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# CONDITIONS FOR CONTINUITY OF RANDOM PROCESSES WITHOUT DISCONTINUITIES OF THE SECOND KIND

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In this paper it is shown that the classical local conditions for sample continuity of random processes without discontinuities of the second kind may be replaced by weaker ones of a simpler form. When such a process is obtained as the limit in distribution in the Skorohod topology, the new conditions may be conveniently restated in terms of the sequence.

**1. Introduction.** Continuity of sample functions is usually discussed under the assumption of separability. The standard results for this case are the sufficient conditions of Loève ([3], pages 515–519). In the particular case of sample functions in $D$-spaces, i.e. spaces of right-continuous functions with left-hand limits, it is possible to weaken the general conditions. In fact, a random process $\xi$ in $D[0, 1]$ is known (see [3], page 515 or [1], page 136) to be a.s. sample continuous if

$$\lim_{h\to 0+} \sup_{t\in[0,1-h]} \frac{1}{h} P(|\xi_{t+h} - \xi_t| > \delta) = 0\,, \qquad \delta > 0\,. \tag{1}$$

Although sample continuity is a local property in the sense that a.s. continuity in some interval $T$ only depends on the distribution of $\{\xi_t, t \in T\}$, it is yet impossible (cf. Section 3) to deduce (1) from the condition

$$\lim_{h\to 0+} \frac{1}{h} P(|\xi_{t+h} - \xi_{t-}| > \delta) = 0\,, \qquad \delta > 0 \text{ and all } t\,. \tag{2}$$

Nevertheless, (2) turns out to be sufficient for sample continuity in $D$-spaces. By symmetry we can replace (2) by

$$\lim_{h\to 0+} \frac{1}{h} P(|\xi_{t+} - \xi_{t-h}| > \delta) = 0\,, \qquad \delta > 0 \text{ and all } t\,, \tag{3}$$

and we shall show that even

$$\liminf_{h\to 0+} \frac{1}{h} P(|\xi_{t+h} - \xi_{t-h}| > \delta) = 0\,, \qquad \delta > 0 \text{ and all } t\,, \tag{4}$$

suffices. Note that (2) can be replaced by the stronger moment condition

$$\lim_{h\to 0+} \frac{1}{h} E\,|\xi_{t+h} - \xi_{t-}|^\alpha = 0 \qquad \text{for all } t\,, \tag{5}$$

where $\alpha > 0$ is arbitrary, and similarly for (3) and (4). Throughout, $t + h$ and

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$t+$ should be replaced by $t$ at closed right end-points, and similarly at closed left end-points.

Further weaking of the conditions is possible. In fact, the event $\{|\xi_{t+h} - \xi_{t-}| > \delta\}$ in (2) may be replaced by events of the type $\{\xi_{t-} \in I_1, \xi_{t+h} \in I_2\}$, where $I_1$ and $I_2$ are any disjoint compact intervals, and similarly for (3) and (4). In particular this means that the moments in (5) may be replaced by truncated moments, provided the truncation is arbitrary. (By this we mean that the random variable $|\xi_{t+h} - \xi_{t-}|^\alpha$ in (5) may be truncated at an arbitrary level, before we take the expectation.) Finally, we permit bi-continuous transformations of "time". All these improvements have resulted in Theorem 1.

In typical applications, the random process $\xi$ under consideration is obtained as a limit in distribution of some sequence $\{\xi_n\}$ of processes. It would then be convenient to use conditions for sample continuity of $\xi$ stated in terms of the distributions of $\xi_1, \xi_2, \cdots$. Such conditions are given in Theorem 2.

In Section 3 we show that, when applicable, our results are always sharper than the classical ones for separable processes. We also prove the impossibility of giving necessary and sufficient conditions for sample continuity in terms of upper bounds on $P(\xi_s \in I_1, \xi_t \in I_2)$.

**2. Main results.** Let $D(a, b)$, $-\infty \leqq a < b \leqq \infty$, denote the space of real-valued right-continuous functions with left-hand limits defined on $(a, b)$, and similarly for semi-closed and closed intervals. For simplicity, our results are stated and proved for open intervals, but they are easily seen to be generally true provided $t + h$ or $t+$ is replaced by $t$ at closed right end-points, and similarly for closed left end-points.

Our random processes $\xi$ are assumed to be measurable with respect to the $\sigma$-algebra generated by all one-dimensional projections $\xi_t = \xi(t)$. Thus the random elements in $D[0, 1]$ coincide with those defined in [1], while the elements in $D[0, \infty)$ (and more generally in $D[a, b)$, $D(a, b]$ or $D[a, b]$) are those of [2], [4]. We now state our main result.

THEOREM 1. *Let $\xi$ be a random process in $D(a, b)$, $-\infty \leqq a \leqq b \leqq \infty$, and let $g$ be an arbitrary strictly increasing and continuous function on $(a, b)$. Then $\xi$ is* a.s. *continuous on $(a, b)$ if any one of the following relations holds for all $t \in g^{-1}(a, b)$ and all disjoint compact intervals $I_1, I_2$:*

(i) $\lim_{h\to 0+} \dfrac{1}{h} P((\xi \circ g)_{t-} \in I_1, (\xi \circ g)_{t+h} \in I_2) = 0,$

(ii) $\lim_{h\to 0+} \dfrac{1}{h} P((\xi \circ g)_{t-h} \in I_1, (\xi \circ g)_{t+} \in I_2) = 0,$

(iii) $\liminf_{h\to 0+} \dfrac{1}{h} P((\xi \circ g)_{t-h} \in I_1, (\xi \circ g)_{t+h} \in I_2) = 0.$

A lemma is needed for the proof.

LEMMA 1. *Let $T = (a, b)$, $-\infty \leqq a < b \leqq \infty$, and let $\eta$ be a $T$-valued random* *variable. Then*

(i) $\sup_{t\in T} \limsup_{h\to 0+} \frac{1}{h} P(t \leqq \eta \leqq t + h) > 0$,

(ii) $\sup_{t\in T} \limsup_{h\to 0+} \frac{1}{h} P(t - h \leqq \eta \leqq t) > 0$,

(iii) $\sup_{t\in T} \liminf_{h\to 0+} \frac{1}{h} P(t - h \leqq \eta \leqq t + h) > 0$.

PROOF. Let $c, d \in T$ be such that $p = P(c < \eta < d) > 0$, and let $G$ be the function defined by

$$G(x) = P(c < \eta \leqq x) - p\frac{x - c}{d - c}, \qquad c < x < d .$$

Note that $G(c+) = G(d-) = 0$. Suppose that (i) is false. Then

$$\sup_{t\in(c,d)} \limsup_{h\to 0+} \frac{1}{h} (G(t + h) - G(t-)) < 0 ,$$

so for each $t \in (c, d)$ there must exist some $\gamma_t > 0$ with

$$G(t + h) - G(t-) < 0 , \qquad h \in (0, \gamma_t) ,$$

and this in turn implies

$$G(t + h) - G(t - k) < 0 , \qquad h, k \in (0, \delta_t) ,$$

for some $\delta_t \in (0, \gamma_t]$. Since any compact sub-interval of $(c, d)$ has a finite covering of intervals $(t - \delta_t, t + \delta_t)$, we thus obtain

$$G(t) - G(s) < 0 , \qquad c < s < t < d ,$$

which yields the contradiction $G(d-) < G(c+)$. This proves (i), while (ii) follows by the symmetric argument.

To prove (iii) we construct a sequence $T_1 \supset T_2 \supset \cdots$ of sub-intervals of $T_0 = (c, d)$ by successive division into equal parts and choosing in each step the part with greatest probability for $\eta$. Thus $|T_n| = 2^{-n}|T_0|$ and $P(\eta \in T_n) \geqq 2^{-n}p$. Let $t \in T$ be defined by $T_n^- \downarrow \{t\}$, and let $h \in (0, |T_0|)$ be arbitrary. If $n \in N$ is such that $2^{-n} \leqq h/|T_0| < 2^{-n+1}$, then

$$\frac{1}{h} P(t - h \leqq \eta \leqq t + h) \geqq \frac{1}{h} P(\eta \in T_n) > 2^{n-1}|T_0|^{-1} 2^{-n}p = \frac{p}{2\,|T_0|} ,$$

and (iii) follows.

For the proof of Theorem 1 (and 2) we shall need a "modulus of continuity" $w_x$, $x \in D[a', b']$, defined by

(6) $$w_x(h) = \inf_{\{t_j\}} \max_{0<j\leqq r} \sup\{|x(u) - x(v)| : u, v \in [t_{j-1}, t_j)\} ,$$

where the infimum extends over finite sets $\{t_j\}$ of points satisfying

$$a' = t_0 < t_1 < \cdots < t_r = b' ; \qquad t_j - t_{j-1} > h , \qquad j = 2, 3, \cdots, r - 1 ,$$

(cf. the definition of $w_x'$ in [1], page 110). Note that $\lim_{h\to 0} w_x(h) = 0$ for all $x \in D[a', b']$, (cf. [2]).

PROOF OF THEOREM 1. Since $\xi$ is continuous whenever $\xi \circ g$ is, it suffices to assume that $g(t) \equiv t$. If $\xi$ is not a.s. continuous in $(a, b)$, then there must exist real $a', b'$ with $a < a' < b' < b$ and disjoint compact intervals $J_1, J_2$ such that $\xi$ with positive probability has at least one jump from $J_1$ to $J_2$ in $T = (a', b')$. Let $A$ denote this event, let $\delta > 0$ be the length of the gap between $J_1$ and $J_2$ and choose $d > 0$ such that $P(w_\xi(d) > \delta/3) < P(A)$, where $w_\xi$ is defined by (6). Note that the event $B = \{A, w_\xi(d) \leqq \delta/3\}$ has then positive probability.

Given $B$, we now choose one of the jumps from $J_1$ to $J_2$ in $T$ at random and denote its position by $\eta$. (The measurability of $\eta$ is easily verified.) By Lemma 1 there exists some $t \in T$ with

$$\limsup_{h\to 0+} \frac{1}{h} P(t \leqq \eta \leqq t + h \mid B) > 0 .$$

Let $I_1, I_2$ be obtained from $J_1, J_2$ by addition of intervals of length $\delta/3$ at each end. By definition of $w_\xi$ we then get for $h < d$ and $t + h < b'$

$$P(\xi_{t-} \in I_1, \xi_{t+h} \in I_2) \geqq P(t \leqq \eta \leqq t + h, B) = P(t \leqq \eta \leqq t + h \mid B)P(B) ,$$

and so

$$\limsup_{h\to 0+} \frac{1}{h} P(\xi_{t-} \in I_1, \xi_{t+h} \in I_2) > 0 .$$

But this contradicts (i) and hence proves that (i) is in fact sufficient. The sufficiency of (ii) and (iii) is proved in the same way.

We now turn to the corresponding limit theorem. Let us write $\xi_n \to_d \xi$ for convergence in distribution of $\xi_n$ to $\xi$ in the sense of [1]. For convenience we refer to [1] for some simple facts about weak convergence in $D$-spaces, though only the space $D[0, 1]$ is treated in [1]. For extensions to the general case, see [2], [4].

THEOREM 2. *Let $\xi, \xi_1, \xi_2, \cdots$ be random processes in $D(a, b)$, $-\infty \leqq a < b \leqq \infty$, such that $\xi_n \to_d \xi$ and let $g$ be an arbitrary strictly increasing and continuous function on $(a, b)$. Then $\xi$ is* a.s. *continuous if*

$$\text{(i)} \qquad \liminf_{h\to 0+} \liminf_{n\to\infty} \frac{1}{h} P((\xi_n \circ g)_{t-h} \in I_1, (\xi_n \circ g)_{t+h} \in I_2) = 0$$

*for all $t \in g^{-1}(a, b)$ and all disjoint compact intervals $I_1, I_2$. If $P(\xi_{t-} \neq \xi_{t+}) = 0$ for all $t \in (a, b)$, then $\xi$ is* a.s. *continuous if anyone of the following relations holds for any $t$, $I_1$ and $I_2$ chosen as above*:

$$\text{(ii)} \qquad \lim_{h\to 0+} \liminf_{n\to\infty} \frac{1}{h} P((\xi_n \circ g)_t \in I_1, (\xi_n \circ g)_{t+h} \in I_2) = 0 ,$$

$$\text{(iii)} \qquad \lim_{h\to 0+} \liminf_{n\to\infty} \frac{1}{h} P((\xi_n \circ g)_{t-h} \in I_1, (\xi_n \circ g)_t \in I_2) = 0 .$$

To use (ii) or (iii) we need an efficient criterion for $P(\xi_{t-} \neq \xi_{t+}) = 0$ to hold at $t$. Such a criterion is given in the following lemma which is also needed for the proof of the first assertion.

LEMMA 2. *Let $\xi, \xi_1, \xi_2, \cdots$ be random processes in $D(a, b)$, $-\infty \leqq a < b \leqq \infty$, with $\xi_n \to_d \xi$ and let $t \in (a, b)$. Then $P(\xi_{t-} \neq \xi_{t+}) = 0$ if and only if*

$$(7) \qquad \liminf_{h,k \to 0+} \liminf_{n\to\infty} P(\xi_n(t-h) \in I_1, \xi_n(t+k) \in I_2) = 0$$

*for all disjoint compact intervals $I_1$ and $I_2$.*

PROOF. Let $J_1$ and $J_2$ be any disjoint compact intervals and let $\delta$, $I_1$ and $I_2$ be defined as in the proof of Theorem 1. For $h, k > 0$, let $A_{hk} \subset D(a, b)$ be the set of functions with at least one jump from $J_1^0$ to $J_2^0$ in $(t-h, t+k)$. Since $A_{hk}$ is obviously open in the Skorohod topology, we get, recalling the definition of $w_\xi$ in (6),

$$\begin{aligned} P(\xi \in A_{hk}) &\leqq \liminf_{n\to\infty} P(\xi_n \in A_{hk}) \\ &\leqq \liminf_{n\to\infty} [P(w_{\xi_n}(h+k) > \delta/3) \\ &\qquad + P(\xi_n \in A_{hk}, w_{\xi_n}(h+k) \leqq \delta/3)] \\ &\leqq \limsup_{n\to\infty} P(w_{\xi_n}(h+k) > \delta/3) \\ &\qquad + \liminf_{n\to\infty} P(\xi_n \in A_{hk}, w_{\xi_n}(h+k) \leqq \delta/3) , \end{aligned}$$

where $w_x$ is defined with respect to some fixed interval $[a', b'] \subset (a, b)$ and where we assume $a' < t-h < t+k < b'$. Now the first term on the right tends to zero as $h, k \to 0$ (this follows easily from Theorem 15.2 in [1]), while the second term at most equals

$$\liminf_{n\to\infty} P(\xi_n(t-h) \in I_1, \xi_n(t+k) \in I_2) ,$$

so if (7) holds we get

$$P(\xi_{t-} \in J_1^0, \xi_{t+} \in J_2^0) = P(\textstyle\bigcap_{h,k} \{\xi \in A_{hk}\}) = \lim_{h,k\to 0+} P(\xi \in A_{hk}) = 0 .$$

Since $J_1$ and $J_2$ were arbitrary, this proves that $P(\xi_{t-} \neq \xi_{t+}) = 0$.

Suppose conversely that $P(\xi_{t-} \neq \xi_{t+}) = 0$, and choose $h, k > 0$ such that

$$(8) \qquad P(\xi_{t-h-} \neq \xi_{t-h+}) = P(\xi_{t+k-} \neq \xi_{t+k+}) = 0 .$$

Then

$$\limsup_{n\to\infty} P(\xi_n(t-h) \in I_1, \xi_n(t+k) \in I_2) \leqq P(\xi_{t-h} \in I_1, \xi_{t+k} \in I_2)$$

for any disjoint compact intervals $I_1$ and $I_2$ (cf. [1], page 124). Now $\xi_{t-h} \to \xi_{t-}$ and $\xi_{t+k} \to \xi_{t+}$ as $h, k \to 0$, which implies $(\xi_{t-h}, \xi_{t+k}) \to_d (\xi_{t-}, \xi_{t+})$, so we get

$$\limsup_{h,k\to 0+} P(\xi_{t-h} \in I_1, \xi_{t+k} \in I_2) \leqq P(\xi_{t-} \in I_1, \xi_{t+} \in I_2) \leqq P(\xi_{t-} \neq \xi_{t+}) = 0 .$$

Since $h$ and $k$ in (8) can be chosen arbitrarily close to zero ([1], page 124), this completes the proof of (7).

PROOF OF THEOREM 2. If (i) holds, then $P(\xi_{t-} \neq \xi_{t+}) = 0$, $t \in (a, b)$, by Lemma 2, and hence

$$((\xi_n \circ g)_{t-h}, (\xi_n \circ g)_{t+h}) \to_d ((\xi \circ g)_{t-h}, (\xi \circ g)_{t+h}) .$$

From (i) we thus obtain

$$\liminf_{h\to 0+} \frac{1}{h} P((\xi \circ g)_{t-h} \in I_1^0, (\xi \circ g)_{t+h} \in I_2^0) = 0 ,$$

and the a.s. continuity of $\xi$ follows by Theorem 1. The second assertion is proved by similar arguments.

**3. Comments on the sharpness.** In this section we compare our results with the classical ones and also discuss the possibilities of further improvements.

First we show that (2) is actually weaker than (1). If we put

$$p_n = n^{-2}(\textstyle\sum_1^\infty k^{-2})^{-1} , \qquad t_n = \sum_{k=n}^\infty p_k - p_n/2 , \qquad n \in N ,$$
$$x_n(t) = (1 - 2\,|t - t_n|/p_n) \vee 0 , \qquad n \in N, t \in [0, 1] ,$$

and let $\xi$ be defined by $P(\xi = x_n) = p_n$, $n \in N$, then (2) is easily verified. But (1) cannot hold since for $t = t_n$ and $h = p_n/2$, $n \in N$, we have

$$h^{-1}P(|\xi_{t+h} - \xi_t| > \delta) = 2 , \qquad \delta \in (0, 1) .$$

It is interesting to notice that even (1) is weaker than the general conditions for separable processes. To see this, suppose that the condition

$$P(|\xi_{t+h} - \xi_t| \geqq g(h)) \leqq q(h) , \qquad a \leqq t < t + h \leqq b ,$$

is satisfied for some non-decreasing functions $g$ and $q$ with

$$\textstyle\sum_n g(2^{-n}) < \infty , \qquad \sum_n 2^n q(2^{-n}) < \infty ,$$

(cf. [3], page 517). Let $\delta > 0$ be arbitrary and choose $n = n(h) \in N$ such that

$$(h2^n + 1)g(2^{-n}) \leqq \delta , \qquad \lim_{h\to 0} h2^n = \infty .$$

Putting $m = [h2^n] + 1$, we get

$$\begin{aligned} \sup_t P(|\xi_{t+h} - \xi_t| > \delta) &\leqq m \sup_t P(|\xi_{t+h/m} - \xi_t| > \delta/m) \\ &\leqq m \sup_t P(|\xi_{t+h/m} - \xi_t| > g(h/m)) \\ &\leqq mq(h/m) \leqq (h2^n + 1)q(2^{-n}) \\ &\sim h2^n q(2^{-n}) = o(h) , \qquad h \to 0 , \end{aligned}$$

and (1) follows. This means that no improvement is attainable by using the general conditions for separable processes.

Similar results hold for the moment conditions. In particular, it is seen that for processes in $D$-spaces the general condition

$$E\,|\xi_{t+h} - \xi_t|^r \leqq c\,|h|/|\log|h||^{1+s} , \qquad a \leqq t < t + h \leqq b ,$$

for some $c > 0$ and $s > r > 0$, (cf. [3], page 519), may be weakened in four respects: (1) The denominator $|\log|h||^{1+s}$ may be omitted, (2) no uniformity in $t$ is required (provided $\xi_t$ is replaced by $\xi_{t-}$), (3) moments may be replaced by truncated moments, and (4) "time" may be transformed bi-continuously.

Finally, we show that the conditions of Theorem 1 are not necessary for sample continuity and that in fact no condition of this type can be both necessary and sufficient. This is hardly surprising since in general the distribution of $\xi$ is not determined by its two-dimensional projections. To prove the above assertion it suffices to construct a pair of random processes $\xi$ and $\eta$ in $D[0, 1]$ such that $\xi$ is a.s. sample continuous while $\eta$ is not, and yet

$$P(\eta_s \in I_1, \eta_t \in I_2) \leqq P(\xi_s \in I_1, \xi_t \in I_2)$$

for arbitrary $s, t \in [0, 1]$ and disjoint compact intervals $I_1, I_2$. In fact, we may even prove the stronger

THEOREM 3. *For each $\varepsilon > 0$ there exist random processes $\xi$ and $\eta$ in $D[0, 1]$ such that $\xi$ is* a.s. *continuous while $\eta$ is not, and moreover,*

$$\left| \frac{P(\eta_s \in B_1, \eta_t \in B_2)}{P(\xi_s \in B_1, \xi_t \in B_2)} - 1 \right| < \varepsilon$$

*for all $s, t \in [0, 1]$ and disjoint Borel sets $B_1, B_2$. (Here $0/0$ is to be interpreted as $1$.)*

PROOF. Let $f$ be the function on $R$ with period 1 satisfying

$$f(x) = ((4|x| - \tfrac{1}{2}) \vee 0) \wedge 1\,, \qquad |x| \leqq \tfrac{1}{2}\,,$$

and put

$$\xi(t) = f(\alpha + \beta t)\,, \qquad t \in [0, 1]\,,$$

where $\alpha$ and $\beta$ are independent random variables such that $\alpha$ is uniformly distributed on $[0, 1]$ while $\beta$ has density $(4x^2)^{-1}$, $x > \frac{1}{4}$. By elementary calculations we get

$$P(\xi_t = 0, \xi_{t+h} = 1 \mid \beta) = g(h\beta)\,, \qquad t, t + h \in [0, 1]\,,$$

where $g$ has period 1 and satisfies

$$g(x) = (|x| - \tfrac{1}{4}) \vee 0\,, \qquad |x| \leqq \tfrac{1}{2}\,.$$

For $h \in (0, 1)$ we thus obtain

$$P(\xi_t = 0, \xi_{t+h} = 1) = Eg(h\beta) = \tfrac{1}{4} \textstyle\int_{\frac{1}{4}}^{\infty} g(hx)x^{-2}\,dx = \frac{h}{4} \int_{\frac{1}{4}}^{\infty} g(y)y^{-2}\,dy = \gamma h\,,$$

where $\gamma$ is a constant $< 1$. If $\zeta$ is defined by

$$\zeta(t) = 1_{\{\gamma t \geqq \alpha\}}\,, \qquad t \in [0, 1]\,,$$

then

$$P(\zeta_t = 0, \zeta_{t+h} = 1) = \gamma h\,, \qquad 0 \leqq t < t + h \leqq 1\,,$$

and hence

$$P(\zeta_s \in B_1, \zeta_t \in B_2) \leqq P(\xi_s \in B_1, \xi_t \in B_2)$$

for any $s, t \in [0, 1]$ and disjoint Borel sets $B_1, B_2$. To prove the assertion it thus suffices to put $\eta = \zeta$ with probability $\varepsilon$ and $\eta = \xi$ with probability $1 - \varepsilon$, independently of $\alpha$ and $\beta$.

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