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A CENTRAL LIMIT THEOREM FOR m-DEPENDENT RANDOM VARIABLES WITH UNBOUNDED m

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For each $k = 1, 2, \cdots$ let $n = n(k)$, let $m = m(k)$, and suppose $y_1^k, \cdots, y_n^k$ is an m-dependent sequence of random variables. We assume the random variables have $(2+\delta)$th moments, that $m^{2+2/\delta}/n \to 0$, and other regularity conditions, and prove that $n^{-\frac{1}{2}}(y_1^k + \cdots + y_n^k)$ is asymptotically normal. An example showing sharpness is given.

Central limit theorems for m-dependent variables (m fixed) have been proved by Hoeffding and Robbins [3], Diananda [2], Orey [5], and Bergstrom [1].

In this paper we prove the following theorem and give an example demonstrating its sharpness.

THEOREM. *For each $k = 1, 2, \cdots$ let $n = n(k)$ and $m = m(k)$ be specified and suppose $y_1^k, \cdots, y_n^k$ is an m-dependent sequence of random variables with zero means. Assume the following*:

(i) *For some $\delta > 0$, $E|y_i^k|^{2+\delta} \leqq M$ for all i and k.*
(ii) $\operatorname{Var}(y_{i+1}^k + \cdots + y_j^k) \leqq (j-i)K$ *for all i, j, and k.*
(iii) $\lim_{k\to\infty} n^{-1} \operatorname{Var}(y_1^k + \cdots + y_n^k)$ *exists and is nonzero. Call the limit v.*
(iv) $\lim_{k\to\infty} m^{2+2/\delta}/n = 0$.

Then $n^{-\frac{1}{2}}(y_1^k + \cdots + y_n^k)$ is asymptotically normal with mean 0 *and variance v.*

PROOF. For each k we choose an integer $p = p(k) > 2m$ so that

$$\lim_{k\to\infty} m/p = 0\,, \qquad \lim_{k\to\infty} p^{2+2/\delta}/n = 0\,. \tag{1}$$

This can be done, for example, by choosing p to be the least integer greater than $m^{\frac{1}{2}}n^{\delta/(4+4\delta)}$ and greater than $2m$. Define $t = t(k)$ and $r = r(k)$ by $n = pt + r$, $0 \leqq r < p$. Then let

$$\begin{aligned}
u_1^k &= y_1^k + \cdots + y_{p-m}^k\,, & x_1^k &= y_{p-m+1}^k + \cdots + y_p^k\,,\\
u_2^k &= y_{p+1}^k + \cdots + y_{2p-m}^k\,, & x_2^k &= y_{2p-m+1}^k + \cdots + y_{2p}^k\,,\\
&\cdots \\
u_t^k &= y_{(t-1)p+1}^k + \cdots + y_{tp-m}^k\,, & x_t^k &= y_{tp-m+1}^k + \cdots + y_{tp}^k\,,\\
R^k &= y_{tp+1}^k + \cdots + y_n^k\,.
\end{aligned} \tag{2}$$

Since the y_i^k are m-dependent and $p > 2m$, $\{x_i^k\}$ and $\{u_i^k\}$ are each independent sequences.

It is easily seen that the difference between $n^{-\frac{1}{2}}(y_1^k + \cdots + y_n^k)$ and

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$n^{-\frac{1}{2}}(u_1^k + \cdots + u_t^k)$ has variance approaching zero so that the asymptotic distributions of these two quantities will be the same, and that

$$\lim_{k\to\infty} \operatorname{Var} n^{-\frac{1}{2}}(u_1^k + \cdots + u_t^k) = v \,. \tag{3}$$

Letting $(B^k)^2 = \operatorname{Var}(u_1^k + \cdots + u_t^k)$ and using the Lyapounov theorem ([4] page 275) to prove the present theorem, it suffices to show that

$$\lim_{k\to\infty} \textstyle\sum_{i=1}^t E|u_i^k|^{2+\delta}/(B^k)^{2+\delta} = 0 \,.$$

By (i), (2), and the Minkowski inequality,

$$E|u_i^k|^{2+\delta} \leqq (p - m)^{2+\delta} M^{2+\delta} \,.$$

Furthermore, (3) implies that for k sufficiently large, $(B^k)^2 = \operatorname{Var}(u_1^k + \cdots + u_t^k) \geqq vn/2$, so that

$$\textstyle\sum_{i=1}^t E|u_i^k|^{2+\delta}/(B^k)^{2+\delta} \leqq \text{const} \dfrac{(p-m)^{1+\delta}}{n^{\delta/2}} \,.$$

But (1) implies that $\lim_{k\to\infty} (p-m)^{1+\delta} n^{-\delta/2} = 0$, and this completes the proof.

If assumption (iv), about the rate at which n increases with m, is simplified to require $n \sim m^\alpha$, then the assumption is that $\alpha > 2 + 2/\delta$. If y_i^k has unbounded moments of all orders, then the assumption is that $\alpha > 2$.

To show that condition (iv) cannot be relaxed, and that n must increase at a sufficient rate so that $m^{2+2/\delta}/n \to 0$, we give an example for which the other conditions are satisfied, but $m^{2+2/\delta}/n \to 1$, and there is no asymptotic normality.

For $k = 1, 2, \cdots$, let $z_1^k, z_2^k, \cdots$ be a sequence of independent identically distributed random variables each with distribution

$$P(z_i^k = 0) = 1 - k^{-1-2/\delta} \,, \qquad P(z_i^k = \pm k^{1/\delta}) = (\tfrac{1}{2}) k^{-1-2/\delta} \,,$$

and let

$$\begin{aligned} y_1^k = \cdots = y_k^k &= z_1^k \\ y_{k+1}^k = \cdots = y_{2k}^k &= z_2^k \\ &\cdots \\ y_{n-k+1}^k = \cdots = y_n^k &= z_t^k \,, \end{aligned}$$

where n is the largest multiple of k less than $k^{2+2/\delta}$ and $t = n/k$. The sequence $y_1^k, \cdots, y_n^k$ is m-dependent with $m = k$. Assumptions (i), (ii), and (iii) of the theorem are readily verified with $M = 1$, $K = 1$, $v = 1$; however, the limit in (iv) is one, not zero.

We consider

$$n^{-\frac{1}{2}}(y_1^k + \cdots + y_n^k) = (k/t)^{\frac{1}{2}}(z_1^k + \cdots + z_t^k) \tag{4}$$

with mean zero and variance one. Letting $w_i^k = (k/t)^{\frac{1}{2}} z_i^k$, $i = 1, \cdots, t$ and letting F^k be the probability distribution function of w_i^k, we apply the Lindeberg criterion ([4] page 280): because $\lim_{k\to\infty} \operatorname{Var} w_i^k = 0$, expression (4) is asymptotically normal only if for all $\varepsilon > 0$

$$\lim_{k\to\infty} \textstyle\sum_{i=1}^t \int_{|w|>\varepsilon} w^2 \, dF^k(w) = 0 \,.$$

But for ε less than $k^{\frac{1}{2}+1/\delta}t^{-\frac{1}{2}}$, which has limit one,

$$\sum_{i=1}^{t} \int_{|w|>\varepsilon} w^2 \, dF^k(w) = \sum_{i=1}^{t} k^{1+2/\delta} t^{-1} k^{-1-2/\delta} = 1 ,$$

so (4) is not asymptotically normal.

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