The Annals of Probability
1973, Vol. 1, No. 2, 338–340

ON THE INTEGRABILITY OF THE SUPREMUM OF ERGODIC RATIOS

BY YVES DERRIENNIC

The Ohio State University[1]

We study the integrability of the supremum of ergodic ratios defined by means of a measure-preserving conservative ergodic transformation of a σ-finite measure space. Our result implies Ornstein's recent result for the supremum of ergodic averages.

D. Ornstein (1971) has proved that, given an invertible measure-preserving ergodic transformation S of a finite measure space, the supremum of ergodic averages $\sup_{n\geqq 1} n^{-1} \sum_{i=0}^{n-1} f \circ S^i$ is integrable if and only if $f \log^+ f$ is integrable. (h^+ is the positive part of h [2].) The purpose of this note is the investigation of the similar property for ergodic ratios. Our result implies Ornstein's theorem although our argument is simpler.

Let T be a measure-preserving, conservative and ergodic transformation of a σ-finite measure space; the measure will be denoted by m. (T is not assumed to be invertible.) Given $g, f \in L_1(m)$, $g > 0$ a.e. and $f \geqq 0$ a.e. let

$$s(f, g) = \sup_{n\geqq 1} \left(\sum_{i=0}^{n-1} f \circ T^i / \sum_{i=0}^{n-1} g \circ T^i\right).$$

THEOREM. *If* $\int f \log^+ (f/g)\, dm < \infty$ *then* $\int g\, s(f, g)\, dm < \infty$. *Conversely if* $\int g[s(f, g) + s(f, g) \circ T]\, dm < \infty$ *then* $\int f \log^+ (f/g)\, dm < \infty$.

The first assertion of the theorem, that follows from Hopf's maximal ergodic lemma, belongs to the common unwritten knowledge. (When the measure is finite, $g = 1$ a.e. see [1] page 678.) Thus we shall be concerned only about the proof of the second. The key step is the following "reverse maximal inequality."

LEMMA. *Given a positive number* a, *let* $A = \{s(f, g) > a\}$. *If* $m(A^c) > 0$ *then* $\int_A f\, dm \leqq a \int_{A \cup T^{-1}A} g\, dm$.

PROOF. We denote also by T the positive contraction of $L_1(m)$ defined by $f \to f \circ T$. Since T is conservative and ergodic we have

$$1_A = \sum_{n=1}^{\infty} (I_A T^*)^n 1_{A^c}$$

where T^* is the conjugate of T, defined on $L_\infty(m)$, 1_A the characteristic function

Received July 14, 1972.

[1] Author's present address is Départment de Mathématiques, Université de Rennes, Avenue du Général Leclerc, 35-Rennes France.

AMS 1970 *subject classifications.* Primary 2870; Secondary 6060.

Key words and phrases. Measure-preserving conservative ergodic transformation, reverse maximal inequality.

of the set A and I_A the operator of multiplication by 1_A. Therefore

$$\begin{aligned}\int_A f\,dm &= \int f \sum_{n=1}^{\infty} (I_A T^*)^n 1_{A^c}\,dm \\ &= \int_{A^c} \sum_{n=1}^{\infty} (TI_A)^n f\,dm \\ &= \sum_{n=1}^{\infty} \int_{A^c \cap [\bigcap_{i=1}^{n} T^{-i}A]} f \circ T^n\,dm\,.\end{aligned}$$

Since the sets $A^c \cap [\bigcap_{i=1}^{k} T^{-i}A] \cap T^{-(k+1)}A^c$, for $k = n, n+1, \cdots$, form a partition of $A^c \cap [\bigcap_{i=1}^{n} T^{-i}A]$ we get

$$\int_A f\,dm = \sum_{n=1}^{\infty} \int_{A^c[\bigcap_{i=1}^{n} T^{-i}A] \cap T^{-(n+1)}A^c} \left(\sum_{i=1}^{n} f \circ T^i\right) dm\,.$$

According to the definition of A, $x \in A^c$ implies $\sum_{i=1}^{n} f \circ T^i(x) \leqq a(\sum_{i=0}^{n} g \circ T^i(x))$ for every n. Hence get

$$\begin{aligned}\int_A f\,dm &\leqq a \sum_{n=1}^{\infty} \int_{A^c \cap [\bigcap_{i=1}^{n} T^{-i}A] \cap T^{-(n+1)}A^c} \left(\sum_{i=0}^{n} g \circ T^i\right) dm \\ &= a\left[\int_A g\,dm + \int_{A^c \cap T^{-1}A} g\,dm\right] \\ &= a \int_{A \cup T^{-1}A} g\,dm\,.\end{aligned}$$

PROOF OF THE THEOREM. Let $a_0 = \inf\{a > 0;\ m\{s(f, g) \leqq a\} > 0\}$. Since $\{f > ag\} \subset \{s(f, g) > a\}$ and $\{s(f, g) > a\} \cup T^{-1}\{s(f, g) > a\} \subset \{s(f, g) + s(f, g) \circ T > a\}$ the lemma implies, for $a > a_0$,

$$\int_{\{f > ag\}} f\,dm \leqq a \int_{\{s(f,g)+s(f,g)\circ T > a\}} g\,dm\,.$$

Then the theorem follows from the simple calculation:

$$\begin{aligned}\int f \log^+ (f/a_0 g)\,dm &= \int_{\{f > a_0 g\}} f(x)\left(\int_{a_0}^{(f/g)(x)} a^{-1}\,da\right) dm(x) \\ &= \int_{a_0}^{\infty} a^{-1}\left(\int_{\{f > ag\}} f\,dm\right) da \\ &\leqq \int_{a_0}^{\infty} \left(\int_{\{s(f,g)+s(f,g)\circ T > a\}} g\,dm\right) da \\ &\leqq \int g[s(f, g) + s(f, g) \circ T]\,dm\,.\end{aligned}$$

COROLLARY. *If there is a constant K such that $g \leqq K(g \circ T)$* a.e. *then $\int gs(f, g)\,dm < \infty$ if and only if $\int f \log^+ (f/g)\,dm < \infty$. In particular if the total measure is finite, $\int s(f, 1)\,dm < \infty$ if and only if $\int f \log^+(f)\,dm < \infty$.*

PROOF. This is a direct consequence of the theorem for $s(f, g) \circ T \leqq (1 + K)s(f, g)$ a.e.

Now we shall give an example showing that the conditions introduced in the theorem are not trivial; especially that the equivalence between $\int g\, s(f, g)\,dm < \infty$ and $\int f \log^+ (f/g)\,dm < \infty$ may fail.

EXAMPLE. The measure space is the interval $[0, 2[$ endowed with the Lebesgue measure. T is defined as follows:

$$\begin{aligned}T(x) &= x + 1 &&\text{if } 0 \leqq x < 1 \\ T(x) &= S(x - 1) &&\text{if } 1 \leqq x < 2\end{aligned}$$

where S is any measure-preserving ergodic transformation of $[0, 1[$. Assume $f = 1$ a.e. and $g(x) = 1$ if $1 \leqq x < 2$. It is easy to check that either $s(1, g)(x) < 3$ or $s(1, g)(x) = 1/g(x)$. Therefore $\int g\, s(1, g)\,dm < \infty$ even if

$\int \log^+ (1/g)\, dm = \infty$. On the other hand, $\int \log^+ (1/g)\, dm < \infty$ does not imply that $\int g[s(1, g) \circ T]\, dm \geqq \int_{[0,1]} (1/g)\, dm$ is finite.

REMARK. D. Ornstein mentions also that $\sup_{n \geqq 1} n^{-1} \sum_{i=0}^{n-1} f \circ T^i$ is not integrable when the total measure is infinite. His proof needs the additional assumption that T is conservative. But this result is a direct consequence of Birkhoff's ergodic theorem. Indeed, $\lim_{n \to \infty} n^{-1} \sum_{i=0}^{n-1} f \circ T^i = 0$ a.e. but not in $L_1(m)$, therefore this supremum cannot be integrable.

REFERENCES

[1] DUNFORD, N. and SCHWARTZ, J. T. (1958). *Linear Operators*. Interscience, New York.
[2] ORNSTEIN, D. (1971). A remark on the Birkhoff ergodic theorem. *Illinois J. Math.* **15** 77–79.

LABORATORIE DE PROBABILITÉS
UNIVERSITÉ DE RENNES
AVENUE DU GÉNÉRAL LECLERC
B. P. 25A
35031 RENNES CEDEX
FRANCE