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OCCUPATION TIMES FOR SMOOTH STATIONARY PROCESSES[1]

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An occupation-time density is identified for a class of absolutely continuous functions $x(t)$ in terms of $x'(t)$ and the number of times that $x(t)$ assumes the values in its range. This result is applied to stationary random processes with a finite second spectral moment. As a by-product, a generalization of Rice's formula for the mean number of crossings is obtained.

0. Introduction. We construct an occupation-time density (OTD) for random processes belonging to a class which includes all stationary processes having a finite second spectral moment λ_2. The problem is similar to that treated by Berman [2] and Orey [12] for certain stationary Gaussian processes with $\lambda_2 = \infty$, but is simpler to deal with in the present context.

The work breaks naturally into two parts. First, in Section 1, using classical results from the theory of functions of a real variable, we give a necessary and sufficient condition for an absolutely continuous (nonrandom) function $x(t)$ to have an OTD, which is given explicitly. Then, in Section 2, we impose probabilistic conditions on a stationary process $x(t, \omega)$ so that the results of Section 1 are applicable to almost every trajectory.

We then obtain a general form of Rice's formula for the mean number of crossings of a level. If $x(t, \omega)$ is Gaussian, with $\lambda_2 < \infty$, the present approach to Rice's formula is less efficient, but perhaps more revealing, than the usual one [4]. Finally, we slightly improve a result of Berman on the multiplicity of the values in the range of a stationary Gaussian process with $\lambda_2 = \infty$.

1. A real variable theorem. Let $x(t)$ be a real-valued, absolutely continuous function on $I = [0, 1]$. We use $\mathscr{B}(I)$ to denote the Borel σ-field in I, $\mathscr{B}$ the Borel σ-field in the real line R, and m for Lebesgue measure ("measurable" will mean *Lebesgue* measurable, unless stated otherwise). The indicator of a set Γ is denoted by I_Γ.

For any subset U of I, $x \in R$, put

$$\nu(x, U) = \#\{t \in U : x(t) = x\},$$

where $\#$ denotes "cardinality of," and is interpreted as ∞ when the indicated set is infinite. For each x, $\nu(x, \cdot)$ is a measure whereas for each interval $U \subset I$, hence for each $U \in \mathscr{B}(I)$, $\nu(\cdot, U)$ is a measurable function. When U is an interval, $\nu(x, U)$ is known as the *Banach indicatrix* of the function $x(t)$ on U. A

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theorem of Banach, together with basic properties of absolute continuity, yields

(1) $$\int_R \nu(x, U)\, dx = \int_U |x'(s)|\, ds$$

for any interval U in I, hence for any $U \in \mathscr{B}(I)$, where $x'(s)$ denotes the derivative of $x(s)$. (If $x(s)$ is only assumed continuous, (1) remains valid if the right side is replaced by the variation of $x(s)$ over U—this is Banach's original result [1].)

THEOREM 1. *The function* $b_t(x) = \int_0^t |x'(s)|^{-1}\nu(x, ds)$ *is finite for almost every* x, *and satisfies*

(2) $$\int_\Gamma b_t(x)\, dx \leqq \int_0^t I_\Gamma(x(s))\, ds \ \ (= m(x^{-1}(\Gamma) \cap [0, t])) \qquad \textit{for every } \Gamma \in \mathscr{B}.$$

Equality obtains in (2) iff

(3) $$m\{t \in I : x'(t) = 0\} = 0\,.$$

Notice $b_t(x) = \sum_{i=1}^n |x'(t_i)|^{-1}$ where $n = \nu(x, [0, t])$ and t_i is the i^{th} hitting time of x.

PROOF. Let $D = \{t \in I : x'(t)$ does not exist$\}$. Then $m(D) = 0$, hence $m(x(D)) = 0$ since an absolutely continuous function carries null sets into null sets. Next, let $N = \{t \in I : x'(t) = 0\}$. By ([6] (17.27)), $m(x(N)) = 0$. It follows from (1) that $m\{x : \nu(x, I) = \infty\} = 0$; hence $m(B) = 0$, where $B = x(D) \cup x(N) \cup \{x : \nu(x, I) = \infty\}$. So $b_t(x)$ is finite for $x \notin B$.

For any $U \in \mathscr{B}(I)$, $\Gamma \in \mathscr{B}$, observe that $\nu(x, x^{-1}(\Gamma) \cap U) = \nu(x, U) I_\Gamma(x)$. Putting this into (1) we find

(4) $$\int_\Gamma \nu(x, U)\, dx = \int_{x^{-1}(\Gamma)\cap U} |x'(s)|\, ds\,, \qquad U \in \mathscr{B}(I),\ \Gamma \in \mathscr{B}.$$

A monotone class argument gives

(5) $$\int_\Gamma \int_0^1 f(s)\nu(x, ds)\, dx = \int_{x^{-1}(\Gamma)} f(s)\, |x'(s)|\, ds$$

for any nonnegative $\mathscr{B}(I)$-measurable function $f(s)$. Take

$$f(s) = I_{N^c \cap [0,t]}(s)\, |x'(s)|^{-1} \quad \text{a.e.}$$

Then (5) becomes

(6) $$\int_\Gamma b_t(x)\, dx = m(x^{-1}(\Gamma) \cap N^c \cap [0, t])\,,$$

and the theorem is proven. □

Clearly $b_t(x)$ is the density of the absolutely continuous component of the measure $m(x^{-1}(\Gamma) \cap [0, t])$. When there is no singular component, i.e. when equality holds in (2), we call $b_t(x)$ an *occupation-time density* (OTD) for the function $x(t)$.

Notes. (i) When (3) is in force, the existence, but not the explicit form, of an OTD is immediate from the Radon–Nikodym theorem: let Γ be a subset of the range of $x(t)$, $m(\Gamma) = 0$. Then $x'(t) = 0$ a.e. on $x^{-1}(\Gamma)$ (see, e.g. [11] page 213), so that $m(x^{-1}(\Gamma)) = 0$ by (3).

(ii) An immediate consequence of (4) with $\Gamma = (x - \varepsilon, x + \varepsilon)$, $U = [0, t]$, is

$$\nu(x, [0, t]) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I_{(0,\varepsilon)}(|x(s) - x|)\, |x'(s)|\, ds \tag{7}$$

for almost every x. This generalizes Lemma 1 of Kac [10] which says that (7) holds for *every* x provided that $x'(t)$ is continuous and neither $x(0)$ nor $x(t)$ equals x. See also Ivanov [8].

2. Stochastic occupation times. Consider a strictly stationary stochastic process $x(t, \omega)$, $t \in R$, over a probability space $(\Omega, \mathscr{F}, P)$, and denote by θ_t the shift transformation of the process ([4] page 149). If the trajectories are continuous, then, by using a suitable function space representation, we may assume that $\{\theta_t \colon t \in R\}$ is a *flow*, i.e. a one-parameter group of measurable, measure-preserving transformations on Ω, with θ_0 the identity, and such that the mapping $(t, \omega) \to \theta_t(\omega)$ of $(R, \mathscr{B}) \times (\Omega, \mathscr{F})$ onto $(\Omega, \mathscr{F})$ is measurable. The definition of shifting entails that $x(t + s, \omega) = x(t, \theta_s \omega)$ for all $s, t \in R$, $\omega \in \Omega$.

We shall apply the results of Section 1 to stationary processes within the following framework:

DEFINITION. (i) An *additive functional* (AF) is a process $\alpha(t, \omega)$ (or $\alpha_t(\omega)$), $t \in R$, $\omega \in \Omega$, for which $\alpha(0) \equiv 0$, $\alpha(t)$ is right continuous and nondecreasing a.s., and, for each $s, t \in R$,

$$\alpha(t + s, \omega) = \alpha(t, \omega) + \alpha(s, \theta_t \omega) \quad \text{a.s.} \tag{8}$$

(For example, $\int_0^t X(\theta_s \omega)\, ds$ is an AF for any bounded, nonnegative random variable X.) If $E\alpha(1) < \infty$, α is an *integrable* AF, and $E\alpha(t) = tE\alpha(1)$, $t \in R$.

(ii) The *Palm measure* of an AF α is defined as:

$$\hat{P}_\alpha(A) = E \int_0^1 I_A \circ \theta \, d\alpha(t)\,, \qquad A \in \mathscr{F}. \tag{9}$$

Obviously, $\hat{P}_\alpha$ is a finite measure iff α is integrable.

(iii) A family of AF's $\beta(x) = (\beta_t(x))$, $x \in R$, is an *occupation-time density* (OTD) for the process $x(t, \omega)$ if $\beta_t(x, \omega)$ is measurable in the pair (x, ω), and there exists a set Ω' such that $P(\Omega') = 1$ and

$$\int_\Gamma \beta_t(x, \omega) \pi(dx) = \int_0^t I_\Gamma(x(s, \omega))\, ds\,, \qquad \Gamma \in \mathscr{B},\ t \in R,\ \omega \in \Omega', \text{ where} \tag{10}$$

$\pi(\Gamma) = P\{x(0) \in \Gamma\}$.

We need two lemmas.

LEMMA 1. *Let* α, β *be integrable* AF'*s. Then* $\hat{P}_\alpha = \hat{P}_\beta$ iff *for almost every* $\omega \in \Omega$, $\alpha(t, \omega) = \beta(t, \omega)$ *for all* $t \in R$.

We omit the proof, which appears in [7]. By a *regular conditional probability given* $x(0)$ (briefly rcp) we mean a family $\{P^x \colon x \in R\}$ of measures on $\mathscr{F}$ for which $x \to P^x(A)$ is measurable for each $A \in \mathscr{F}$, and

$$P\{A, x(0) \in \Gamma\} = \int_\Gamma P^x(A) \pi(dx)\,, \qquad \Gamma \in \mathscr{B},\ A \in \mathscr{F}. \tag{11}$$

Clearly P^x is a probability measure a.e. $[\pi]$. (The conventional notation for $P^x(A)$ is $P(A \mid x(0) = x)$.) We can now state

LEMMA 2. *Let $\beta(x)$, $x \in R$, be a family of* AF's *such that $\beta_t(x, \omega)$ is (x, ω)-measurable. Then $\beta(x)$ is an* OTD iff *the family of Palm measures $\hat{P}_{\beta(x)}$ is an* rcp.

PROOF. Suppose $\beta(x)$ is an OTD. For $\Gamma \in \mathscr{B}$ we have

$$\begin{aligned} \int_\Gamma \hat{P}_{\beta(x)}(A)\pi(dx) &= \int_\Gamma E \int_0^1 I_A \circ \theta_t d\beta_t(x)\pi(dx) \\ &= E \int_0^1 I_A \circ \theta_t I_\Gamma(x(t))\, dt & \text{by (10)} \\ &= P\{A, x(0) \in \Gamma\} & \text{(stationarity).} \end{aligned}$$

Thus $\{\hat{P}_{\beta(x)}\}$ is an rcp. Conversely, consider the additive functionals $\gamma_\Gamma(t) = \int_\Gamma \beta_t(x)\pi(dx)$ and $\delta_\Gamma(t) = \int_0^t I_\Gamma(x(s))\, ds$. The above computation, done in reverse, shows that γ_Γ, δ_Γ have the same Palm measure, hence (Lemma 1) coincide on a set Ω_Γ of probability 1. Letting Γ run through the family of intervals with rational endpoints we obtain a set Ω' for which (9) holds. □

Let $\{P^x : x \in R\}$, $\{Q^x : x \in R\}$ be two rcp's. If $\mathscr{F}$ is separable, as we can and do assume (since $x(t)$ is continuous), then $P^x = Q^x$ a.e. $[\pi]$. In some cases—notably Gaussian—there is a natural choice of an rcp dictated by joint densities.

Let us now assume that the stationary process $x(t, \omega)$ has a quadratic mean derivative $\dot{x}(t, \omega)$. (This is equivalent to the requirement that the second spectral moment λ_2 be finite.) As Doob ([5] page 536) has shown, taking a "standard modification" of the original process, we may assume the trajectories are absolutely continuous and have sample derivative $\dot{x}(t, \omega)$—all with probability one. In applying Section 1, we may remove the restriction that t be in I: all the definitions and results extend immediately to R. For the trajectory $x(t, \omega)$ we define $\nu_t(x, \omega) = \nu(x, (0, t], \omega)$ when $t \geqq 0$, and $\nu_t(x, \omega) = -\nu(x, (t, 0], \omega)$ when $t < 0$, with $\nu(x, U)$ as in Section 1. Obviously $\nu_t(x, \omega)$ is an AF for each $x \in R$.

THEOREM 2. *If*

$$P\{\dot{x}(0) = 0\} = 0\,, \tag{12}$$

then the one-dimensional distribution $\pi(dx)$ is absolutely continuous, and there is an OTD *for the process. The converse is also true. Moreover, under* (12),

(i) *the additive functionals*

$$\beta_t(x, \omega) = (p(x))^{-1} \int_0^t |\dot{x}(s)|^{-1}\, d\nu_s(x, \omega) \tag{13}$$

serve as an OTD *($p(x)$ being a density for $\pi(dx)$);*

(ii) $d\hat{P}_{\nu(x)} = p(x)\, |\dot{x}(0)|\, dP^x$ a.e. $[\pi]$, *where $\{P^x, x \in R\}$ is an* rcp;

(iii) $E\nu_1(x) = p(x)E^x\, |\dot{x}(0)|$ a.e. $[\pi]$ *(E^x denotes integration with P^x).*

PROOF. If (12) holds, $P\{\dot{x}(t) = 0\} = 0$ for each $t \in R$; hence it follows from Fubini's theorem that $m\{t : \dot{x}(t) = 0\} = 0$ a.s. Now, by Theorem 1, for almost every ω the AF $b_t(x, \omega) = \int_0^t |\dot{x}(s, \omega)|^{-1} d\nu_s(x, \omega)$ satisfies (2) with equality. Taking expectations shows that $p(x) = Eb_1(x)$ is a density for $\pi(dx)$ and, since $\pi(dx) = p(x)dx$, the AF $\beta_t(x) = (p(x))^{-1}b_t(x)$ satisfies (10).

For the converse, let $\pi(dx) = p(x)\,dx$ and $\beta_t(x)$ be an OTD. Then with probability one, the trajectory $x(t, \omega)$ cannot spend positive time in a set of Lebesgue measure zero. By Theorem 1, $m\{t\colon \dot{x}(t) = 0\} = 0$ a.s., and a Fubini argument gives $P\{\dot{x}(t) = 0\} = 0$ for almost every t, hence for every t by stationarity. In particular, (12) holds.

By Banach's theorem (Section 1), $\nu_t(x) < \infty$ for almost every x (hence a.e. $[\pi]$) if we exclude a set of probability zero. Another Fubini argument allows us to conclude that, for almost every x, $\nu_t(x, \omega) < \infty$ a.s. For such x, we can "invert" (13) to obtain

$$\nu_t(x) = p(x) \int_0^t |\dot{x}(s)|\, d\beta_s(x) \quad \text{a.s.} \tag{14}$$

Putting $P^x = \hat{P}_{\beta(x)}$ gives an rcp by Lemma 2, and (ii) and (iii) follow immediately. □

We conclude this section with several remarks.

(a) If we write (1) for the trajectory $x(t, \omega)$ with $U = [0, 1]$, and take the expectation of both sides, then, with the sole assumption that $\lambda_2 < \infty$, we get

$$\int_R E\nu_1(x)\, dx = E\, |\dot{x}(0)| < \infty$$

since $\dot{x}(0) \in L^2$. Thus $E\nu_1(x) < \infty$ a.e., a fact which does not seem to be in the literature (see [4] page 201).

(b) Because the density $p(x)$ is only determined almost everywhere, the exceptional x-sets in (ii) and (iii) are to a certain extent unremovable unless, of course, further restrictions are imposed. (See (d) below.)

(c) Let $x(t, \omega)$ be Gaussian, with standard normal one-dimensional distributions. Then $x(0)$ and $\dot{x}(0)$ are independent, and

$$p(x)E^x\, |\dot{x}(0)| = (\lambda_2^{\frac{1}{2}}/\pi)\exp(-x^2/2)\,,$$

where P^x refers to the usual Gaussian rcp. It can be shown that $(\lambda_2^{\frac{1}{2}}/\pi)^{-1}\hat{P}_{\nu(0)}$ is the so-called "horizontal window" probability $P(\cdot \mid x(0) = 0hw)$ defined in [14]. It is immediate that, under P^0, $x(t)$ is distributed as the sum of two Gaussian processes: $x_1(t)$ having mean zero and covariance $r(t-s) - r(t)r(s) - \lambda_2^{-1}r'(t)r'(s)$ where $r(t) = Ex(t)x(0)$, and $x_2(t) = \lambda_2^{-\frac{1}{2}}r'(t)\xi$, where ξ is standard normal and independent of the process $x_1(t)$. Slepian's [14] decomposition of the "horizontal-window" process now follows directly from (ii).

(d) Let $B = B(\omega)$ be defined as in the proof of Theorem 1, relative to the trajectory $x(t, \omega)$. Then $m(B(\omega)) = 0$ a.s.; in particular, there cannot be a tangency to any $x \notin B(\omega)$. Hence with probability one, $\nu_t(x) = \alpha_t(x)$, for almost every x, where $\alpha_t(x)$ is the number of "genuine" crossings; such a crossing occurs at t_0 if $x(t_0) = x$ and $x(t) - x$ changes sign on every neighborhood of t_0.

With ν replaced by α in Theorem 2, part (iii) is then a general form of Rice's formula ([4] Chapter 10) for the mean number of crossings. By using polygonal approximations to $x(t)$, it follows that $E\alpha_1(x) \leqq \liminf_{h\to 0} E\alpha_1(x+h)$ for every x. In general, one must impose further restrictions to obtain $E\alpha_1(x) = p(x)E^x\,|\dot{x}(0)|$ for a given x. (See e.g. [8] Theorem 3.)

(e) From Theorem 1, every Gaussian process having absolutely continuous trajectories has an OTD as in Section 1. This gives a partial answer to a conjecture of Orey [12].

3. On a theorem of Berman. Let $x(t, \omega)$, $0 \leqq t \leqq 1$, be as in Section 2, except we now assume that $x(t)$ is Gaussian, mean 0, and $\lambda_2 = \infty$ (equivalently, $t^{-2}(1 - r(t)) \to \infty$ as $t \to 0$ where $r(t)$ is the covariance). The paths are no longer absolutely continuous (see below) and the results of Section 2 do not apply. However, we recall Banach's theorem (Section 1): a continuous function on an interval U has bounded variation if and only if $\nu(x, U)$ is integrable over R. We can apply this to obtain, and slightly improve, the following result of Berman [2], amended in [3]:

THEOREM (Berman). *Suppose, for almost every $\omega \in \Omega$, $x(t, \omega)$ has an* OTD *$b_t(x)$, which admits a version continuous in x for every rational t. Then the set of values y where $x(\cdot, \omega)$ crosses y at most finitely often is of category one in the range of $x(\cdot, \omega)$ for almost all ω.*

We will show that the conclusion remains true without assuming the existence of an OTD. The proof goes as follows. Let P denote the distribution of $x(t, \omega)$ in the space $C[0, 1]$ of continuous real-valued functions on $[0, 1]$; measurable sets are those in the completion under P of the usual product σ-field. Let G consist of paths with a (finite) derivative at some fixed t. Then G is a measurable subgroup; hence $P(G) = 0$ or $P(G) = 1$—see [9]. If $P(G) = 1$, $h^{-1}(x(h) - x(0))$ converges in law, which is impossible since $E[h^{-1}(x(h) - x(0))]^2$ diverges by our hypothesis about $r(t)$. By stationarity and a simple Fubini argument, $x(t, \omega)$ is non-differentiable a.e. on $[0, 1]$ a.s. In particular, $x(t, \omega)$ is of unbounded variation on every subinterval of $[0, 1]$ a.s. It is known that for each y fixed, $\alpha_1(y, \omega) = \nu_1(y, \omega)$ a.s. Banach's theorem now implies that the integral of $\alpha_1(y, \omega)$ diverges over every subinterval of $[0, 1]$. In particular, the closure of $\{y : \alpha_1(y, \omega) \leqq k\} \cap x([0, 1], \omega)$ has no interior for every $k \geqq 1$ a.s. (see [13] for the details). Noting that

$$B = \{y : \alpha_1(y, \omega) < \infty\} = \textstyle\bigcup_{k\geqq 1} \{y : \alpha_1(y, \omega) \leqq k\}$$

we are finished. (Of course, as a subset of B, $\{y : \nu_1(y, \omega) < \infty\}$ is also of category one in $x([0, 1], \omega)$ a.s.) □

The argument above extends immediately to any real Gaussian process on $[0, 1]$ with stationary increments, continuous paths, and for which $t^{-2}\sigma^2(t) \to \infty$ as $t \to 0$ where $\sigma^2(t)$ is the incremental covariance. (See [3] for conditions under which B is actually nowhere dense in $x([0, 1], \omega)$ a.s.)

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