > (1-\varepsilon)\rho_k$.

Now, for $k > K_1$, we define $A_k = (1-\varepsilon)^{k-K_1}\rho_{K_1} < \rho_k$ and $M(A) = (\rho^2/A^{1/p} - 2B)$, so that $M(A) \to \infty$ when $A \to 0$. Take $K_2 \geq K_1$ such that $M(A_{K_2}) > 2p\alpha_{K_1}/K_1$ and $[(k+1)/k]^p < 1 + 2p/k$ for any $k > K_2$. We show in the rest of the proof that this implies $\rho_k > (1-\varepsilon)A_{K_2}$ for any $k > K_2$.

The proof is by induction on k. It is true for $k = K_2 + 1$. Assume that it is true for k, and consider the two cases $\rho_k > A_{K_2}$, $A_{K_2} \geq \rho_k > (1 - \varepsilon)A_{K_2}$. The first one gives $\rho_{k+1} > (1-\varepsilon)A_{K_2}$, and only the second has to be considered. Inequality (A.1) implies $\max_{\mathbf{x}\in\mathscr{X}} d_k(\mathbf{x}) > k\rho^2/(\alpha_k A_{K_2}^{1/p})$, which gives $d_k(\mathbf{x}_{k+1}) > kM(A_{K_2})/\alpha_k$ when (2.6) is used, see (A.4), or when (3.10) is used, see (A.5). Then, (6.5) gives

$$\begin{aligned}\rho_{k+1}-\rho_k &> \rho_k\left(\frac{\alpha_k}{\alpha_{k+1}}\right)^p\left[1+\frac{M(A_{K_2})}{\alpha_k}\right]-\rho_k\\ &\ge \rho_k\frac{\alpha_k^{p-1}}{\alpha_{k+1}^p}\left\{M(A_{K_2})-\left[\left(\frac{(k+1)}{k}\right)^p-1\right]\alpha_k\right\}\\ &> \rho_k\frac{\alpha_k^{p-1}}{\alpha_{k+1}^p}\left[M(A_{K_2})-\frac{2p\alpha_{K_1}}{K_1}\right]>0,\end{aligned}$$

so that $\rho_{k+1} > \rho_k > (1-\varepsilon)A_{K_2}$. □

LEMMA 2. *Assume that* A1–A2 *are satisfied*, *that* $\{\alpha_k/k\}$ *is nonincreasing*, $\alpha_k \nearrow \infty$, *and that* $(\alpha_k/k)\log\alpha_k \to 0$. *Then*, *the sequence generated by* (2.6) *satisfies*

$$\exists \underline{D} \quad \text{such that } \forall k \ge K_0,\ \frac{\alpha_k}{k}\max_{\mathbf{x}\in\mathscr{X}} d_k(\mathbf{x}) < \underline{D},$$

where K_0 *is defined in* A2. *If*, *moreover*, A3 *and* A4 *are satisfied*, *the same is true for the sequence generated by* (3.10).

PROOF. Define $\bar{d}_k = \max_{\mathbf{x}\in\mathscr{X}} d_k(\mathbf{x})$. We show first that $\alpha_i\bar{d}_i/i > D > 2B$ for $i = k, k+1, \dots, k+n_k$ and k larger than some K_1, implies $n_k < k$. Then we show by contradiction that $\alpha_k\bar{d}_k/k$ is bounded for all k larger than $2K_1$.

When (2.6) is used, A1 implies $\alpha_k\bar{d}_k/k \le \alpha_k d_k(\mathbf{x}_{k+1})/k + 2B$, see (A.4), and thus $\alpha_i d_i(\mathbf{x}_{i+1})/i > (1-\varepsilon)D$ for $i = k, k+1, \dots, k+n_k$, where $\varepsilon = 2B/D < 1$. The same is true under A3 and A4 when (3.10) is used, see (A.5). Using (6.5), $\alpha_k \nearrow$ and Lemma 1, we get

$$\begin{aligned}\det\mathbf{I}_{k+n_k+1} &> \left(\frac{k}{k+n_k+1}\right)^p \det\mathbf{I}_k\prod_{i=k}^{k+n_k}\left[1+\frac{(1-\varepsilon)D}{\alpha_i}\right]\\ &> \Gamma\left(\frac{k}{k+n_k+1}\right)^p\left[1+\frac{(1-\varepsilon)D}{\alpha_{k+n_k}}\right]^{n_k+1}\left(\frac{\alpha_k}{k}\right)^p.\end{aligned}$$

Therefore, since $\alpha_k \nearrow$, for k large enough

$$\begin{aligned}\log\det\mathbf{I}_{k+n_k+1} &> \log\Gamma + p\log\left(\frac{\alpha_k}{\alpha_{k+n_k}}\right)+p\log\left(\frac{\alpha_{k+n_k}}{k+n_k}\right)+p\log(1/2)\\ &\quad+(n_k+1)\frac{(1-\varepsilon)D}{2\alpha_{k+n_k}}.\end{aligned}\tag{6.6}$$

and also, from A1, $\log\det\mathbf{I}_{k+n_k+1} < \log\bar{D}$ for some $\bar{D}$. This gives

$$\frac{n_k(1-\varepsilon)D}{2(k+n_k)} < \frac{\alpha_{k+n_k}}{k+n_k}\left[\log\frac{\bar{D}}{\Gamma}-p\log\frac{\alpha_{k+n_k}}{k+n_k}+p\log\alpha_{k+n_k}+p\log 2\right]$$

and, since $(\alpha_k/k)\log\alpha_k \to 0$ and $\alpha_k \nearrow \infty$, $\exists K_1$ such that $\forall k \ge K_1$, $n_k(1-\varepsilon)D/[2(k+n_k)] < (1-\varepsilon)D/4$, that is, $n_k < k$.

We show now that $\forall n > 2K_1$, $\alpha_n \bar{d}_n/n \leq 2D/[1 - 1/(2K_1)]$. Assume that $\alpha_n \bar{d}_n/n > 2D/[1 - 1/(2K_1)]$ for some $n > 2K_1$. Direct calculation using the definition (2.7) of $d_k(\mathbf{x})$ gives $d_{k+1}(\mathbf{x}) \leq [(k+1)/k]d_k(\mathbf{x})$ for any $\mathbf{x}$, and thus, since $\alpha_k/k \searrow$, $\alpha_{k+1}\bar{d}_{k+1}/(k+1) \leq (\alpha_{k+1}/\alpha_k)\alpha_k \bar{d}_k/k \leq [(k+1)/k]\alpha_k \bar{d}_k/k$. Therefore, $\alpha_m \bar{d}_m/m > (m/n)2D/[1 - 1/(2K_1)] > D$ for $m = \lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1, \dots, n$ which contradicts the first part of the lemma. □

LEMMA 3. *Assume that* A1–A2 *are satisfied and that* $\alpha_k \to \infty$. *Then, the sequence generated by* (2.6) *is such that* $\max_{\mathbf{x} \in \mathscr{X}} d_k(\mathbf{x})/k \to 0$. *If, moreover,* A3 *and* A4 *are satisfied, the property is also valid for the sequence generated by* (3.10).

PROOF. From Pázman (1974) [see also Proposition V.5 in Pázman (1986)], $d_k(\mathbf{x}_{k+1})/k \to 0$. When (2.6) is used, A1 implies that, for $k > K_0$, $\max_{\mathbf{x} \in \mathscr{X}} d_k(\mathbf{x})/k < d_k(\mathbf{x}_{k+1})/k + 2B/\alpha_k$, which tends to zero when $k \to \infty$. The same is true under A3 and A4 when (3.10) is used. □

PROOF OF THEOREM 2.3. We use notation similar to Section 2, and define the objective $H_k(\xi) = (\alpha_k/k) \log \det \mathbf{I}(\xi) + F(\xi)$. Its first and second-order directional derivatives at ξ_k in the direction of $\xi_{\mathbf{x}}$ are given by $\nabla H_k(\xi_k, \xi_{\mathbf{x}}) = (\alpha_k/k)(d_k(\mathbf{x}) - p) + \eta(\bar{\theta}, \mathbf{x}) - F(\xi_k)$, $\nabla^2 H_k(\xi_k, \xi_{\mathbf{x}}) = (\alpha_k/k)(1 - [d_k(\mathbf{x}) - 1]^2 - p)$. In a first part (i), we give a lower bound on the increase $H_k(\xi_{k+1}) - H_k(\xi_k)$. In part (ii), we show that there exists an infinite subsequence $\{\xi_{k_s}\}$ such that $F(\xi_{k_s}) \to \eta(\bar{\theta}, \mathbf{x}^*)$. In part (iii) we show that $F(\xi_k) \to \eta(\bar{\theta}, \mathbf{x}^*)$. Finally, in part (iv) we show that A6 implies $\{\xi_k\} \xrightarrow{w} \xi_{\mathbf{x}^*}$.

(i) We can write

$$H_k(\xi_{k+1}) = H_k(\xi_k) + \frac{1}{k+1} \nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}}) + \frac{1}{2(k+1)^2} \nabla^2 H_k(\alpha),$$

for some $\alpha \in [0, 1/(k+1)]$, where

$$\begin{aligned} \nabla^2 H_k(\alpha) &= \nabla^2 H_k[(1-\alpha)\xi_k + \alpha \xi_{\mathbf{x}_{k+1}}, \xi_{\mathbf{x}_{k+1}}] \\ &= \frac{\alpha_k}{k} \left(1 - \left[\frac{d_k(\mathbf{x}_{k+1})}{\alpha d_k(\mathbf{x}_{k+1}) + (1-\alpha)} - 1 \right]^2 - p \right), \end{aligned}$$

so that

$$\nabla^2 H_k(\alpha) \geq \nabla^2 H_k(0) \geq \frac{\alpha_k}{k}(1 - [d_k(\mathbf{x}_{k+1}) - 1]^2 - p). \tag{6.7}$$

Define $\mathscr{S}_k(A) = \{\xi \in \Xi(\mathscr{X}) : (\alpha_k/k)d_k(\mathbf{x}_{k+1}) \leq A\}$. One has from (6.7): $\exists K_1$ such that $\forall k > K_1$, $\xi_k \in \mathscr{S}_k(A) \Rightarrow \nabla^2 H_k(\alpha) \geq -A^2 k/\alpha_k$, and thus

$$H_k(\xi_{k+1}) \geq H_k(\xi_k) + \frac{1}{k+1} \nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}}) - \frac{A^2}{k\alpha_k}. \tag{6.8}$$

Consider now the case where $\xi_k \notin \mathscr{S}_k(A)$. Define $\nabla H'_k = (k/\alpha_k)\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}})$, $\nabla^2 H'_k = (k/\alpha_k)\nabla^2 H_k(\xi_k, \xi_{\mathbf{x}_{k+1}})$. For A large enough, A1 implies $\exists K_2$ such that

$\forall k > K_2$, $\nabla H'_k > d_k(\mathbf{x}_{k+1})/2$ and $\nabla^2 H'_k > -2d_k^2(\mathbf{x}_{k+1})$. Therefore, $\forall k > K_2$, $\xi_k \notin \mathscr{S}_k(A)$ implies

$$H_k(\xi_{k+1}) \geq H_k(\xi_k) + \frac{1}{k+1}\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}})\left[1 - \frac{4}{k+1}, \frac{k}{\alpha_k}\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}})\right].$$

Now, from Lemma 3 and A1, and since $\alpha_k \to \infty$, $\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}})/\alpha_k$ tends to zero when $k \to \infty$. Therefore, we have the following: $\exists K_3$ such that $\forall k > K_3$, (6.8) is satisfied if $\xi_k \in \mathscr{S}_k(A)$ and

$$H_k(\xi_{k+1}) \geq H_k(\xi_k) + \frac{1}{2(k+1)}\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}}) \tag{6.9}$$

otherwise.

(ii) Assume now that $\exists \varepsilon$ such that $\forall k$, $F(\xi_k) < \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon$. This implies $\nabla H_k(\xi_k, \xi_{\mathbf{x}^*}) > \varepsilon - p\alpha_k/k$, and thus $\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}}) \geq \nabla H_k(\xi_k, \xi_{\mathbf{x}^*}) > \varepsilon - p\alpha_k/k$. Since $\alpha_k/k \to 0$,

$$\exists K_4 \text{ such that } \forall k > K_4, \quad \nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}}) > \varepsilon/2. \tag{6.10}$$

Inequalities (6.8) and (6.9) then give

$$H_k(\xi_{k+1}) \geq H_k(\xi_k) + \frac{\varepsilon}{2(k+1)} - \frac{A^2}{k\alpha_k} \qquad \text{for } \xi_k \in \mathscr{S}_k(A),$$
$$H_k(\xi_{k+1}) \geq H_k(\xi_k) + \frac{\varepsilon}{4(k+1)} \qquad \text{for } \xi_k \notin \mathscr{S}_k(A).$$

Therefore, since $\alpha_k \to \infty$, $\exists K_5$ such that $\forall k > K_5$ and $\forall \xi_k$,

$$H_k(\xi_{k+1}) \geq H_k(\xi_k) + \frac{\varepsilon}{4(k+1)}. \tag{6.11}$$

Consider now $H_{k+1}(\xi_{k+1}) = H_k(\xi_{k+1}) + [\alpha_{k+1}/(k+1) - \alpha_k/k] \log\det \mathbf{I}_{k+1}$. There are two cases. If $\log\det \mathbf{I}_{k+1} \leq 0$, then, since $\alpha_k/k \searrow$, $H_{k+1}(\xi_{k+1}) \geq H_k(\xi_{k+1})$. Otherwise, since $\alpha_k/k \searrow 0$ and $\det \mathbf{I}_k$ is bounded: $\exists K_6$ such that $\forall k > K_6$, $(\alpha_k/k)\log\det \mathbf{I}_{k+1} < \varepsilon/8$. Also, since $\alpha_k \nearrow$,

$$\left(\frac{\alpha_{k+1}}{k+1} - \frac{\alpha_k}{k}\right)\log\det \mathbf{I}_{k+1} > -\frac{\alpha_k}{k(k+1)}\log\det \mathbf{I}_{k+1} > -\frac{\varepsilon}{8(k+1)}.$$

Inequality (6.11) thus implies for $k > \max\{K_5, K_6\}$

$$H_{k+1}(\xi_{k+1}) \geq H_k(\xi_{k+1}) - \frac{\varepsilon}{8(k+1)} > H_k(\xi_k) + \frac{\varepsilon}{8(k+1)}. \tag{6.12}$$

This in turn implies that $H_k(\xi_k) \to \infty$, which is impossible from A1. Therefore, there exists an infinite subsequence $\{\xi_{k_s}\}$ such that $F(\xi_{k_s}) \to \eta(\bar{\theta}, \mathbf{x}^*)$.

(iii) We show now that $F(\xi_k) \to \eta(\bar{\theta}, \mathbf{x}^*)$. First note that $F(\xi_{k+1}) - F(\xi_k) = [\eta(\bar{\theta}, \xi_{k+1}) - F(\xi_k)]/(k+1)$, so that A1 implies

$$\text{(6.13)} \qquad \forall \varepsilon,\ \exists K_7 \quad \text{such that } \forall k > K_7,\ F(\xi_{k+1}) > F(\xi_k) - \frac{\varepsilon}{2}.$$

We use Lemma 1: $\det \mathbf{I}_k > \Gamma(\alpha_k/k)^p$ for k larger than some K, and thus $(\alpha_k/k)\log\det\mathbf{I}_k > (\alpha_k/k)\log\Gamma + p(\alpha_k/k)\log(\alpha_k/k)$. On the other hand, $\det\mathbf{I}_k$ is bounded and $\alpha_k/k \to 0$ gives

$$\text{(6.14)} \quad \forall \varepsilon,\ \exists K_8 \quad \text{such that } \forall k > K_8,\ H_k(\xi_k) - \frac{\varepsilon}{4} < F(\xi_k) < H_k(\xi_k) + \frac{\varepsilon}{4}.$$

Using part (ii) of the proof, we get: $\forall k > \max\{K_5, K_6\}$,

$$\text{(6.15)} \qquad [F(\xi_k) < \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon] \Rightarrow [H_{k+1}(\xi_{k+1}) > H_k(\xi_k)],$$

see (6.12). Take $k_s > \max\{K_5, \dots, K_8\}$ and such that $F(\xi_{k_s}) \geq \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon$. Consider the sequence $F(\xi_k)$ for $k \geq k_s$, and define $\mathscr{K}_\varepsilon = \{k \geq k_s : F(\xi_k) \geq \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon\}$. We show that for any two consecutive k_i, k_j in $\mathscr{K}_\varepsilon$, $k_j > k_i \geq k_s$, $F(\xi_k) > \eta(\bar{\theta}, \mathbf{x}^*) - 2\varepsilon$, $\forall k \in \{k_i, k_i + 1, \dots, k_j\}$. First, (6.13) gives $F(\xi_{k_i+1}) > \eta(\bar{\theta}, \mathbf{x}^*) - 3\varepsilon/2$. If $F(\xi_{k_i+1}) \geq \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon$, then $k_i + 1 = k_j \in \mathscr{K}_\varepsilon$. Otherwise, $F(\xi_{k_i+1}) < \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon$, (6.15) implies $H_{k_i+2}(\xi_{k_i+2}) > H_{k_i+1}(\xi_{k_i+1})$, (6.14) implies $F(\xi_{k_i+2}) > H_{k_i+2}(\xi_{k_i+2}) - \varepsilon/4$ and $H_{k_i+1}(\xi_{k_i+1}) > F(\xi_{k_i+1}) - \varepsilon/4$, which give altogether $F(\xi_{k_i+2}) > \eta(\bar{\theta}, \mathbf{x}^*) - 2\varepsilon$. We can repeat the same arguments, and either $k_i + 2 \in \mathscr{K}_\varepsilon$ or $F(\xi_{k_i+2}) < \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon$. In the second case, $H_{k_i+3}(\xi_{k_i+3}) > H_{k_i+2}(\xi_{k_i+2}) > H_{k_i+1}(\xi_{k_i+1})$ and $F_{k_i+3}(\xi_{k_i+3}) > H_{k_i+1}(\xi_{k_i+1}) - \varepsilon/4 > F_{k_i+1}(\xi_{k_i+1}) - \varepsilon/2 > \eta(\bar{\theta}, \mathbf{x}^*) - 2\varepsilon$. By induction, $F(\xi_k) > \eta(\bar{\theta}, \mathbf{x}^*) - 2\varepsilon$, $\forall k \geq k_s$.

(iv) Define $\mathscr{C}_k(\beta) = \{\mathbf{x}_i \notin \mathscr{B}(\mathbf{x}^*, \beta), i = 1, \dots, k\}$, and assume that $\exists \beta > 0$ such that $\limsup_{k\to\infty} \#\mathscr{C}_k(\beta)/k > \gamma > 0$, where $\#\mathscr{C}$ denotes the number of elements of $\mathscr{C}$. From A6, this implies that there exists $\varepsilon > 0$ such that $\limsup_{k\to\infty}(1/k)\sum_{i=1}^{k}[\eta(\bar{\theta}, \mathbf{x}^*) - \eta(\bar{\theta}, \mathbf{x}_i)] > \varepsilon\gamma$, which contradicts the fact that $F(\xi_k) \to \eta(\bar{\theta}, \mathbf{x}^*)$. We thus have $\forall \beta > 0$, $\lim_{k\to\infty} \#\mathscr{C}_k(\beta)/k = 0$, and $\xi_k \xrightarrow{w} \xi_{\mathbf{x}^*}$. $\square$

PROOF OF THEOREM 3.1. We already showed that $\lambda_{\min}(\mathbf{M}_k) \to \infty$. Using (A.2), we get $\lambda_{\min}(\mathbf{M}_k)/[\log\lambda_{\max}(\mathbf{M}_k)]^{1+\delta} > \lambda_{\min}(\mathbf{M}_k)/(\log kL)^{1+\delta}$, and thus, for some $K_1 > 0$,

$$\text{(6.16)} \qquad \frac{\lambda_{\min}(\mathbf{M}_k)}{[\log\lambda_{\max}(\mathbf{M}_k)]^{1+\delta}} > \frac{\lambda_{\min}(\mathbf{M}_k)}{(2\log k)^{1+\delta}} \qquad \forall k > K_1.$$

Lemma 1 and (A.3) give

$$\text{(6.17)} \qquad \frac{\lambda_{\min}(\mathbf{M}_k)}{[\log\lambda_{\max}(\mathbf{M}_k)]^{1+\delta}} > \frac{\Gamma}{L^{p-1}}\ \frac{\alpha_k^p}{k^{p-1}(2\log k)^{1+\delta}}$$

for $k > \max\{K_0, K_1\}$, so that

$$\frac{\lambda_{\min}(\mathbf{M}_k)}{[\log\lambda_{\max}(\mathbf{M}_k)]^{1+\delta}} > \frac{\Gamma}{L^{p-1}}\ \alpha^{p-1}\ \frac{\alpha_k}{(2\log k)^{1+\delta}} \to \infty \quad \text{as } k \to \infty.$$

Therefore, from (3.11), $\hat{\theta}^k \to \bar{\theta}$ almost surely as $k \to \infty$.

Using the boundedness condition (A.5) and arguments similar to the proof of Theorem 2.1, we get $H(\xi_k) \to H(\xi^*)$ almost surely. □

PROOF OF THEOREM 3.2. As in Theorem 3.1, $\lambda_{\min}(\mathbf{M}_k) \to \infty$ and (6.16) is satisfied. Since $(\alpha_k/k)\log\alpha_k \searrow$ and $\alpha_k \to \infty$, $(\alpha_k/k)\log\alpha_k \searrow l \geq 0$. If $l = 0$, Lemma 2 applies, and (A.1) gives $\lambda_{\min}(\mathbf{M}_k) > \rho^2\alpha_k/\underline{D}$ for $k > K_0$. Therefore, for $k > \max\{K_0, K_1\}$

$$\frac{\lambda_{\min}(\mathbf{M}_k)}{[\log\lambda_{\max}(\mathbf{M}_k)]^{1+\delta}} > \frac{\rho^2}{\underline{D}}\,\frac{\alpha_k}{(2\log k)^{1+\delta}} \to \infty \quad \text{as } k \to \infty.$$

If $l > 0$, Lemma 1 and (A.3) give (6.17) for $k > \max\{K_0, K_1\}$. Inequality $(\alpha_k/k)\log\alpha_k \geq l$ then gives

$$\begin{aligned}\frac{\lambda_{\min}(\mathbf{M}_k)}{[\log\lambda_{\max}(\mathbf{M}_k)]^{1+\delta}} &> \frac{\Gamma l^{p-1}}{L^{p-1}}\,\frac{\alpha_k}{(\log\alpha_k)^{p-1}(2\log k)^{1+\delta}}\\ &> \frac{\Gamma l^{p-1}}{L^{p-1}}\,\frac{\alpha_k}{(\log\alpha_k)^{p-1}2^{1+\delta}(\log\alpha_k + \log\log\alpha_k - \log l)^{1+\delta}}\end{aligned}$$

which tends to infinity when $k \to \infty$. Condition (3.11) is thus satisfied for any $l \geq 0$ and $\hat{\theta}^k \to \bar{\theta}$ almost surely, $k \to \infty$.

The only modification required in the proof of Theorem 2.3, due to the fact that the sequence is generated now by (3.10) instead of (2.6), concerns the construction of a lower bound on $\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}})$ in (ii); see (6.10).

Assume that $\exists\varepsilon$ such that: $\forall k$, $F(\xi_k) < \eta(\bar{\theta}, \mathbf{x}^*) - \varepsilon$. It implies $\nabla H_k(\xi_k, \xi_{\mathbf{x}^*}) > \varepsilon - p\alpha_k/k$, and thus $\max_{\mathbf{x}\in\mathscr{X}} \nabla H_k(\xi_k, \xi_{\mathbf{x}}) > \varepsilon - p\alpha_k/k$. Strong consistency of $\hat{\theta}^k$, A3 and A4 then imply that there exists K such that $\forall k > K$, $|\max_{\mathbf{x}\in\mathscr{X}} \nabla H_k(\xi_k, \xi_{\mathbf{x}}) - \nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}})| < \varepsilon/4$. Now, since $\alpha_k/k \to 0$, $p\alpha_k/k < \varepsilon/4$ starting with some k and thus, $\exists K_4$ such that $\forall k > K_4$, $\nabla H_k(\xi_k, \xi_{\mathbf{x}_{k+1}}) > \varepsilon/2$, which coincides with (6.10). The rest of the proof is similar to that of Theorem 2.3. □

APPENDIX

CONSEQUENCES OF A1–A4. A2 implies that the components of $\mathbf{f}(\cdot)$ are independent, and thus, there exist $\rho > 0$ such that the ball $\mathscr{B}(\mathbf{0}, \rho)$ of radius ρ centered at the origin $\mathbf{0}$ is included in the convex closure of $\mathscr{G} \cup (-\mathscr{G})$, where $\mathscr{G} = \{\mathbf{f}(\mathbf{x}),\ \mathbf{x} \in \mathscr{X}\}$. Therefore, A1 and A2 imply that for any positive definite matrix $\mathbf{M}$,

$$\max_{\mathbf{x}\in\mathscr{X}} \mathbf{f}^\top(\mathbf{x})\mathbf{M}^{-1}\mathbf{f}(\mathbf{x}) \geq \frac{\rho^2}{\lambda_{\min}(\mathbf{M})}, \tag{A.1}$$

where $\lambda_{\min}(\mathbf{M})$ denotes the minimum eigenvalue of $\mathbf{M}$. A1 implies

$$\exists L > 0 \quad \text{such that } \forall k,\ \lambda_{\max}(\mathbf{I}_k) < L, \tag{A.2}$$

where $\mathbf{I}_k = \mathbf{M}_k/k$ and, for any $\mathbf{M}$, $\lambda_{\max}(\mathbf{M})$ denotes the maximum eigenvalue of $\mathbf{M}$. This in turn implies that

$$\lambda_{\min}(\mathbf{I}_k) \geq \frac{\det \mathbf{I}_k}{L^{p-1}}, \tag{A.3}$$

where the inequality is strict when $\mathbf{I}_k$ is positive definite, that is, when $k \geq K_0$ as defined in A2. A1 also implies

$$\exists B > 0 \quad \text{such that } \forall \mathbf{x} \in \mathscr{X},\ -B \leq \eta(\bar{\theta}, \mathbf{x}) \leq B. \tag{A.4}$$

A3 and A4 imply that the same property is true for any $\hat{\theta}^k$:

$$\exists B > 0 \quad \text{such that } \forall \hat{\theta}^k \in \Theta,\ \forall \mathbf{x} \in \mathscr{X},\ -B \leq \eta(\hat{\theta}^k, \mathbf{x}) \leq B. \tag{A.5}$$

REFERENCES

ALLISON, B. J., TESSIER, P. J. C. and DUMONT, G. A. (1995). Comparison of suboptimal dual adaptive controllers. In *Proceedings of the 3rd European Control Conference* 36–43. Rome.

ALSTER, J. and BLANGER, P. R. (1974). A technique for dual adaptive control. *Automatica* **10** 627–634.

ASTRÖM, K. J. and WITTENMARK, B. (1989). *Adaptive Control*. Addison-Wesley, Reading, MA.

BAR-SHALOM, Y. (1981). Stochastic dynamic programming: caution and probing. *IEEE Trans. Automatic Control* **26** 1184–1195.

BAR-SHALOM, Y. and TSE, E. (1974). Dual effect, certainty equivalence, and separation in stochastic control. *IEEE Trans. Automatic Control* **19** 494–500.

BOZIN, A. S. and ZARROP, M. B. (1991). Self tuning optimizer–convergence and robustness properties. In *Proceedings of the 1st European Control Conference* 672–677. Hermés, Paris.

FEL'DBAUM, A. A. (1960a). Dual control theory. I. *Avtomat. i Telemek.* **21** 1240–1249.

FEL'DBAUM, A. A. (1960b). Dual control theory. II. *Avtomat. i Telemek.* **21** 1453–1464.

FEL'DBAUM, A. A. (1961a). The theory of dual control. III. *Avtomat. i Telemek.* **22** 3–16.

FEL'DBAUM, A. A. (1961b). The theory of dual control. IV. *Avtomat. i Telemek.* **22** 129–142.

GINEBRA, J. and CLAYTON, M. K. (1995). Response surface bandits. *J. Roy. Statist. Soc. Ser. B* 771–784.

KULCSÁR, C., PRONZATO, L. and WALTER, E. (1996). Dual control of linearly parameterised models via prediction of posterior densities. *European J. Control* **2** 135–143.

LAI, T. L. and WEI, C. Z. (1982). Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems. *Ann. Statist.* **10** 154–166.

LAI, T. L. and WEI, C. Z. (1987). Asymptotically efficient self-tuning regulators. *SIAM J. Control Optim.* **25** 466–481.

LJUNG, L. (1977). Analysis of recursive stochastic algorithms. *IEEE Trans. Automat. Control* **22** 551–575.

PÁZMAN, A. (1974). A convergence theorem in the theory of *D*-optimum experimental designs. *Ann. Statist.* **2** 216–218.

PÁZMAN, A. (1986). *Foundations of Optimum Experimental Design*. Reidel, Dordrecht.

PRONZATO, L. and WALTER, E. (1993). Experimental design for estimating the optimum point in a response surface. *Acta Appl. Math.* **33** 45–68.

PUKELSHEIM, F. (1993). *Optimal Experimental Design*. Wiley, New York.

SILVEY, S. D. (1980). *Optimal Design*. Chapman & Hall, London.

WELLSTEAD, P. E. and ZARROP, M. B. (1991). *Self–Tuning Sytems*. Wiley, Chichester.

WITTENMARK, B. (1975). An active suboptimal controller for systems with stochastic parameters. *Automat. Control Theory Appl.* **3** 13–19.

WITTENMARK, B. and ELEVITCH, C. (1985). An adaptive control algorithm with dual features. In *Prep. 7th IFAC/IFORS Symposium on Identification and System Parameter Estimation* 587–592, York.

WU, C.-F. and WYNN, H. P. (1978). The convergence of general step–length algorithms for regular optimum design criteria. *Ann. Statist.* **6** 1273–1285.

WYNN, H.P. (1970). The sequential generation of D-optimum experimental designs. *Ann. Math. Statist.* **41** 1655–1664.

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