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# ABSOLUTE CONTINUITY OF MEASURES CORRESPONDING TO DIFFUSION PROCESSES IN BANACH SPACE[1]

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This paper is concerned with the stochastic processes in Banach space arising from the solutions of stochastic integral equations. It is shown that under certain assumptions the corresponding measures of two such stochastic processes are equivalent. The Radon-Nikodym derivative is also given.

**1. Introduction.** Let $B$ be a real separable Banach space with norm $\|\cdot\|$. Gross constructs a subset $H$ of $B$ which is a real separable Hilbert space with norm $|\cdot|$ and inner product $\langle\ ,\ \rangle$, such that $\|\cdot\|$ is measurable over $H$. Thus any real separable Banach space $B$ can be regarded as a couple $(H, B)$. Such a couple is called an abstract Wiener space. It carries a family of Wiener measures. For more details see Gross [1]. It is known that a real separable Banach space $B$ with a Gaussian measure $m$ is an abstract Wiener space $(H, B)$ and $m$ is a Wiener measure [2].

Let $(H, B)$ be an abstract Wiener space. The dual space $B^*$ of $B$ is imbedded in $B$ in the natural way $B^* \subset H^* \approx H \subset B$.

In this paper we assume that there is an increasing sequence $Q_n$ of finite dimensional projections such that $Q_n(B) \subset B^*$ and $Q_n$ converges strongly to the identity both in $B$ and in $H$. Let $\Omega$ denote the space of continuous functions $\omega$ from $[0, \infty)$ into $B$ with $\omega(0) = 0$. Then there is a unique probability measure $\mathscr{P}$ on the $\sigma$-field generated by the coordinate functions such that $W(t, \omega) = \omega(t)$ is the process with Wiener measures of $(H, B)$ as its transition probabilities. $W(t)$ is called a Wiener process in $B$. $\mathscr{E}$ will denote the expectation with respect to $(\Omega, \mathscr{P})$.

Let $A$ be a map from $[0, 1] \times B$ into $\mathscr{B}(B, B)$ ($\mathscr{B}(E, F)$ denotes the Banach space of all bounded operators from $E$ into $F$ with operator norm $\|\cdot\|_{E,F}$) such that $(A(t, x) - I)(B) \subset B^*$ for all $t \in [0, 1]$ and all $x \in B$. Let $\sigma$ be a map from $[0, 1] \times B$ into $B$. Assume that $A$ and $\sigma$ satisfy the following conditions.

(*) (i) For each $x$ in $B$ the maps $A(\cdot, x) - I: [0, 1] \to \mathscr{B}(B, B^*)$ and $\sigma(\cdot, x): [0, 1] \to B$ are continuous.

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(ii) There exists a constant $c$ such that for all $t$ in $[0, 1]$ and $x$, $y$ in $B$

$$\|A(t, x) - A(t, y)\|_2 \leqq c\|x - y\|$$
$$\|\sigma(t, x) - \sigma(, y)\| \leqq c\|x - y\|$$
$$\|A(t, x) - I\|_2^2 \leqq c(1 + \|x\|^2)$$
$$\|\sigma(t, x)\|^2 \leqq c(1 + \|x\|^2)\,,$$

where $\|T\|_2$ denotes the Hilbert–Schmidt norm of $T|_H$. Henceforth if $T$ is an operator of $B$ such that $T(H) \subset H$ then $T|_H$ denotes the restriction of $T$ as an operator of $H$. We are concerned with the stochastic integral equation

$$X(t) = x + \int_0^t A(s, X(s))\, dW(s) + \int_0^t \sigma(s, X(s))\, ds\,, \qquad x \in B \;\; and \;\; 0 \leqq t \leqq 1\,. \tag{1}$$

In Theorem 5.1 [3] we showed the existence and the uniqueness of a non-anticipating continuous solution of (1) under conditions slightly different from (*). The only difference is that in [3] we assumed $\sigma(t, x) \in H$ for all $t$ and $x$, $\sigma(\cdot, x) : [0, 1] \to H$ is continuous and

$$|\sigma(t, x) - \sigma(t, y)| \leqq c\|x - y\|\,, \qquad |\sigma(t, x)|^2 \leqq c(1 + \|x\|^2)\,.$$

However, the proof of Theorem 5.1 goes equally well under the conditions (*). Thus the stochastic integral equation (1) under the conditions (*) has a unique non-anticipating solution.

In this paper we consider two stochastic integral equations with diffusion coefficients $(A, \sigma_1)$ and $(A, \sigma_2)$ and show that the measures corresponding to the solutions are equivalent under some additional assumptions in $A$, $\sigma_1$ and $\sigma_2$ (Section 2). The proof depends on the finite dimensional approximation (Section 3), the known result for the finite dimensional space (e.g. page 97 of [4]) and a lemma from Skorohod (page 100 of [5]).

**2. Main theorem.** Let $(E, |\cdot|_E)$ be a Banach space. $\mathscr{L}^2_{\text{n.a.}}(E)$ will denote the space consisting of all non-anticipating (w.r.t. $W(t)$) stochastic processes $\xi(t)$ with state space $E$ such that $\int_0^1 \mathcal{E}|\xi(t)|_E^2\, dt < \infty$. Three particular spaces $\mathscr{L}^2_{\text{n.a.}}(B^*)$, $\mathscr{L}^2_{\text{n.a.}}(H)$ and $\mathscr{L}^2_{\text{n.a.}}(R)$ will be considered here. Let $\xi \in \mathscr{L}^2_{\text{n.a.}}(B^*)$ be simple, i.e. there exists a sequence of real numbers $0 < t_1 < t_2 < \cdots < t_k < 1$ such that

$$\xi(t) = \xi(t_j) \qquad \text{if} \quad t_j \leqq t < t_{j+1}\,.$$

($t_0 = 0$, $t_{k+1} = 1$ by convention). Define

$$J_\xi(t) = \sum_{i=0}^{j-1} (\xi(t_i), W(t_{i+1}) - W(t_i)) + (\xi(t_j), W(t) - W(t_j))\,, \qquad t_j \leqq t < t_{j+1}\,,$$

where ( , ) is the natural pairing between $B^*$ and $B$. It is shown in [3] that there exists a linear operator $J_{(\cdot)}$ from $\mathscr{L}^2_{\text{n.a.}}(H)$ into $\mathscr{L}^2_{\text{n.a.}}(R)$, denoted by $J_\xi(t) = \int_0^t (\xi(s), dW(s))$, such that

(i) $J_\xi$ has continuous sample paths,

(ii) $J_\xi$ is a martingale,
(iii) $\mathscr{P}\{\sup_{0\leqq s\leqq t} |J_\xi(s)| > \alpha\} \leqq \alpha^{-2}\mathscr{E}|J_\xi(t)|^2$,
(iv) $\mathscr{E}J_\xi(t) = 0$ and $\mathscr{E}|J_\xi(t)|^2 = \mathscr{E}\int_0^t |\xi(s)|^2\, ds$.

Let $\mathscr{C}$ denote the Banach space of all continuous functions $f$ from $[0, 1]$ into $B$ with the sup norm $||f||_\infty = \sup_{0\leqq t\leqq 1} ||f(t)||$. If $X(t)$ is the solution of (1) then $\Theta_X$ denotes the mapping from $\Omega$ into $\mathscr{C}$ defined by $\Theta_X(\omega)(t) = X(t, \omega)$ and $\mu_X$ denotes the corresponding probability measure in $\mathscr{C}$. Let $(A, \sigma_1)$ and $(A, \sigma_2)$ satisfy the conditions (*) and $X_j$, $j = 1, 2$ be the solution of

$$X_j(t) = x + \int_0^t A(s, X_j(s))\, dW(s) + \int_0^t \sigma_j(s, X_j(s))\, ds\,. \tag{2}$$

We have the following

THEOREM. *Suppose $A(t, x)|_H$ is self-adjoint and there is a positive constant $\delta$ such that $A(t, x)|_H \geqq \delta I$ for all $t$ and all $x$. If $\sigma_2(t, x) - \sigma_1(t, x) \in H$ for all $t$ and all $x$ and $|\sigma_2(t, x) - \sigma_1(t, x)|^2 \leqq \alpha(1 + ||x||^2)$ for some constant $\alpha$ independent of $t$ and $x$. Then $\mu_{X_1}$ and $\mu_{X_2}$ are equivalent. Moreover, the Radon–Nikodym derivative of $\mu_{X_2}$ with respect to $\mu_{x_1}$ is given by*

$$\frac{d\mu_{X_2}}{d\mu_{X_1}}(\Theta_{X_1}) = \exp\{\int_0^1 (\sigma(t, X_1(t)), dW(t)) - \tfrac{1}{2}\int_0^1 |\sigma(t, X_1(t))|^2\, dt\}\,,$$

*where $\sigma(t, x) = A^{-1}(t, x)(\sigma_2(t, x) - \sigma_1(t, x))$.*

REMARK 1. $\sigma(t, X_1(t))$ is obviously a non-anticipating stochastic process with state space $H$. Moreover,

$$\begin{aligned}|\sigma(t, x)|^2 &= |A^{-1}(t, x)(\sigma_2(t, x) - \sigma_1(t, x))|^2\\ &\leqq \delta^{-2}|\sigma_2(t, x) - \sigma_1(t, x)|^2 \leqq \delta^{-2}\alpha^2(1 + ||x||^2)\,.\end{aligned}$$

Hence $\mathscr{E}|\sigma(t, X_1(t))|^2 \leqq \delta^{-2}\alpha^2(1 + \mathscr{E}||X_1(t)||^2)$. It is shown in [3] that

$$\sup_{0\leqq t\leqq 1} \mathscr{E}||X_1(t)||^2 < \infty\,.$$

Thus $\int_0^1 \mathscr{E}||X_1(t)||^2\, dt < \infty$ and $\sigma(t, X_1(t))$ is stochastic process belong to $\mathscr{L}^2_{\text{n.a.}}(H)$. The stochastic integral $\int_0^1 (\sigma(t, X_1(t)), dW(t))$ is then defined in the above way.

REMARK 2. In case $B$ is finite dimensional the assumption on $\sigma_2 - \sigma_1$ is superfluous because $B = H$ and the $B$-norm is equivalent to $H$-norm.

**3. Finite dimensional approximation.** In this section we consider the stochastic integral equation (1) with diffusion coefficients $(A, \sigma)$ satisfying the conditions (*). We assume also that $A$ satisfies the conditions in the Theorem. This additional assumption in $A$ will make the finite dimensional approximation possible. Define $K(t, x) = A(t, x) - I$ and $A_n(t, x) = I + Q_n K(t, x) Q_n$. Let $B_n = Q_n B$ so that $B_n$ is finite dimensional. It is obvious that $A_n$, as an operator of $B_n$, is invertible. In fact, $A_n$ is self-adjoint and $\langle A_n x, x\rangle \geqq \delta\langle x, x\rangle$ for all $x$ in $B_n$. Let $\sigma_n(t, x) = Q_n\sigma(t, x) \in B_n$. Let $X(t)$ be the solution of (1) and $X_n(t)$ be the solution of the following stochastic integral equation

$$X_n(t) = Q_n x + \int_0^t A_n(s, X_n(s))\, dQ_n W(s) + \int_0^t \sigma_n(s, X_n(s))\, ds\,. \tag{3}$$

Clearly $X_n(t)$ lies in the finite dimensional space $B_n$. Let $\mathfrak{A}$ be the Banach space of all non-anticipating stochastic processes $X(t)$ with state space $B$ with the norm

$$|||X||| = \{\sup_{0\leqq t\leqq 1} \mathscr{E}(||X(t)||^2)\}^{\frac{1}{2}} < \infty .$$

PROPOSITION 1. *$X_n \to X$ in $\mathfrak{A}$ as $n \to \infty$.*

PROOF. Clearly we have

$$(4) \qquad ||X_n(t) - X(t)||^2 \leqq 3||Q_n x - x||^2 + 3||Y_n(t)||^2 + 3||Z_n(t)||^2 ,$$

where

$$Y_n(t) = \int_0^t A_n(s, X_n(s))\, dQ_n W(s) - \int_0^t A(s, X(s))\, dW(s)$$

and

$$Z_n(t) = \int_0^t \sigma_n(s, X_n(s))\, ds - \int_0^t \sigma(s, X(s))\, ds .$$

For the sake of clarity we prove this Proposition by three steps.

STEP 1. Let $U_n(t) = \int_0^t Q_n K(s, X_n(s)) Q_n\, dQ_n W(s) - \int_0^t K(s, X(s))\, dW(s)$. Then $Y_n(t) = Q_n W(t) - W(t) + U_n(t)$. Hence

$$||Y_n(t)||^2 \leqq 2||Q_n W(t) - W(t)||^2 + 2||U_n(t)||^2 .$$

Using Proposition 3.1 [3] and making a complicated computation, we can obtain eventually the inequality

$$\mathscr{E}(||U_n(t)||^2) \leqq 2\beta^2 c^2 \int_0^t \mathscr{E}(||X_n(s) - X(s)||^2)\, ds + c_1(n) ,$$

where $\beta$ is a constant such that $||x|| \leqq \beta|x|$ for all $x$ in $H$, $c$ is the constant in conditions (*) and

$$(5) \qquad c_1(n) = 2\beta^2 \int_0^1 \mathscr{E}(||Q_n K(s, X(s)) Q_n - K(s, X(s))||_2^2)\, ds$$

Therefore,

$$(6) \qquad \mathscr{E}(||Y_n(t)||^2) \leqq 2|||Q_n W - W||| + 4\beta^2 c^2 \int_0^t \mathscr{E}(||X_n(s) - X(s)||^2\, ds + 2c_1(n) .$$

Similar computation yields easily that

$$(7) \qquad \mathscr{E}(||Z_n(t)||^2) \leqq 2c^2 \int_0^t \mathscr{E}(||X_n(s) - X(s)||^2)\, ds + c_2(n) ,$$

where

$$(8) \qquad c_2(n) = 2 \int_0^1 \mathscr{E}(||(Q_n - I)\sigma(s, X(s))||^2)\, ds .$$

STEP 2. Let $g_n(t) = \mathscr{E}(||X_n(t) - X(t)||^2)$. Taking $\mathscr{E}$ in (4) and using (6) and (7), we obtain immediately that

$$g_n(t) \leqq c_3(n) + 6c^2(1 + 2\beta^2) \int_0^t g_n(s)\, ds ,$$

where $c_3(n) = 3||Q_n x - x||^2 + 6|||Q_n W - W||| + 6c_1(n) + 3c_2(n)$. By Gronwall's Lemma, we conclude that

$$g_n(t) \leqq c_3(n) \exp\{6c^2(1 + 2\beta^2)\} \qquad 0 \leqq t \leqq 1 .$$

Hence $|||X_n - X||| = \sup_{0\leqq t\leqq 1} g_n(t) \leqq c_3(n) \exp\{6c^2(1 + 2\beta^2)\}$. To complete the proof it suffices to show

STEP 3.

$$c_3(n) \to 0 \quad \text{as} \quad n \to \infty .$$

Obviously, $||Q_n x - x||^2 \to 0$ as $n \to \infty$ since $Q_n \to I$ strongly in $B$. It is easy to find out that

$$|||Q_n W - W|||^2 = \int_B ||Q_n x - x||^2 p_1(dx) ,$$

where $p_1(dy)$ is the Wiener measure with parameter 1. By the Principle of Uniform Boundedness there is a constant $\lambda < \infty$ such that $||Q_n||_{B,B} \leqq \lambda$ for all $n$. Hence $||Q_n x - x||^2 \leqq (2\lambda^2 + 1)||x||^2$ which is integrable with respect to $p_1(dx)$. Moreover, $||Q_n x - x|| \to 0$ for each $x \in B$. Thus by Lebesgue's dominated convergence theorem $|||Q_n W - W||| \to 0$ as $n \to \infty$. Lebesgue's dominated convergence theorem can be applied also to $c_1(n)$ and $c_2(n)$ (see (5) and (8)) to see that they converge to 0 as $n \to \infty$. Therefore $c_3(n) \to 0$ as $n \to \infty$ and the Proposition is completely proved.

**4. Proof of the Theorem.** Recall that $X_j$ ($j = 1, 2$) is the solution of (2). Let $X_{j,n}$ ($j = 1, 2$) be the finite dimensional approximation of $X_j$ given by the previous section, i.e., $X_{j,n}$ is the solution of the following equation.

$$X(t) = Q_n x + \int_0^t A_n(s, X(s))\, dQ_n W(s) + \int_0^t \sigma_{j,n}(s, X(s))\, ds ,$$

where $A_n(s, x) = I + Q_n K(s, x) Q_n$ and $\sigma_{j,n}(s, x) = Q_n \sigma_j(s, x)$, $j = 1, 2$. From Proposition 1, $X_{j,n} \to X_j$ in $\mathfrak{A}$ as $n \to \infty$, $j = 1, 2$. Let $\mu_{X_{j,n}}$ be the measure in $\mathscr{C}$ corresponding to $X_{j,n}$. From the finite dimensional result we know that $\mu_{X_{1,n}}$ and $\mu_{X_{2,n}}$ are equivalent and the Radon–Nikodym derivative of $\mu_{X_{2,n}}$ with respect to $\mu_{X_{1,n}}$ is given by

$$\frac{d\mu_{X_{2,n}}}{d\mu_{X_{1,n}}}(\Theta_{X_{1,n}}) = \exp\{\int_0^1 (\sigma^{(n)}(t, X_{1,n}(t)), dQ_n W(t)) - \tfrac{1}{2} \int_0^1 |\sigma^{(n)}(t, X_{1,n}(t))|^2\, dt\}$$

where $\sigma^{(n)}(t, x) = A_n^{-1}(t, x)(\sigma_{2,n}(t, x) - \sigma_{1,n}(t, x))$.

Now, we will state two propositions whose proofs are rather long and tiresome, but nevertheless elementary, hence omitted. We need only to make use of the conditions (*), the hypothesis of the Theorem and equality (4) of Theorem 3.2 [3].

PROPOSITION 2. *If* $f_n = \int_0^1 (\sigma^{(n)}(t, X_{1,n}(t)), dQ_n W(t))$ *and* $f = \int_0^1 (\sigma(t, X_1(t)), dW(t))$ *then* $\mathscr{E}(|f_n - f|^2) \to 0$ *as* $n \to \infty$. *Hence* $f_n \to f$ *in the mean as* $n \to \infty$.

PROPOSITION 3. *If* $h_n = \int_0^1 |\sigma^{(n)}(t, X_{1,n}(t))|^2\, dt$ *and* $h = \int_0^1 |\sigma(t, X_1(t))|^2\, dt$ *then* $\mathscr{E}(|h_n - h|^2) \to 0$ *as* $n \to \infty$. *Hence* $h_n \to h$ *in the mean as* $n \to \infty$.

It follows from the above two propositions that $\log\{(d\mu_{X_{2,n}}/d\mu_{X_{1,n}})(\Theta_{X_{1,n}})\}$ converges in the mean to $\log\{(d\mu_{X_2}/d\mu_{X_1})(\Theta_{X_1})\}$ as $n \to \infty$. By interchanging $X_1$ with $X_2$ it is evident that $\log\{(d\mu_{X_{1,n}}/d\mu_{X_{2,n}})(\Theta_{X_{2,n}})\}$ converges in the mean to $\log\{(d\mu_{X_1}/d\mu_{X_2})(\Theta_{X_2})\}$ as $n \to \infty$. To finish the proof we simply apply Lemma 4 on page 100 of Skorohod [5].

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