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BOOK REVIEW

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WILLIAM FELLER, *An Introduction to Probability Theory and its Applications* **2**, 2nd ed. John Wiley and Sons Inc., 1971. xxiv + 669 pp. $15.95.

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Volume 1 of this work has long been established as a classic in mathematical treatises and has gone through three quite different editions. It is elementary in the sense that it begins at the beginning and presupposes little background. [When it first came out it was much used as a textbook for a first course in probability; however, it is becoming increasingly difficult to do so as the audience has widened but its mathematical preparedness has not.] In fact, it deals only with a countable sample space so that all random variables and distributions are necessarily of the discrete type. This of course does not prevent Feller from going off at various deep ends and discussing some of the most up-to-date topics of interest. The restriction was deliberately imposed in order to waste no time on dull generalities (often indiscriminately referred to as "measure theory") and to proceed at once to significant results. It is clear that many basic concepts cannot even be defined in the restricted context, and any serious exposition of stochastic processes in continuous time would be out of the question. But the latter is precisely where the action is at these days. Feller himself made fundamental contributions in the nineteen-fifties to this area called by him "generalized diffusions" and by others "Feller processes" (precursor of Doob-Hunt-Meyer processes); albeit largely in an analytic form, without "paths," or rather with these only lurking in the background. It was therefore the general expectation that he would take this up in Volume 2 as a main theme. This would be a fitting sequel to his extremely successful popularization of Markov chain theory in Volume 1, acknowledged by himself in the Preface to the first edition. It must have come as a surprise to many that this is not what he did in Volume 2. Although there are recurrent and persistent allusions to bigger and better things to come (witness on page 333 "we are not at this juncture interested in developing a systematic theory"), general discussion of such processes is mostly relegated to heuristic descriptions, preparatory material and footnotes—not too copious. When asked why he chose not to expound his own theory of diffusion, Feller had responded by invoking the future promise of Volume 3, 4, $\cdots$. Alas, time has stopped too soon for him and his readers.

What he did here, in his characteristic vigor and verve, may be described in his own words as "a systematic exploitation and development of the best available techniques" for analytic problems arising from probability and its applications. A glance at the Contents shows this determination. The volume begins with three chapters on exponential, uniform, and other related special distributions. Simple as they are, they could not even be defined in a discrete sample space as permitted by Volume 1, so now he pounced and pounded on them with true vengeance. The old-fashioned probabilist will find much to delight him here, and may read no further to get his money's worth. [There are some combinatorialists who would never, but never, mention a random variable nor penetrate the depth of this alien notion.] The super-modern, however, should be required to do a minor thesis on these three chapters before they engage in their dissertations, in order to learn that probability theory does not have to begin with some projective limits! Next comes a brief interlude of "probability measures and space" which occupies only sixty-six pages, about ten per cent of the volume. Contrast this with the rather standard practice in advanced probability texts and courses where the tail of measure and integration wags the oft-impoverished dog. On the other hand, the brief treatment is of course not adequate to any launching of the paths (see below). A survey of the rest of the book is then given in Chapter 6 containing these topics: stable and infinitely divisible distributions, processes with independent increments, renewal processes, random walks, queuing. They constitute the main fare. By contrast, "general Markov chains" and "martingales" take only ten pages, and the last-named object is included only as an honored but by-passed presence (see footnote on page 209). For it is an interesting historical note that although Feller was an open admirer of Lévy and Doob, he had never reasoned with a path in continuous time nor used a martingale in his own work. Any possibly overlooked exception to this observation will only serve to prove the rule. On the other hand, analytical tools such as the Laplace and Fourier transforms, and the convolution operator method originated with Lindeberg, as well as combinatorial arguments of various sorts, are fully developed and applied to a large number of particular problems. Connections with classical analysis and concrete examples are stressed. Whereas the author insists on the simplifying and unifying aspects of his approaches (among which, a simple renewal derivation of Cramér's famous risk estimate), the reader is more immediately struck with the rich variety of the material. Several advanced courses on analytical probability can be built around portions of the book. But from the point of view of modern stochastic processes it is strange indeed that there is no discussion to speak of about sample function properties, not even for Poisson or Brownian motion processes. In the latter case the continuity of paths is tossed off to Wiener and Lévy in one sentence on page 181. As for processes with independent increments including the stable and infinitely divisible (see in particular Chapter 17), it may be proper in this new issue honoring Paul Lévy to quote the reviewer's remarks in his review of Doob's *Stochastic Processes* (see *Bull. Amer. Math. Soc.* **60** (1954) page 198).

> This theory, one of the crowning achievements in modern probability, is a natural generalization of the addition of independent random variables from the discrete to the continuous parameter case. It was further developed in his 1937 book. ...It is clear from Lévy's writing that he has always regarded the subject as one belonging to a (continuous-parameter) process and it was under this guidance that he was led by his extraordinary intuition to the discovery of all the main facts of the theory. That Khintchine and later authors chose the more formal analytical approach must partly be due to the fact that at the time the foundations of stochastic processes were hardly laid... and that mathematicians endowed with less intuition feared to tread the ground broken by Lévy.

As a matter of fact, the processes mentioned above are all special cases of Feller processes (possessing an additional property of spatial homogeneity), and more stochastic treatments following the footsteps of Lévy can now be found in the books Dynkin, Itô–McKean, Gihman–Skorohod [also mimeographed lecture notes by Itô], aside from Doob's book under review above, which is listed in the present book as "of historical interest."

As is familiar to readers of Volume 1, Feller's style is vivid and personally involved. His discontent with the unlearned, unkempt or unregenerate in the field is evident. He has made enormous efforts to present his favorite topics in the best light as he saw it. The undaunted reader soon learns to treat such Fellerian phrases as "a glance at... shows" with the same respect due the celebrated Laplacian "it is easy to see." As an excellent example, this reviewer can recommend Section XI. 9, "Renewal theory on the whole line" to anyone who wishes to test his wits on an impressive new proof of the last of all renewal limit theorems.

This second edition contains over forty more pages and a fair number of new vignettes. The curiosity-seeker can find a convenient list of the latter in the review by David Kendall (*Math. Gazette*, **56** (1972) page 65). More important to the average user is perhaps the rewriting of many passages to make the reading easier, intended by the author. Several reviewers of the first edition of Volume 2 have spotted a good number of errors. Some of these were corrected or got rid of by rewriting, but other obvious ones remain. For example, the "double" on page 13, line 12 is wrong, contradicts (4.3), and makes the waiting time paradox doubly paradoxical. Feller was probably thinking of a Poisson process in the whole line but he never said so. As another example of a minute slip, why the "$>$" on page 201, line 9? This is the kind of difficulty which will not "deter the layman," as the author well realized, but only a serious student (it can be fixed easily). He was also aware of the touchy question of assigning credits.

But in spite of his spirited defense in the Preface to the first edition, many grumbles have been heard since its appearance. One dissatisfied creditor even complained about the excessive credits received by another author! Let me therefore mention that Feller had apparently forgotten to assign a Tauberian theorem (page 443) to Heinz König although this was pointed out by a reviewer (Spitzer). He had also dropped an appropriate footnote giving reference to the short proof of recurrence of random walks on page 203 of the first edition, perhaps because another reviewer (Orey) had inadvertently thought it was new. These records are set straight here to arouse further interest in this new edition which is a substantial revision of the old, and not meant to detract from its overwhelming merits. Nothing can detract from a book which has so many good things to offer.

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