WHICH FUNCTIONS OF STOPPING TIMES ARE STOPPING TIMES?\footnote{Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, USAF, under Grant No. AFOSR-71-2100. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.}

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Some functions of stopping times are necessarily stopping times, but others need not be. For example, the sum $\tau_1 + \tau_2$ of two stopping times is, while for stochastic processes in continuous time, the product $\tau_1 \cdot \tau_2$ need not be. Determined here for each positive integer $n$ are those functions $\phi$ for which $\phi(\tau)$ is a stopping time for all $n$-tuples of stopping times $\tau = \tau_1, \cdots, \tau_n$.

1. Introduction. If $\tau$ is a stop rule and $\phi$ maps the set $T$ of positive reals into itself, then $\phi(\tau)$ may or may not be a stop rule. Which are the (measurable) $\phi$ that transform all stop rules into stop rules? Among them are certainly those for which $\phi(t) \geq t$ for all $t$. The only others are those which, for some $c$, satisfy

$$\phi(t) \geq t \quad \text{for} \quad t \leq c,$$

$$= c \quad \text{for} \quad t > c.$$

With the convention that $c$ can be $\infty$, (1) characterizes the (measurable) $\phi$ that carry stop rules into stop rules. What positive-valued (measurable) functions $\phi$ of two positive variables carry each pair of stop rules into stop rules? Just those $\phi$ for which $\phi(s, t)$ satisfies Condition (1) in $s$ for each $t$ and in $t$ for each $s$. The principle extends immediately from two to any finite or infinite number of arguments.

This characterization of the $\phi$'s that transform stop rules for discrete-time stochastic processes into stop rules was jointly reported in [1] (Section 2.9). This note reports that the same characterization holds for strict stop rules when the processes have a time-parameter that is continuous. Slight variants of the characterization hold when wide-sense stop rules are admitted.

2. Notation. $\mathcal{F}^* = [\mathcal{F}_t, 0 \leq t < \infty]$ is an increasing family of $\sigma$-fields of subsets of a set $U$, and $\tau$ is a stop rule for $\mathcal{F}^*$, that is, a map of $U$ into the nonnegative reals such that $(\tau \leq t) \in \mathcal{F}_t$ for each $t \geq 0$.

Lemma 1. For each $t \geq 0$, each Borel-subset $B$ of the closed interval $[0, t]$, and each stop rule $\tau$ for $[\mathcal{F}_t, 0 \leq t < \infty]$, (2)

$$\tau^{-1}(B) \in \mathcal{F}_t.$$
PROOF. Verify: (i) If \( B = [0, s] \) for an \( s \leq t \), then (2) holds; (ii) The set of \( B \) for which (2) holds is closed under countable unions; and (iii) If (2) holds for \( B \subset [0, t] \), then it holds for the complement of \( B \) in \([0, t]\). This implies that (2) holds for all Borel \( B \subset [0, t] \).

**Theorem 1.** A necessary and sufficient condition on a Borel-measurable, real-valued function \( \phi \) of a nonnegative real variable that, for every increasing family of sigma fields, \( \mathcal{F}^* = \{ \mathcal{F}_t : 0 \leq t < \infty \} \), and every stop rule \( \tau \) for \( \mathcal{F}^* \), \( \phi \circ \tau \) is a stop rule for \( \mathcal{F}^* \) is that, for some positive \( c \), possibly infinite, \( \phi \) satisfies Condition (1).

**Proof.** For the sufficiency, what must be verified is that, for each \( t \),

\[
(\phi \circ \tau \leq t) \in \mathcal{F}_t.
\]

The left-hand side of (3) is \( \tau^{-1}(B) \) where \( B = \phi^{-1}[0, t] \) is plainly a Borel set. Suppose first that \( t < c \). Then \( B \subset [0, t] \), and Lemma 1 yields the desired conclusion. If \( t \geq c \), then \( B \supset (c, \infty) \), which implies that

\[
B = B^* \cup (c, \infty),
\]

where \( B^* \), being \( B \cap [0, c] \), satisfies

\[
B^* \subset [0, t].
\]

In view of (4), \( \tau^{-1}(B) \) is the union of \( \tau^{-1}(B^*) \) with \( (\tau > c) \). Since \( \tau^{-1}(B^*) \) is in \( \mathcal{F}_t \), according to (5) and Lemma 1, and \( (\tau > c) \), being in \( \mathcal{F}_t \), is certainly in \( \mathcal{F}_t \), so is their union, namely \( \tau^{-1}(B) \).

If \( \phi \) does not satisfy the condition, then there is a stop rule \( \tau \) for a \( \mathcal{F}^* = \{ \mathcal{F}_t : 0 \leq t < \infty \} \) where each \( \mathcal{F}_t \) is either the trivial field or else the field of all subsets of a two-element set \( U \), that is, a set of cardinality 2. For if \( \phi \) does not satisfy the condition, then there exist \( t_0, t_1 \) such that \( \phi(t_0) < t_0, \phi(t_1) \neq \phi(t_0) \) and \( t_1 > \phi(t_0) \). For \( U \), one may take, for example, two paths \( w_0 \) and \( w_1 \) which agree until time \( \phi(t_0) \), but not thereafter, and for \( \mathcal{F}_t \) the trivial, or the universal, field according as \( t \leq \phi(t_0) \) or \( t > \phi(t_0) \). As is easily verified, if \( \tau(w_i) = t_i \), then \( \tau \) is a stop rule for \( \mathcal{F}^* \), but \( \phi(\tau) \) is not.

Turn now to the problem of determining those positive-value measurable functions \( \phi \) of two positive variables that carry each pair of stop rules into a stop rule. For a necessary condition on \( \phi \) let \( \tau_1, \tau_2 \) be a constant stop rule. Then \( \varphi(s, \tau) \) must be a stop rule for each stop rule \( \tau \). Hence, according to Theorem (1), for each \( s, \varphi(s, \tau) \) must satisfy (1) in \( t \). Similarly, for each \( t, \varphi(s, \tau) \) satisfies (1) in \( s \). To see that this condition is sufficient, two preliminary lemmas are needed.

**Lemma 2.** Let \( \tau = (\tau_1, \tau_2) \) be a pair of stop rules for an increasing family of sigma fields, \( \{ \mathcal{F}_s : 0 \leq s < \infty \} \), \( t \) a positive number, \( I \) the closed interval \([0, t] \), and \( B \) a Borel subset of the square \( I \times I \). Then

\[
\tau^{-1}(B) \in \mathcal{F}_t.
\]
PROOF. According to Lemma 1, if \( B \) is the Cartesian product of two Borel subsets \( B_i \) of \( I \), then

\[
\tau^{-1}(B) = \tau_1^{-1}(B_1) \cap \tau_2^{-1}(B_2) \in \mathcal{F}_r;
\]

and since the set of \( B \) for which (6) holds is a \( \sigma \)-field, the proof is evident.

**Lemma 3.** Let \( \phi \) satisfy (1) in \( s \) for each \( t \) and in \( t \) for each \( s \), and let \( r \) be a positive number. Let \( A = [(s, t) : \phi(s, t) \leq r] \) and

\[
A_1 = A \cap \{s \leq r \text{ and } t \leq r\}; \\
A_2 = A \cap \{s \leq r < t\}; \\
A_3 = A \cap \{t \leq r < s\}; \\
A_4 = A \cap \{r < s \text{ and } r < t\}.
\]

Then \( A = A_1 \cup A_2 \cup A_3 \cup A_4 \), and: (i) \( A_1 \) is a subset of \( I \times I \) where \( I = [0, r] \); (ii) \( A_2 = \alpha \times (r, \infty) \) for some subset \( \alpha \) of \( [0, r] \); (iii) \( A_3 = (r, \infty) \times \alpha \) for some subset \( \alpha \) of \( [0, r] \); (iv) \( A_4 \) is either empty or equal to \( (r, \infty) \times (r, \infty) \). Moreover, if \( \phi \) is Borel, so is each \( A_i \) and \( \alpha \).

**Proof.** Plainly, (i) is trivial. Suppose that \( (s, t) \) is in \( A_2 \). To verify (ii), it is only necessary to check that for the same \( s \) and \( t' > r \), \( (s, t') \) is in \( A_2 \). By hypothesis, \( \phi(s, t) \leq r \), \( s \leq r \). Since for this \( s \), \( \phi \) satisfies (1) in \( t \), \( \phi(s, t') = c = \phi(s, t') \) for all \( t' \), and \( \alpha \) for all \( t' > r \). Hence, \( \phi(s, t') \leq c \leq r \) for all \( t' > r \), which implies that each such \( (s, t') \) is in \( A_2 \). The proof of (iii) is obtained from the proof of (ii) by interchanging the roles of \( s \) and \( t \). Since the proof of (iv) is quite similar, it need not be given. It is trivial that each \( A_i \) is Borel if \( \phi \) is Borel. To see that each \( \alpha \) is Borel, recall that the measurability of a rectangle implies the measurability of each of its sides (e.g., see [2] Section 32, Problem 4).

**Lemma 4.** The condition that \( \phi \) satisfy (1) in \( s \) for each \( t \) and in \( t \) for each \( s \) is sufficient for \( \phi \circ \tau \) to be a stop rule whenever \( \tau = (\tau_1, \tau_2) \) is a pair of stop rules.

**Proof.** What must be seen is that for each positive \( r \), \( \{\phi \circ \tau \leq r\} = \tau^{-1}(A) \in \mathcal{F}_r \), where \( A \) is as in Lemma 3. In the notation of Lemma 3, \( \tau^{-1}(A) = \bigcup \tau^{-1}(A_i) \). Hence, it is only necessary to show that for each \( i \), \( \tau^{-1}(A_i) \subseteq \mathcal{F}_r \). Since, \( A_i \) is a Borel subset of the square \( [0, r] \times [0, r] \), Lemma 2 applies. Since, by Lemma 3, \( A_2 = \alpha \times (r, \infty) \), where \( \alpha \) is a Borel subset of \( [0, r] \), \( \tau_2^{-1}(A_2) = \tau_2^{-1}(\alpha) \cap \tau_2^{-1}(r, \infty) \). As Lemma (1) implies that \( \tau_2^{-1}(\alpha) \subseteq \mathcal{F}_r \), and since \( \tau_2^{-1}(r, \infty) \subseteq \mathcal{F}_r \), for any stop rule \( \tau_2 \), their intersection, namely \( \tau^{-1}(A_i) \), is in \( \mathcal{F}_r \). The proofs that \( \tau^{-1}(A_i) \) and \( \tau^{-1}(A_i) \) are in \( \mathcal{F}_r \) are quite analogous.

As is easily checked, the argument given for functions \( \phi \) of two variables applies to functions of any finite or denumerable number of variables.

3. A variation. A very similar query to the one answered by Theorem 1 is to ask which Borel \( \phi \) transform all wide-sense stop rules into (strict sense) stop rules? As usual, \( \tau \) is a wide sense stop rule for \( \{\mathcal{F}_t : 0 \leq t < \infty\} \) if, for each \( t, (\tau \leq t) \) is in \( \mathcal{F}_{t+} \), where \( \mathcal{F}_{t+} = \bigcap_{t>1} \mathcal{F}_t \). Among these \( \phi \) are certainly those for which \( \phi(t) > t \) for all \( t \). Now the only others are those which, for
some $c$, satisfy

(9) \[ \phi(t) > t \quad \text{for} \quad t < c \]
\[ = c \quad \text{for} \quad t \geq c. \]

Plainly, if there is a critical finite $c$, then $\phi(c)$ must be $c$. This contrasts with Condition (1) where $\phi(c)$ could exceed $c$.

As before, a Borel function of $n$ variables, where $n$ may be the smallest infinite cardinal, transforms every $n$-tuple of wide-sense stop rules into a stop rule if, and only if, $\varphi$ satisfies Condition (9) in each variable separately. There is no need to give the arguments, for they are but slight variants of those given above.

REFERENCES


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