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# A NOTE ON FINE'S AXIOMS FOR QUALITATIVE PROBABILITY[1]

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Fine gives axioms on a binary relation $\precsim$ on a field of events, with $A \precsim B$ interpreted as "$A$ is (subjectively) no more probable than $B$," sufficient to guarantee the existence of an order-preserving probability measure and an additive order-preserving probability measure. It is noted that one of Fine's axioms, that the order topology have a countable base, can be replaced by the more appealing axiom that there is a countable order-dense subset.

Let $A$ be a set and $\precsim$ a binary relation on $A$. We shall say that a real-valued function $f$ on $A$ is an *order-preserving function* if for all $a, b \in A$,

$$a \precsim b \Leftrightarrow f(a) \leqq f(b) . \tag{1}$$

Suppose $\mathscr{F}$ is a field of events and $\precsim$ is interpreted as "is (subjectively) no more probable than." If $P$ is an order-preserving function on $(\mathscr{F}, \precsim)$, we shall call it a *measure of qualitative probability*. (Fine (1971) calls $P$ a qualitative probability.) We give an alternate proof of a theorem of Fine which gives conditions on $(\mathscr{F}, \precsim)$ sufficient for the existence of a qualitative probability measure and we state an alternate set of conditions. Fine also gives conditions on $(\mathscr{F}, \precsim)$ sufficient to guarantee that there is an *additive qualitative probability measure*, i.e. a qualitative probability measure $P$ such that for all $A, B \in \mathscr{F}$ with $A \cap B = \emptyset$,

$$P(A \cup B) = P(A) + P(B) . \tag{2}$$

An alternative to Fine's theorem follows easily from our results.

To state Fine's theorem on qualitative probability, we introduce some definitions. Given a simple (linear) order $(A, \precsim)$, let $a \prec b$ hold if and only if $\sim[b \precsim a]$. If $a \in A$, the *open rays* at $a$ are $\{b \in A : a \prec b\}$ and $\{b \in A : b \prec a\}$. If $a$ and $b$ are in $A$, the *open interval* $(a, b)$ is $\{c \in A : a \prec c \prec b\}$. The *order topology* $\mathscr{T}(A, \precsim)$ is the topology having as base all open intervals and open rays.

If $\mathscr{F}$ is a field of events and $\precsim$ is a binary relation on $\mathscr{F}$, standard axioms for the existence of a qualitative probability measure on $(\mathscr{F}, \precsim)$ are the following conditions, which are required to hold for arbitrary $A, B, C \in \mathscr{F}$.

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C1. $(\mathscr{F}, \precsim)$ is a simple order.

C2. $\emptyset \precsim A$.[3]

C3. $A \precsim B$, $C \cap (A \cup B) = \emptyset \Leftrightarrow A \cup C \precsim B \cup C$, $C \cap (A \cup B) = \emptyset$.

THEOREM 1 (Fine). *If $\mathscr{F}$ is a field of events and $\precsim$ is a binary relation on $\mathscr{F}$ satisfying* C1, C2 *and* C3, *then there is a (not necessarily additive) qualitative probability measure on $(\mathscr{F}, \precsim)$ if and only if*

C4. *The order topology $\mathscr{T}(\mathscr{F}, \precsim)$ has a countable base.*

Fine's result holds in a more general setting and with fewer assumptions and it follows from a well-known theorem characterizing the representation (1) for binary relations on arbitrary sets. To see this, let us recall that if $\precsim$ is a binary relation on a set $A$, we say $B \subseteq A$ is *order-dense* if for every $a, b \in A$ such that $a, b \notin B$ and $a \prec b$, there is $c \in B$ such that $a \prec c \prec b$. The reader should note that order-denseness is different from denseness, which requires the existence of $c$ even if $a$ or $b$ is in $B$. To see the difference, note that the odd integers are order-dense in the integers, but not dense. (Denseness in the order sense should be distinguished from denseness in the topological sense, which is studied in the context of order-preserving functions by Fleischer (1961).) The improvement in Fine's Theorem is contained in the following result.

THEOREM 2. *Suppose $A$ is a set and $\precsim$ is a binary relation on $A$ satisfying* C1. *Then the following are equivalent*:

(i) *There is an order preserving function on $(A, \precsim)$.*

(ii) C4. *The order-topology $\mathscr{T}(A, \precsim)$ has a countable base.*

(iii) C4′. *$(A, \precsim)$ has a countable order-dense subset.*

REMARK. In a Corollary to Theorem 1, Fine observes that under C1, C2, and C3, the existence of a countable dense subset of $(\mathscr{F}, \precsim)$ implies the existence of a qualitative probability measure and calls this sufficient condition "more appealing" than C4. In the same sense, we find the existence of a countable order-dense subset of $(\mathscr{F}, \precsim)$ a more appealing condition than C4.

The equivalence of (i) and (iii) in Theorem 2 was apparently first proved by Milgram (1939). Birkhoff (1948) also offered a proof, though it has some gaps. A recent version of the proof can be found in Krantz, *et al.* (1971, Theorem 2, page 40). Theorem II of Debreu (1954) asserts that (ii) implies the existence of an order-preserving function continuous in the order-topology. Thus, that (ii) implies (i) is a Corollary of Debreu's Theorem. Fleischer (1961) notes that (i) implies (ii) because "Second countability in the order topology is characteristic for real sets."[4]

We shall furnish a direct proof of the equivalence of (ii) and (iii). This, together with the Krantz, *et al.* proof of the equivalence of (i) and (iii), will

[3] If we want to require $P(X) = 1$ for $X = \bigcup \{A: A \in \mathscr{F}\}$, then we must add the axiom $\emptyset \prec X$.

[4] The author thanks Professor Peter C. Fishburn for drawing his attention to the results of Milgram, Debreu and Fleischer.

provide a direct proof of Theorem 2. To prove that (iii) implies (ii), suppose $B$ is a countable order-dense subset of $(A, \precsim)$. Let us say an open interval $(a, b)$ is a *gap* with *lower and upper endpoints* $a$ and $b$ respectively if for all $c \in A$, $\sim[a \prec c]$ or $\sim[c \prec b]$, i.e. if $(a, b) = \emptyset$. Let $A_1^*$ be the set of lower endpoints of gaps, $A_2^*$ the set of upper endpoints of gaps and $A^* = A_1^* \cup A_2^*$. Then $A^*$ is countable. To see this, note that $A_1^* - B$ can be mapped into $B$ in a one-to-one fashion. For if $(a, b)$ is a gap and $a \notin B$, then $b \in B$. Thus $A_1^* - B$ is countable. So is $A_2^* - B$. Finally,

$$A^* = (A_1^* - B) \cup (A_2^* - B) \cup (A^* \cap B) .$$

(This proof of the countability of $A^*$ is the same as one of Krantz, *et al.* (1971, page 41).) A countable base $\mathscr{B}$ for the topology $\mathscr{T}(A, \precsim)$ is now given by the collection of all $\{a\}$ such that $a \in A^*$ plus the collection of all open intervals $(a, b)$ such that $a, b \in B$.

Finally, to prove that (ii) implies (iii), suppose $\mathscr{B}$ is a countable base for the topology $\mathscr{T}(A, \precsim)$. By the Axiom of Choice, there is a function $c: \mathscr{B} \to A$ such that $c(0) \in 0$ for each $0 \in \mathscr{B}$; $c$ picks one representative from each set in $\mathscr{B}$. (The proof of the equivalence of (i) and (iii) also uses the Axiom of Choice.) Let

$$B = \{c(0): 0 \in \mathscr{B}\} .$$

Clearly $B$ is countable. We show it is order-dense. Suppose $a \prec b$ and $a, b \notin B$. If $(a, b)$ is a gap, then $\{a\}$ must be in the base $\mathscr{B}$, and so $c(\{a\}) = a$ is in $B$, which is a contradiction. Thus there is $u \in A$ such that $a \prec u \prec b$. Similarly $(a, u)$ is not a gap, so there is $v \in A$ such that $a \prec v \prec u \prec b$. If $(v, u)$ is a gap, then $v \in B$ or $u \in B$, as desired. If $(v, u)$ is not a gap, then there is $w \in (v, u)$. Since $\mathscr{B}$ is a base, there is a set $0 \in \mathscr{B}$ such that $w \in 0 \subseteqq (v, u)$. In particular, it follows that $v \precsim c(0) \precsim u$, so $a \prec c(0) \prec b$. This completes the proof of Theorem 2.

Finally, we note that if $\mathscr{F}$ is a field of events and $\precsim$ is a binary relation on $\mathscr{F}$, then by the Corollary to Theorem 2 of Fine (1971), sufficient conditions on $(\mathscr{F}, \precsim)$ for the existence of an additive qualitative probability measure are given by conditions C1 to C4 and an additional condition, C5, which asserts the existence of almost uniform partitions. (For the exact statement of C5, the reader is referred to Fine's paper.) It is easy to see that the same proof implies that sufficient conditions are also given by C1 to C3, C4′ and C5.[5]

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[5] For a further discussion of conditions sufficient to guarantee the existence of qualitative or additive qualitative probability measures, the reader is referred to Professor Fine's book, Fine (1973).

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