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SUBGROUPS OF PATHS AND REPRODUCING KERNELS[1]

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The following generalizations of certain theorems due to G. Kallianpur and to Jamison and Orey are proved for an arbitrary Gaussian measure P on a space of real functions: if the reproducing kernel Hilbert space H is infinite dimensional then $P(H) = 0$; if a subgroup G of the space of real functions (under addition) is measurable with respect to the P-completion of the Borel product sigma-algebra, then $P(G) = 0$ or $P(G) = 1$ and in the latter case $H \subset G$.

Interesting relationships exist between the reproducing kernel Hilbert space H of (any) real and Gaussian random function X and the X-induced probability measure P defined on sets of the completed Borel product sigma-algebra $\mathscr{F}$. Among these relationships are the following:

(a) $P(H) = 0$ if H has infinite dimension.

(b) $P(G) > 0$ implies $H \subset G$, for G an additive subgroup of functions.

(c) $G \in \mathscr{F}$ implies $P(G) = 0$ or $P(G) = 1$, for G an additive subgroup of functions.

While some of these results have been known or suspected in special cases (see [2] for specific reference to Cameron and Graves), only recently have results approaching the generality of (a), (b), (c) been achieved. In particular the earliest rigorous proof of (a) in a general setting is Kallianpur's [3]. In this paper we prove (b) and (c) by methods which are of independent interest in the study of Gaussian measures. An outline of the way in which these methods yield the strongest form of (a) will be presented.

Throughout this paper $X = (X(t), t \in T)$ is a real Gaussian family of random variables indexed by a set T. Without loss of generality we understand these to be the coordinate evaluation functions defined by $(X(t))(f) = f(t)$, for every $t \in T, f \in R^T$. We let $\mathscr{F}_0$ stand for the product Borel sigma-algebra of R^T sets, and denote by P the unique probability measure defined on $\mathscr{F}_0$ with respect to which X has the desired Gaussian law. The P-completion of $\mathscr{F}_0$ will be denoted by $\mathscr{F}$. Let H be the reproducing kernel Hilbert space of the covariance function $\Gamma(s, t) = \int_{R^T} (X(s) - m(s))(X(t) - m(t))\, dP$, $(s, t) \in T \times T$, where $m(t) = \int_{R^T} X(t)\, dP$, $t \in T$, (see [5] page 84). As is well known, H is isometrically isomorphic to a linear subspace $L^\sim$ of $L_2(P)$ under the linear extension of the association $\Gamma(t, \cdot) \leftrightarrow X(t) - m(t)$, $t \in T$, first to finite real linear combinations and thence to H

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(equivalently $L_2(P)$) limits. If $h \in H$ we let its companion in $L^\sim$ under this isomorphism be denoted by $h^\sim$.

PROPOSITION 1. *If $A \in \mathscr{F}^\sim$, $P(A) > 0$, $h \in H$, then for all r in a real neighborhood of zero $rh \in A \ominus A$ (the set of differences).*

PROOF. We may suppose $||h|| = 1$. The Gaussian random functions $Y = X - h^\sim h$, $h^\sim h$ are uncorrelated and therefore mutually independent. Applying to $A \in \mathscr{F}^\sim$ a standard result of product measures ([5] page 74) we conclude that almost every Y-section is measurable, and if $P(A) > 0$ then for at least one $y \in R^T$ (it's all we need) the event $\{y + h^\sim h \in A\}$ has positive probability with respect to the standard normal distribution of $h^\sim$. Therefore the Lebesgue measure of $B = \{r: y + rh \in A\}$ is positive. A result of Steinhaus ([6] page 99) applies to say that $B \ominus B$ (the set of differences) contains an open interval about zero. This implies the result for $A \ominus A$. □

COROLLARY 1. *If $G \in \mathscr{F}^\sim$ and G is a group under pointwise sum, then $P(G) > 0$ implies $H \subset G$.*

PROOF. Suppose $h \in H$. If $P(G) > 0$ then from Proposition 1 there is an integer $n > 0$ for which $h/n \in G \ominus G$. If G is a group, $G = G \ominus G$, so $h/n \in G$. By summing n of these elements h/n, conclude that since G is a group $h \in G$. □

PROPOSITION 2. *If $A \in \mathscr{F}^\sim$ and $A = A \oplus H$ (the set of sums) then $P(A) = 0$ or $P(A) = 1$.*

PROOF. Since $\mathscr{F}^\sim$ is the completed sigma-algebra generated by $\{h^\sim : h \in H\}$ we may choose a (possibly empty or finite, but at most enumerable) orthonormal set $\{h_k : k \in K\} \subset H$ such that A belongs to the completion of the sigma-algebra generated by $\{h_k^\sim : k \in K\}$, (see [5] page 81). Arguing as in Proposition 1, the Gaussian random functions $Y_n = X - \sum_{k=1}^{k=n} h_k^\sim h_k$ and $\sum_{k=1}^{k=n} h_k^\sim h_k$ are defined for $\{h_1, \cdots, h_n\} \subset \{h_k : k \in K\}$. If dimension H is zero, then K is necessarily empty, but in this case X is almost surely a constant and therefore $P(A) = 0$ or $P(A) = 1$. If dimension H is positive but finite, then for $n =$ dimension H, Y_n is almost surely constant (in fact it equals m) and if $A = A \oplus H$ then $A = (X \in A) = (Y_n + \sum_{k=1}^{k=n} h_k^\sim h_k \in A) = (Y_n \in A)$ almost surely, and the last of these has probability either zero or one. Finally, if dimension H is infinite, we may choose K an infinite set, and $A = (X \in A) = (X_n \in A)$ almost surely for every n. This places A in the completed tail sigma-algebra of the independent sequence $\{h_k^\sim : k \geqq 1\}$. By the ordinary zero or one law ([5] page 128) we conclude $P(A) = 0$ or $P(A) = 1$. □

COROLLARY 2. *If G is a group under pointwise sum and $G \in \mathscr{F}^\sim$, then $P(G) = 0$ or $P(G) = 1$. In the latter case $H \subset G$.*

PROOF. If G is a group and $G \in \mathscr{F}^\sim$, then by Corollary 1, $P(G) = 0$ or $H \subset G$. If $H \subset G$ then, H being a group, $G = G \oplus H$ and so by Proposition 2, $P(G) = 0$ or $P(G) = 1$. In the case $P(G) = 1$, we have $H \subset G$ by Corollary 1. □

In [4], a strengthening of Proposition 2 has been shown to imply the equivalence-singularity dichotomy for Gaussian measures. Jamison and Orey [1] have proved a special case of Corollary 2 when $T = [0, 1]$, X has continuous sample functions almost surely, and $m \equiv 0$. In their paper they raise the question of whether the condition of sample continuity may be removed. This is answered affirmatively by Corollary 2. Closer to Corollary 2 are twin results due to G. Kallianpur [2]. The first is Corollary 2 under the additional hypotheses that T is a separable metric space, Γ is continuous, and G is closed under multiplication by rationals. The second removes the hypothesis of closure under multiplication by rationals, but requires $G \in \mathscr{F}_0$.

When H has infinite dimension and $m \equiv 0$, Kallianpur has in [3] proved under mild smoothness and separability hypotheses that $H \in \mathscr{F}$ and $P(H) = 0$. The methods of this paper apply also to the proof of (a). Suppose $\{t_k : k \geqq 1\} \subset T$ and $\{h_k : k \geqq 1\} \subset H$ is an infinite orthonormal sequence derived from $\{\Gamma(t_k, \cdot) : k \geqq 1\}$ by Gram-Schmidt orthonormalization in H (such a choice is always possible since $\{\Gamma(t, \cdot) : t \in T\}$ span H). Then for all $j \geqq 1$, if $m \equiv 0$,

$$P(X(t_j) = \limsup_{n\to\infty} \textstyle\sum_{k=1}^{k=n} h_k^{\sim} h_k(t_j)) = 1 \qquad \text{and}$$

$$\{\limsup_{n\to\infty} \textstyle\sum_{k=1}^{k=n} h_k^{\sim} h_k \in H\} \subset \{\sum_{k=1}^{\infty} h_k^{\sim 2} < \infty\}\,.$$

Therefore the set in R^T, on which X agrees on $\{t_k : k \geqq 1\}$ with an element of H, has probability zero. Therefore $P(H) = 0$. The case $m \not\equiv 0$ is similar.

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