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ASYMPTOTICS OF RANDOMLY STOPPED SEQUENCES WITH INDEPENDENT INCREMENTS

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Let S_n, $n = 1, 2, \cdots$, be a sequence of sums of independent, identically distributed random variables X_i such that $P\{X_i > y\}$ is a regularly varying function of y at infinity. Let N be a stopping time for S_n with finite mean. A necessary and sufficient condition is given that

$$\lim_{y\to\infty} P\{S_N > y\}/P\{X_1 > y\} = EN .$$

Examples further illustrate the role of this condition.

1. Introduction. Let S_n, $n = 1, 2, \cdots$ be a sequence of sums of independent, identically distributed random variables X_i defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}_n$ denote the subfield of $\mathscr{F}$ generated by $S_1, \cdots, S_n$. A stopping time for S_n is a positive integer-valued random variable N such that for each n, $\{N \leqq n\} \in \mathscr{F}_n$.

A family of equations relates the successive moments of the stopped process S_N and the moments of N whenever the appropriate moments are finite. For instance, if $E|S_n| < \infty$ and $EN < \infty$ then $ES_N = EX_1EN$. Such moment identities appeared in Wald's classical work in sequential analysis. Their general validity for martingales and for processes with stationary, independent increments has been shown by Chow, Robbins and Teicher (1965), (1966), Brown (1969), and Hall (1970).

If S_n has zero mean, the second moment grows linearly with time, $ES_n^2 = kn$. Wald's equation $ES_N^2 = kEN$ says that this linear growth is preserved under random stopping. The equation $ES_N = EX_1EN$ is a similar statement about the mean when not zero.

We find that the linear relation governing the asymptotic property of the process distribution is also preserved under random stopping. If $P\{X_1 > y\}$ varies regularly as $y \to \infty$ then (see for instance Feller (1966)),

$$P\{S_n > y\} \sim P\{X_1 > y\}n . \tag{1}$$

If in addition EN is finite then

$$P\{S_N > y\} \sim P\{X_1 > y\}EN \tag{2}$$

is equivalent to condition (8).

In Section 3 the condition $EN < \infty$ is shown not to imply (2) in general.

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However, a result of Monroe (1972) shows that for strictly stable sequences (2) is true whenever $EN < \infty$.

The expression $a \wedge b$ will denote the minimum of a and b. The statement $f(y) \sim g(y)$ will mean that $\lim_{y\to\infty} f(y)/g(y) = 1$.

2. An asymptotic formula. We assume throughout that $P\{X_1 > y\}$ varies regularly at ∞, i.e. $P\{X_1 > y\} \sim y^{-p}L(y)$, $p \geqq 0$, where L is a slowly varying function at ∞ : $L(yx) \sim L(y)$ for any $x > 0$. As an initial step we prove statement (2) for the bounded stopping time $N \wedge n$,

LEMMA. *For each* $n = 1, 2, \cdots$,

$$P\{S_{N\wedge n} > y\} \sim P\{X_1 > y\}E\{N \wedge n\}.$$

PROOF. If $n = 1$ we have $1 = 1$. Assume that the lemma is true with n replaced by $n - 1$:

$$P\{S_N > y, N < n - 1\} + P\{S_{n-1} > y, N \geqq n - 1\} \sim P\{X_1 > y\}E\{N \wedge (n - 1)\},$$

i.e.

$$P\{S_N > y, N < n\} + P\{S_{n-1} > y, N \geqq n\} \sim P\{X_1 > y\}E\{N \wedge (n - 1)\}. \tag{3}$$

We wish to show

$$P\{S_N > y, N < n\} + P\{S_n > y, N \geqq n\} \sim P\{X_1 > y\}E\{N \wedge n\}. \tag{4}$$

The difference between relations (3) and (4), since $E\{N \wedge n\} = E\{N \wedge (n - 1)\} + P\{N \geqq n\}$, is

$$P\{S_n > y, N \geqq n\} - P\{S_{n-1} > y, N \geqq n\} \sim P\{X_1 > y\}P\{N \geqq n\}. \tag{5}$$

We rewrite the left side of (5) as

$$P\{S_n > y, S_{n-1} \leqq y, N \geqq n\} - P\{S_{n-1} > y, S_n \leqq y, N \geqq n\}$$

and show that

$$\lim_{y\to\infty} P\{S_n > y, S_{n-1} \leqq y, N \geqq n\}/P\{X_1 > y\} = P\{N \geqq n\}, \tag{6}$$

and

$$\lim_{y\to\infty} P\{S_{n-1} > y, S_n \leqq y, N \geqq n\}/P\{X_1 > y\} = 0. \tag{7}$$

The technique used by Feller to prove (1) also proves (6). For any $\varepsilon > 0$,

$$\begin{aligned} P\{S_n > y, S_{n-1} \leqq y, N \geqq n\} &= P\{S_{n-1} + X_n > y, S_{n-1} \leqq y, N \geqq n\} \\ &\geqq P\{S_{n-1} > y(1 + \varepsilon), S_{n-1} \leqq y, N \geqq n\}P\{|X_n| < y\varepsilon\} \\ &\quad + P\{|S_{n-1}| > y\varepsilon, S_{n-1} \leqq y, N \geqq n\}P\{X_n > y(1 + \varepsilon)\}. \end{aligned}$$

We have used the independence of X_n from the event $\{N > n\}$ which, with its complement, is measurable with respect to $\mathscr{F}_{n-1}$. The first term on the right is zero and

$$\lim_{y\to\infty} P\{|S_{n-1}| < y\varepsilon, N \geqq n\}P\{X_n > y(1 + \varepsilon)\}/P\{X_1 > y\} = P\{N \geqq n\}(1 + \varepsilon)^{-p}.$$

We have, then, the $\geqq$ half of (6). On the other hand, for any $\varepsilon > 0$,

$$P\{S_n > y, S_{n-1} \leqq y, N \geqq n\} \leqq P\{y \geqq S_{n-1} > y(1-\varepsilon), N \geqq n\} + P\{X_n > y(1-\varepsilon)\}P\{S_{n-1} \leqq y, N \geqq n\} + P\{y \geqq S_{n-1} > y\varepsilon, N > n\}P\{X_n > y\varepsilon\} .$$

Let us deal first with the last term, omitting the $\{N \geqq n\}$ and recalling (1):

$$\lim_{y\to\infty} P\{y \geqq S_{n-1} > y\varepsilon\}P\{X_n > y\varepsilon\}/P\{X_1 > y\} = (n-1)(\varepsilon^{-p}-1)\lim_{y\to\infty} P\{X_n > y\varepsilon\} = 0 .$$

By a similar computation,

$$\lim_{y\to\infty} P\{S_n > y, S_{n-1} \leqq y, N \leqq n\}/P\{X_1 > y\} \leqq ((1-\varepsilon)^{-p}-1)(n-1) + (1-\varepsilon)^{-p}P\{N \geqq n\} .$$

Since $\varepsilon > 0$ was arbitrary we have the $\leqq$ half of (6).

To prove (7) we observe that for any $\varepsilon > 0$,

$$\lim_{y\to\infty} P\{S_n > y\}/P\{X_1 > y\} = n ,$$

whereas

$$\lim_{y\to\infty} P\{S_{n-1} > y(1+\varepsilon)\}/P\{X_1 > y\} = (n-1)(1+\varepsilon)^{-p} .$$

For a small fixed ε and large enough y, then,

$$P\{S_n > y\} \geqq P\{S_{n-1} > y(1+\varepsilon)\} .$$

It follows that

$$\lim_{y\to\infty} P\{S_{n-1} > y, S_n \leqq y\}/P\{X_1 > y\} \leqq \lim_{y\to\infty} P\{y < S_{n-1} \leqq y(1+\varepsilon)\}/P\{X_1 > y\} = 1-(1+\varepsilon)^{-p} .$$

Since ε can be chosen arbitrarily small, the limit is zero.

REMARK. Statement (5) still holds if we replace $\{N \geqq n\}$ by any event measurable with respect to $\mathscr{F}_{n-1}$.

THEOREM. *Let S_n, $n = 1, 2, \cdots$, be a sequence of sums of independent, identically distributed random variables X_i. Suppose that $P\{X_1 > y\}$ is regularly varying at ∞. Let N be a stopping time for S_n such that*

(8) $$\liminf_{n\to\infty} \limsup_{y\to\infty} [P\{S_N > y, N \geqq n\} - P\{S_n > y, N \geqq n\}]/P\{X_1 > y\} = 0 .$$

Then

(9) $$P\{S_N > y\} \sim P\{X_1 > y\}EN .$$

If $EN < \infty$ then "lim inf" *may be replaced by* "lim" *in* (8), *and* (8) *and* (9) *are equivalent.*

PROOF. For each fixed n, and then as $n \to \infty$ on an appropriate subsequence,

$$\begin{aligned}
&\limsup_{y\to\infty} P\{S_N > y\}/P\{X_1 > y\} \\
&\quad = \limsup_{y\to\infty} [P\{S_N > y, N < n\} + P\{S_N > y, N \geqq n\} \\
&\qquad + (P\{S_n > y, N \geqq n\} - P\{S_n > y, N \geqq n\})]/P\{X_1 > y\} \\
&\quad = \limsup_{y\to\infty} P\{S_{N\wedge n} > y\}/P\{X_1 > y\} \qquad (10)\\
&\qquad + \limsup_{n\to\infty} [P\{S_N > y, N \geqq n\} - P\{S_n > y, N \geqq n\}]/P\{X_1 > y\} \\
&\quad = \lim_{n\to\infty} E(N \wedge n)\,,
\end{aligned}$$

by the Lemma and (8). As $n \to \infty$, $N \wedge n$ increases to N almost surely, and $E(N \wedge n)$ goes to EN. If $EN < \infty$ and the limit in (8) is not 0, calculation (10) shows (9) to be false.

A similar argument gives us the

COROLLARY. *If* $\limsup_{y\to\infty} P\{S_N > y\}/P\{X_1 > y\} < \infty$ *and* $EN < \infty$ *then*

$$\lim_{n\to\infty} \limsup_{y\to\infty} [P\{S_N > y, N \geqq n\} - P\{S_n > y, N \geqq n\}]/P\{X_1 > y\} = M < \infty\,,$$

and

$$\limsup_{y\to\infty} P\{S_N > y\}/P\{X_1 > y\} = EN + M\,.$$

3. Examples. Wald's equations hold whenever the appropriate moments of N and S_n are fininite. However, finiteness of EN is not sufficient for condition (9) to hold. In the following example $P\{X_1 > y\}$ varies regularly and $EN < \infty$ but $P\{S_N > y\}/P\{X_1 > y\} \to \infty$. Then the Theorem says that (8) fails.

Let S_n be a sequence of sums of independent, identically distributed, symmetric X_i such that $P\{X_1 > y\} \sim y^{-4-\varepsilon}$ for some $\varepsilon > 0$. Then $ES_n = 0$ and $ES_n^4 < \infty$. Denote by k_2, k_3, k_4 the second, third, and fourth moments of X_1. Let N be a stopping time for S_n such that $EN < \infty$, whereas $EN^2 = \infty$. For each $n < \infty$, Theorems 2 and 7 of Chow, Robbins, and Teicher (1965) give us the Wald equations

$$\begin{aligned}
ES_N^2 &= k_2 EN\,,\\
E(S_{N\wedge n})^4 &= 6k_2 E\{(N \wedge n)S^2_{N\wedge n}\} \\
&\quad + 4k_3 E\{(N \wedge n)S_{N\wedge n}\} + k_4 E\{N \wedge n\} - 3k_2 E\{N \wedge n)^2\}\,.
\end{aligned}$$

Schwarz's inequality gives

$$\begin{aligned}
3k_2 E\{(N \wedge n)^2\} &\leqq 6k_2 E\{(N \wedge n)^2\}^{\frac12} E\{S^4_{N\wedge n}\}^{\frac12} \\
&\quad - E\{S^4_{N\wedge n}\} + 4k_3 E\{(N \wedge n)^2\}^{\frac12} E\{(S_{N\wedge n})^2\}^{\frac12} + k_4 E\{N \wedge n\}\,.
\end{aligned}$$

Then

$$\begin{aligned}
3k_2 E\{(N \wedge n)^2\}^{\frac12} &\leqq 6k_2 E\{S^4_{N\wedge n}\}^{\frac12} + 4k_3 E\{S^2_{N\wedge n}\}^{\frac12} \\
&\quad + k_4 E\{N \wedge n\}/E\{(N \wedge n)^2\}^{\frac12}\,.
\end{aligned}$$

As n increases the left side goes to infinity. On the right, the second term is bounded, the third goes to zero, and we conclude that $E\{S^4_{N\wedge n}\}$ goes to infinity.

Therefore $ES_N^4 = \infty$ and, since X_i is symmetric, $\limsup_{y\to\infty} y^{4+\varepsilon}P\{S_N > y\} = \infty$.

The following theorem of Monroe implies that for a strictly stable process X_t, $\lim_{y\to\infty} P\{X_T > y\}/P\{X_1 > y\} = ET$ if T is any stopping time such that ET is finite. By "strictly stable" is meant that X_t and $t^{1/\alpha}X_1$ have the same distribution, where α is the index of the process, $0 < \alpha \leqq 2$.

THEOREM (Monroe (1972)). *Let X_t be a strictly stable process and let T be a stopping time such that $ET < \infty$. Then the distribution of X_T is in the domain of normal attraction of the distribution of X_{ET}.*

A similar result for a strictly stable sequence S_n follows from evaluation of X_t at the integers.

Monroe's theorem enables us to construct an example which shows that condition (8) cannot be replaced by

$$\liminf_{n\to\infty} \lim_{y\to\infty} P\{S_N > y,\ N \geqq n\}/P\{X_1 > y\} = 0 .$$

Let S_n be a strictly stable sequence, as above, with index α. Let $N = \min\{n: S_n < n^p\}$. Then $0 < EN \leqq \sum_{n=1}^{\infty} nP\{S_{n-1} \geqq (n-1)^p\} = \sum_{n=1}^{\infty} nP\{X_1 > (n-1)^{p-1/\alpha}\}$, since $S_n/n^{1/\alpha}$ has the same distribution as X_1. Using the property of the stable distribution, $P\{X_1 > y\} \sim y^{-\alpha}$, we find that the sum converges if p is, for instance, $4/\alpha$. Monroe's theorem implies that

$$\lim_{y\to\infty} P\{S_N > y\}/P\{X_1 > y\} = EN .$$

Now we can compute, since $N^p > S_N$,

$$\begin{aligned}\lim_{n\to\infty} \lim_{y\to\infty} & P\{S_N > y,\ N \geqq n\}/P\{X_1 > y\} \\ &= \lim_{n\to\infty} \lim_{y\to\infty} ENP\{N \geqq n \mid S_N > y\} = EN .\end{aligned}$$

Because $EN < \infty$, our Theorem implies that also

$$\lim_{n\to\infty} \lim_{y\to\infty} P\{S_n > y,\ N \geqq n\}/P\{X_1 > y\} = EN .$$

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