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ARTICLES

ON MARKOV PROCESSES WITH RANDOM STARTING TIME

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The paper deals with Markov processes which have both random starting and terminal times. Such processes were suggested by G. A. Hunt, were constructed by L. L. Helms (under the name Markov processes with creation and annihilation) and were treated also by M. Nagasawa and the author. The paper contains a new existence proof by a way of constructing such a process from its given associated semigroup of kernels $\tilde{P}_t$, $t \geqq 0$, and its (Markov) transition function. This construction is more general than that given by L. L. Helms (in terms of the Markov transition function and the creation measure) and is also more convenient as far as perturbation theory of Markov processes is concerned. Indeed more general relations between this theory and creation of mass processes are established. Finally an application to solving the Cauchy problem in partial differential equations is indicated.

1. Introduction. Lately several authors have dealt with generalized Markov processes in which both the starting time and the terminal time are random variables. These processes were first suggested by G. A. Hunt [6] and constructed by Helms [3] (see also [10] and [8]). Such a process can be regarded probabilistically as a perturbation of some Markov process (in the usual sense) and indeed relations were found between these processes and the theory of perturbation of infinitesimal generators of semigroups of operators on some Banach space X. In [3] Helms has constructed such processes, which he termed "Markov processes with creation and annihilation," starting from a transition function P_t, $t \geqq 0$, and a creation measure $\phi_x(ds, dy)$ satisfying several conditions. The process was constructed so that ϕ_x determines the starting time and the starting position in the state space E_∇ of a particle whose motion is described by the process. P_t, $t \geqq 0$, and ϕ_x determined, in that construction, a family of kernels $\tilde{P}_t$, $t \geqq 0$, on E_∇ by

$$\tilde{P}_t(x, A) = P_t(x, A) + \int_0^t \int_{E_\nabla} P_s(y, A)\phi_x(ds, dy) . \tag{1.1}$$

A process denoted $\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ on a σ-finite measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$ was constructed to satisfy the relation

$$\tilde{P}_t(x, A) = \mathscr{P}_x(\xi_t \in A) \tag{1.2}$$

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and to have transition function P_t, $t \geqq 0$, on E_∇. The purpose of this paper is to construct such processes starting directly from the transition function P_t, $t \geqq 0$, and a family of kernels $\tilde{P}_t$, $t \geqq 0$, satisfying some mild conditions which certainly hold in the case of [3]. The family $\tilde{P}_t$, $t \geqq 0$, will be related to the process just as before through (1.2). The construction is easier than that in [3] and turns out to be more general and to give better results in connection with perturbation theory.

Section 2 deals with the construction of such processes. Section 3 deals with relations to perturbation theory and partial differential equations.

2. Construction of the processes. We start from a sub-Markov transition function P_t, $t \geqq 0$, on $E \times \mathscr{E}$, where E is a locally compact space having a countable basis for its topology, $\mathscr{E}$ its σ-algebra of Borel subsets. Let $E_\nabla = E \cup \{\nabla\}$, ∇ being an isolated point or the point of infinity according to whether E is compact or not. Let $\mathscr{E}_\nabla$ be the σ-algebra of subsets of E_∇ generated by $\mathscr{E}$. Extend P_t, $t \geqq 0$, to a Markov transition function on $E_\nabla \times \mathscr{E}_\nabla$ by letting $P_t(x, \{\nabla\}) = 1 - P_t(x, E)$, $P_t(\nabla, A) = \delta_\nabla(A)$, where $\delta_\nabla(A)$ is 1 or 0 according to whether $\nabla \in A$ or not. Extend E_∇ further by putting $E_{\nabla,\Delta} = E_\nabla \cup \Delta$, Δ being an isolated point not in E_∇ and let $\mathscr{E}_{\nabla,\Delta}$ be the σ-algebra generated by $\mathscr{E}_\nabla$. A Markov process with creation and annihilation (of mass) $\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ on a σ-finite measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$ having transition function P_t, $t \geqq 0$, and a state space $E_{\nabla,\Delta}$ was defined in [3]. Mainly it should satisfy the following two conditions. (i) Each $\omega \in \Omega$ is a function from $[0, \infty)$ into $E_{\nabla,\Delta}$ satisfying the two conditions $\omega(s) = \Delta$ implies $\omega(t) = \Delta$ for $t \leqq s$ and $\omega(s) = \nabla$ implies $\omega(t) = \nabla$ for $t \geqq s$. (ii) For each $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$, the following Markov property holds.

$$(2.1) \qquad \mathscr{P}_x(\xi_{s+t} \in A \mid \mathscr{F}_t \cap \{\xi_t \in E_\nabla\}) = P_s(\xi_t, A) \quad \text{a.e.} \quad \mathscr{P}_x \quad \text{on} \quad \{\xi_t \in E_\nabla\}$$

where for each $t \geqq 0$, ξ_t is a function from Ω into $E_{\nabla,\Delta}$ defined by $\xi_t(\omega) = \omega(t)$ and where $\mathscr{F}_t = \sigma(\xi_s : s \leqq t)$.

A particle whose motion is described by such a process starts in the precreation state Δ and is then transferred to the state space E_∇ at a random time, after which its motion is controlled by the Markov transition function P_t, $t \geqq 0$. Thus the motion in E_∇ is that of a Markov process with a random starting time.

THEOREM 2.1. *Let P_t, $t \geqq 0$, be a Markov transition function on $E_\nabla \times \mathscr{E}_\nabla$. Let $\tilde{P}_t$, $t \geqq 0$, be a family of kernels on $E_\nabla \times \mathscr{E}_\nabla$ satisfying*

(i) *$\tilde{P}_t(x, E_\nabla) < \infty$ for each $t > 0$.*
(ii) *For each $t \geqq s > 0$, $\tilde{P}_t \geqq \tilde{P}_s P_{t-s}$.*

Then there exists a Markov process with creation and annihilation $\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ on a σ-finite measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$, $x \in E_\nabla$, having transition function P_t, $t \geqq 0$, for which $\tilde{P}_t(x, A) = \mathscr{P}_x(\xi_t \in A)$, $A \in \mathscr{E}_\nabla$.

PROOF. Let $\Omega_0 = E_{\nabla,\Delta}^{[0,\infty)}$ and

$$\Upsilon = \{(t_1 \cdots t_n) : 0 < t_1 < \cdots < t_n < \infty, n \geqq 1\}.$$

For each $\tau \in \Upsilon$, define a measure $\mathscr{P}_x^\tau$ on the product σ-algebra $\mathscr{E}^\tau_{\nabla,\Delta}$ as follows. If $\tau = (t_1 \cdots t_n) \in \Upsilon$, let $\tau_k = (t_k \cdots t_n)$ and let $A_i \in \mathscr{E}_{\nabla,\Delta}$, $i = 1, \cdots, n$. Consider the following cases:

Case 1. If $A_i \in \mathscr{E}_\nabla$, $i = 1, \cdots, n$, then let

$$\mathscr{P}_x^\tau(A_1 \times \cdots \times A_n) = \int_{A_n} \cdots \int_{A_1} \tilde{P}_{t_1}(x, dy_1) P_{t_2 - t_1}(y_1, dy_2) \cdots P_{t_{n-1}}(y_{n-1}, dy_n) .$$

Case 2. If $A_i = \Delta$, $1 \leqq i \leqq k - 1 < n$, $A_i \in \mathscr{E}_\nabla$, $i \geqq k$ then

$$\mathscr{P}_x^\tau(A_1 \times \cdots \times A_n) = \mathscr{P}_x^{\tau_k}(A_k \times \cdots \times A_n) - \mathscr{P}_x^{\tau_{k-1}}(E_\nabla \times A_k \times \cdots \times A_n) .$$

Case 3. If $A_i = \Delta$, $1 \leqq i \leqq n$, then

$$\mathscr{P}_x^\tau(A_1 \times \cdots \times A_n) = \lim_{t \to \infty} \tilde{P}_t(x, E_\nabla) - \tilde{P}_{t_n}(x, E_\nabla) .$$

Case 4. If there exist $k \geqq i \geqq 1$ such that $A_k = \Delta$ and $A_i \in \mathscr{E}_\nabla$, then $\mathscr{P}_x^\tau(A_1 \times \cdots \times A_n) = 0$.

Clearly $\mathscr{P}_x^\tau$ is well-defined and nonnegative. In Case 2 this follows from condition (ii) and in Case 3 from the fact that $\tilde{P}_t(x, E_\nabla)$ is increasing with t which again follows from (ii).

Since any rectangle set in $\mathscr{E}^\tau_{\nabla,\Delta}$ can be expressed uniquely as a finite disjoint union of the above four types of sets, we can extend $\mathscr{P}_x^\tau$ to the collection of all rectangle sets in $\mathscr{E}^\tau_{\nabla,\Delta}$. A standard procedure allows us to extend $\mathscr{P}_x^\tau$ first to the algebra of finite disjoint union of rectangle sets and then (using Carathéodory's theorem) to a σ-additive measure $\mathscr{P}_x^\tau$ on $\mathscr{E}^\tau_{\nabla,\Delta}$. Let us show next that the measures $\mathscr{P}_x^\tau$, $\tau \in \Upsilon$, are consistent. Indeed fix $\tau = (t_1 \cdots t_n) \in \Upsilon$ let $A_i \in \mathscr{E}_{\nabla,\Delta}$, $i = 1 \cdots n$, and set $A_j = E_{\nabla,\Delta}$. We have to consider the following cases (a) $j = 1$ (b) $1 < j < n$ (c) $j = n$. Under each case it is enough to consider only sets of the types (1)—(4).

Case a. If $A_2 \subset E_\nabla$ then

$$\begin{aligned}
&\mathscr{P}_x^\tau(E_{\nabla,\Delta} \times A_2 \times \cdots \times A_n) \\
&\qquad = \mathscr{P}_x^\tau(E_\nabla \times A_2 \times \cdots \times A_n) + \mathscr{P}_x^\tau(\Delta \times A_2 \times \cdots \times A_n) \\
&\qquad = \mathscr{P}_x^\tau(E_\nabla \times A_2 \times \cdots \times A_n) + \mathscr{P}_x^{\tau_2}(A_2 \times \cdots \times A_n) \\
&\qquad\quad - \mathscr{P}_x^\tau(E_\nabla \times A_2 \times \cdots \times A_n) \\
&\qquad = \mathscr{P}_x^{\tau_2}(A_2 \times \cdots \times A_n) ;
\end{aligned}$$

The case $A_2 = \Delta$ is even easier since

$$\begin{aligned}
\mathscr{P}_x^\tau(E_{\nabla,\Delta} \times \Delta \times \cdots \times A_n) &= \mathscr{P}_x^\tau(E_\nabla \times \Delta \times A_n) + \mathscr{P}_x^\tau(\Delta \times \Delta \times \cdots \times A_n) \\
&= \mathscr{P}_x^\tau(\Delta \times \Delta \times \cdots \times A_n) \\
&= \mathscr{P}_x^{\tau_2}(\Delta \times \cdots \times A_n)
\end{aligned}$$

since $\Delta \times \Delta \times \cdots \times A_n$ can be of the types (2)—(4) and the equality holds in each of these cases.

Case b. Follows from the semigroup property of P_t, $t \geqq 0$ and from Case a.

Case c. If $A_{n-1} \subset E_\nabla$ then again use the semigroup property of P_t, $t \geqq 0$. If $A_{n-1} = \Delta$ then

$$(2.2)\qquad \begin{aligned} &\mathscr{P}_x^\tau(A_1 \times \cdots \times \Delta \times E_{\nabla,\Delta}) \\ &\quad = \mathscr{P}_x^\tau(A_1 \times \cdots \times \Delta \times E_\nabla) + \mathscr{P}_x^\tau(A_1 \times \cdots \times \Delta \times \Delta)\,. \end{aligned}$$

Since we consider only sets of types (1)—(4) there are two possibilities. Either all of $A_i = \Delta$, $i = 1, \cdots, n-1$, in which case the right-hand side of (2.2) equals

$$\begin{aligned} &= \tilde{P}_{t_n}(x, E_\nabla) - \tilde{P}_{t_{n-1}}(x, E_\nabla) + \lim_{t\to\infty} \tilde{P}_t(x, E_\nabla) - \tilde{P}_{t_n}(x, E_\nabla) \\ &= \mathscr{P}_x^{(t_1,\cdots,t_{n-1})}(A_1 \times \cdots \times \Delta)\,. \end{aligned}$$

The other case is that there exists an $i < n-1$ such that $A_i \subset E_\nabla$ in which case the right-hand side of (2.2) is zero. But in this case also $\mathscr{P}_x^{(t,\cdots,t_{n-1})}(A_1 \times \cdots \times \Delta)$ is zero so the desired equality holds again.

Thus $(\mathscr{P}_x^\tau, \mathscr{E}_{\nabla,\Delta}^\tau)$ constitutes a projective system of regular measures for each $x \in E_\nabla$. To prove the existence of a projective limit let us consider first a special case where $\lim_{t\to\infty} \tilde{P}_t(x, E_\nabla) < \infty$. In this case each of the measures $\mathscr{P}_x^\tau$ is a finite measure on a compact space and so by [9] the projective limit $(\mathscr{P}_x, \Omega_0)$ exists. In the general case define for each $c \geqq 0$ a family $\tilde{P}_t^c$ of kernels on $E_\nabla \times \mathscr{E}_\nabla$ by

$$(2.3)\qquad \begin{aligned} \tilde{P}_t^c(x, A) &= \tilde{P}_t(x, A) && t \leqq c \\ &= \tilde{P}_c P_{t-c}(x, A) && t > c \end{aligned}$$

and let $\mathscr{P}_x^{\tau,c}$ be defined just as $\mathscr{P}_x^\tau$ but with $\tilde{P}_t$ replaced by $\tilde{P}_t^c$. Then clearly $\tilde{P}_t^c \leqq P_t$ and $\tilde{P}_t^c(x, E_\nabla) < \infty$ for each $t \geqq 0$. Thus $\tilde{P}_t^c$, $t \geqq 0$, satisfy (i) of the theorem. As for (ii) we must show that for $s < t$, $\tilde{P}_t^c \geqq \tilde{P}_s^c P_{t-s}$. Indeed if $c < s < t$ then by (2.3) $\tilde{P}_s^c P_{t-s} = \tilde{P}_c P_{s-c} P_{t-s} = \tilde{P}_c P_{t-c} = \tilde{P}_t^c$. While if $s < c < t$ again by (ii) $\tilde{P}_t^c = \tilde{P}_c P_{t-c} \geqq \tilde{P}_s P_{c-s} P_{t-c} = \tilde{P}_s P_{t-s}$. Thus $\tilde{P}_t^c$, for each $c > 0$, satisfies all the assumptions of the theorem and in addition $\lim_{t\to\infty} \tilde{P}_t^c(x, E_\nabla) = \lim_{t\to\infty} \tilde{P}_c P_{t-c}(x, E_\nabla) \leqq \tilde{P}_c(x, E_\nabla) < \infty$. Thus by the special case there exists a projective limit $(\mathscr{P}_x^c, \Omega_0)$ of the systems $(\mathscr{P}_x^{\tau,c}, \mathscr{E}_{\nabla,\Delta}^\tau)$. Thus for each $c > 0$ we have defined a measure $\mathscr{P}_x^c$ on $(E_{\nabla,\Delta}^{[0,\infty)}, \mathscr{E}_{\nabla,\Delta}^{[0,\infty)})$. Let us show next that the measures $\mathscr{P}_x^c$, $c > 0$ are increasing with c. It is enough to show that for each $t > 0$, the measures $\tilde{P}_t^c(x, \cdot)$ are increasing with c. Indeed fix $t > 0$ and let $0 < c_1 < c_2$. There are three cases (1) $t \leqq c_1$, in this case $\tilde{P}_t^{c_1}(x, A) = \tilde{P}_t(x, A) = \tilde{P}_t^{c_2}(x, A)$. (2) $c_1 < t \leqq c_2$, in this case by (ii) $\tilde{P}_t^{c_1}(x, A) = \tilde{P}_{c_1} P_{t-c_1}(x, A) \leqq \tilde{P}_t(x, A) = \tilde{P}_t^{c_2}(x, A)$. (3) $t > c_2$, in this case again by (ii) $\tilde{P}_t^{c_1}(x, A) = \tilde{P}_{c_1} P_{t-c_1}(x, A) = \tilde{P}_{c_1} P_{c_2-c_1} P_{t-c_2}(x, A) \leqq \tilde{P}_{c_2} P_{t-c_2}(x, A) = \tilde{P}_t^{c_2}(x, A)$. It is clear by construction that if $\tilde{P}_t^c$ are increasing with c then so are $P_x^{\tau,c}$ and thus also the projective limit $\mathscr{P}_x^c$ is increasing with c. Thus the measure $\mathscr{P}_x = \lim_{c\to\infty} \mathscr{P}_x^c$ is well defined on Ω_0 with the product topology. $\mathscr{P}_x$ is regular and is clearly the projective limit of the system $(\mathscr{P}_x^\tau, \mathscr{E}_{\nabla,\Delta}^\tau)$. Define a subset $\Omega \subset \Omega_0$ consisting of the functions $\omega(t) \in \Omega_0$ satisfying $\omega(t) = \nabla$ implies $\omega(s) = \nabla$ for $s \geqq t$ and $\omega(t) = \Delta$ implies $\omega(s) = \Delta$ $s \leqq t$. Define a process $\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ on a measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$ by letting $\xi_t(\omega) = \omega(t)$ for each $\omega \in \Omega$, $t \geqq 0$ and $\mathscr{F}_t = \sigma(\xi_s : s \leqq t)$ (the σ-algebra generated by the ξ_s, $s \leqq t$), $\mathscr{F} = \sigma(\xi_s : s \geqq 0)$. $\mathscr{P}_x$ is a σ-finite measure. Indeed

$$\Omega = \bigcup_{j=1}^{\infty} \{\xi(j) \in E_\nabla\} \cup \omega_\Delta$$

where ω_Δ is the point in Ω all of whose coordinates equal Δ. Further $\mathscr{P}_x(\xi(j) \in E_\nabla) = \tilde{P}_j(x, E_\nabla) < \infty$ for each $j \geqq 1$, and $\mathscr{P}_x(\omega_\Delta) = \lim \mathscr{P}_x^c(\omega_\Delta)$

$$\mathscr{P}_x^c(\omega_\Delta) \leqq \lim_{j\to\infty} \tilde{P}_x^c(\xi(j) = \Delta) = \lim_{n\to\infty} (\lim_{t\to\infty} \tilde{P}_t^c(x, E_\nabla) - \tilde{P}_n^c(x, E_\nabla)) = 0 .$$

Thus it remains to prove only the Markov property (2.1).

It is enough to show that for any $t_1 < \cdots < t_n < t$, any $A_i \in \mathscr{E}_{\nabla,\Delta}$, $i = 1, \cdots, n$ and any $A \in \mathscr{E}_\nabla$

$$\mathscr{P}_x(\{\xi_{s+t} \in A\} \cap D) = \int_D P_s(\xi_t, A)\, d\mathscr{P}_x$$

where $D = \{\xi_{t_1} \in A_1, \cdots, \xi_{t_n} \in A_n, \xi_t \in E_\nabla\}$. But this follows immediately from the definition of $\mathscr{P}_x^\tau$, $\tau = (t_1, \cdots, t_n, t, t+s)$. This completes the proof.

REMARK 2.2. Actually condition (ii) is also a necessary condition for the existence of a Markov process with creation and annihilation satisfying (1.2). Indeed we only have to notice that if there exists a Markov process with creation and annihilation $\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ on $(\Omega, \mathscr{F}, \mathscr{P}_x)$ then by the Markov property

$$\begin{aligned}\tilde{P}_{s+t}(x, A) &= \mathscr{P}_x(\xi_{s+t} \in A) \geqq \mathscr{P}_x(\xi_s \in E_\nabla, \xi_{s+t} \in A) \\ &= \int_{\{\xi_s \in E_\nabla\}} \mathscr{P}_x(\xi_{s+t} \in A \mid \xi_s \in E_\nabla)\, d\mathscr{P}_x \\ &= \tilde{P}_s P_t(x, A) .\end{aligned}$$

Regularity properties of the process, the strong Markov property and quasi-left continuity can be proved as in [3].

The following is a special case of the theorem.

COROLLARY 2.3. *Let P_t, $t \geqq 0$, be a Markov transition function on $E_\nabla \times \mathscr{E}_\nabla$ and let $\tilde{P}_t$, $t \geqq 0$, be a finite quasi-transition function on $E_\nabla \times \mathscr{E}_\nabla$ satisfying $\tilde{P}_t \geqq P_t$; then the same results of Theorem* 2.1 *hold.*

Another corollary of the theorem is the main theorem in [3]. For that denote by $\mathscr{B} = \mathscr{B}[0, \infty)$ the σ-algebra of Borel subsets of $[0, \infty)$. We then have

COROLLARY 2.4. *Let P_t, $t \geqq 0$, be a Markov transition function on $E_\nabla \times \mathscr{E}_\nabla$. Let $\phi_x(ds, dy)$, $x \in E_\nabla$, be a measure on $\mathscr{B} \times \mathscr{E}_\nabla$ satisfying condition* (B) *of* [3], i.e. (a) *For each $A \in \mathscr{B} \times \mathscr{E}_\nabla$, $\phi_.(A)$ is measurable,* (b) *$\phi_x(M \times [0, t)) < \infty$ for each $x \in E_\nabla$, $t > 0$, and* (c) *$\phi_x(M \times \{0\}) = 0$. Let $\tilde{P}_t$, $t \geqq 0$, be a family of kernels on $E_\nabla \times \mathscr{E}_\nabla$ defined as follows, for $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$*

$$\tilde{P}_t(x, A) = P_t(x, A) + \int_0^t \int_{E_\nabla} P_s(y, A)\phi_x(ds, dy) . \tag{2.4}$$

Then P_t, $\tilde{P}_t$, $t \geqq 0$, determine a Markov process with creation and annihilation $(\xi_t, \mathscr{F}_t : t \geqq 0)$ defined on a σ-finite measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$ satisfying $\mathscr{E}_x f(\xi_t) = \tilde{P}_t f(x)$, where $\mathscr{E}_x$ denotes expected values relative to P_x.

PROOF. We only have to show that $\tilde{P}_t$, $t \geqq 0$ defined by (2.4) satisfies (i)—(ii) of Theorem 2.1. (i) is clear since $\tilde{P}_t(x, E_\nabla) = P_t(x, E_\nabla) + \phi_x((0, t) \times E_\nabla) < \infty$ by (b). As for (ii) notice first that for any f bounded and $\mathscr{E}_\nabla$ measurable on E_∇ $\tilde{P}_t f(x) = P_t f(x) + \int_0^t \int_{E_\nabla} P_{t-r} f(y) \phi_x(dr, dy)$.

Thus for $s \leqq t$

$$\begin{aligned}\tilde{P}_s P_{t-s}(x, A) &= P_s P_{t-s}(x, A) + \int_0^s \int_{E_\nabla} P_{s-r} P_{t-s}(y, A) \phi_x(dr, dy) \\ &\leqq P_t(x, A) + \int_0^t \int_{E_\nabla} P_{t-r}(y, A) \phi_x(ds, dy) \\ &= \tilde{P}_t(x, A) .\end{aligned}$$

Thus condition (ii) holds and the rest follows from the theorem.

3. Relations to perturbation theory of Markov processes. As we mentioned earlier the method of constructing a Markov process with creation and annihilation through its transition function P_t, $t \geqq 0$, and the family $\tilde{P}_t$, $t \geqq 0$ is also more convenient for applications to the theory of perturbations of infinitesimal generators. Indeed let $\{T_t : t \geqq 0\}$ be a strongly continuous family of contraction operators on $\mathscr{C}_0(E)$. Let P_t, $t \geqq 0$, be the sub-Markov family of kernels on $E \times \mathscr{E}$ associated with it, extended to $E_\nabla \times \mathscr{E}_\nabla$. Extend $\{T_t : t \geqq 0\}$ to $\mathscr{C}_0(E_\nabla)$ by $T_t f(\nabla) = 0$. Let $(X_t, \mathscr{G}_t : t \geqq 0)$ be a Markov process on the probability space $(\Omega, \mathscr{G}, \mathscr{P}_x)$, $x \in E_\nabla$, having P_t, $t \geqq 0$, as its transition function. Let A be its infinitesimal generator, and let $\mathscr{D}$ be its domain in $\mathscr{C}_0(E_\nabla)$. Perturbation theory deals with the following type of problem, (for definitions, see [1], [4]). Given another closed operator B with domain containing $\mathscr{D}$, under what conditions will $A + B$ be an infinitesimal generator of a semigroup of bounded operators on $\mathscr{C}_0(E_\nabla)$? From a probabilistic point of view we can continue and ask is there some Markov process (maybe in a general sense) corresponding to the new semigroup and how is it related to the original Markov process. [3], [8] and [10] gave some partial results to this problem. Using our construction, we can generalize these results in the following way. If $A + B$ generates a semigroup of bounded operators $\{S_t : t \geqq 0\}$ on $\mathscr{C}_0(E_\nabla)$ and if $S_t \geqq T_t$ then clearly if we denote by $\tilde{P}_t$, $t \geqq 0$, the semigroup of kernels on $E_\nabla \times \mathscr{E}_\nabla$ associated with $\{S_t : t \geqq 0\}$, then $\tilde{P}_t$, $t \geqq 0$ satisfies all the conditions of Corollary 2.4 and thus there is a Markov process with creation and annihilation associated with it by (1.2). This was indeed the case in [3] and [7]. Using a theorem of Phillips [1], [4] we can, for example, state the following result.

THEOREM 3.1. *Let $\{T_t : t \geqq 0\}$ be a strongly continuous contraction semigroup of operators on $\mathscr{C}_0(E)$ with $T_0 = I$, let P_t, $t \geqq 0$ be its corresponding semigroup of kernels on $E \times \mathscr{E}$ extended to $E_\nabla \times \mathscr{E}_\nabla$. Let A with domain $\mathscr{D}$ in $\mathscr{C}_0(E_\nabla)$ be its infinitesimal generator. Let B be a nonnegative closed operator whose domain contains $\mathscr{D}$ and which satisfies the conditions that for each $t > 0$ there is a constant $K_t < \infty$ such that $\|BT_t\| \leqq K_t$ where K_t can be chosen so that it has a finite integral over* $(0, 1]$. *Then*

(i) *there exists a strongly continuous semigroup $\{S_t : t \geqq 0\}$ of operators on $\mathscr{C}_0(E_\nabla)$ with $S_t \geqq T_t$, whose infinitesimal generator is $A + B$ and*

(ii) *there exists Markov process with creation of mass* $(\xi_t, \mathscr{F}_t: t \geqq 0)$ *defined on a σ-finite measure space* $(\Omega, \mathscr{F}, \mathscr{P}_x)$ *having transition function* P_t, $t \geqq 0$, *such that if* $\tilde{P}_t$, $t \geqq 0$, *is the family of kernels on* $E_\nabla \times \mathscr{E}_\nabla$ *corresponding to* $\{S_t: t \geqq 0\}$, *then*

(a) $S_t f(x) = \mathscr{E}_x f(\xi_t)$ *for each* $f \in \mathscr{C}_0(E_\nabla)$.
(b) $\lim_{t\to 0} \tilde{P}_t(x, A) = \delta_x(A)$ ($\delta_x(A) = 1$ *or* 0 *according to whether* $x \in A$ *or not*).
(c) $\lim_{t\to 0+} [\mathscr{E}_x f(\xi_t) - f(x)]/t = Af(x) + Bf(x), f \in 0$.

In other words we found a process whose infinitesimal generator is $A + B$. This process is related to the original process by having the same transition function. Obviously this gives us a probabilistic method for solving the following Cauchy problem in partial differential equations. Given $f \in \mathscr{C}_0(E_\nabla)$ find a solution $W(t, x)$, $t \in [0, \infty)$, $x \in E_\nabla$, satisfying

$$\frac{\partial W}{\partial t} = AW + BW \tag{3.1}$$

$$W|_{t=0} = f\,.$$

The above results thus imply that if we have a probabilistic solution for (3.1) with $B = 0$ of the form $W(t, x) = E_x f(X_t)$, for some Markov process $(X_t, \mathscr{G}_t, \mathscr{G}, \theta_t, P_x)$. Then we can find a probabilistic solution $W(t, x) = \mathscr{E}_x f(\xi_t)$ of (3.1) for some Markov process with creation of mass $(\xi_t, \mathscr{F}_t, \mathscr{F}, \theta_t, \mathscr{P}_x)$. The relations between these two processes are as mentioned above.

A well-known special case is the case where $E = R^N$ (the N-dimensional Euclidean space), $A = \frac{1}{2}\sum_{i=1}^N (\partial^2/\partial x_i^2)$ and $B = u \times I$ for some $u \in \mathscr{C}_0(E_\nabla)$ (see [3], [10]). In this case there are other probabilistic interpretations of (3.1) such as, for example, branching Markov processes; see [11].

As noted in [8], B need not be nonnegative. It can satisfy a weaker comparison criterion. Theorems similar to Theorem 3.1 can easily be formulated using different theorems from the theory of perturbations of semigroups, (see [7]).

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