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BOUNDARY CROSSING PROBABILITIES ASSOCIATED WITH MOTOO'S LAW OF THE ITERATED LOGARITHM[1]

BY MICHAEL J. WICHURA

University of Chicago

For certain recurrent diffusion processes, Motoo has given an integral test which allows one to determine whether an increasing function belongs to the upper or lower class relative to the process at hand. We show that a refinement of his methods yields asymptotic estimates for the tail probabilities of the time of last crossing of an upper class function g in cases where the speed measure of the process has sufficiently thin tails and the curve g increases sufficiently slowly. Similar results are derived for certain extremal processes and for non-decreasing stable processes.

1. Statement of results: diffusions. Throughout this section, our terminology and notation is, for the most part, that of the second half of D. Freedman's recent book on Brownian motion and diffusion.

Let $(X(t))_{t\geqq 0}$ be a diffusion process in natural scale with state space $I = (-\infty, \infty)$. Let $g: [0, \infty) \to (0, \infty)$ be an increasing function which diverges to ∞. Motoo (1959) showed that if the total mass, $m(I)$, of the speed measure, m, for X is finite, then

$$(1.1) \qquad \int^\infty (1/g(t))\, dt < \infty \quad \text{implies} \quad P_x\{X(t) \geqq g(t) \text{ i.o. as } t \uparrow \infty\} = 0 \quad \text{for all } x \in I;$$

$$(1.2) \qquad \int^\infty (1/g(t))\, dt = \infty \quad \text{implies} \quad P_x\{X(t) \geqq g(t) \text{ i.o. as } t \uparrow \infty\} = 1 \quad \text{for all } x \in I.$$

When (1.1) (respectively (1.2)) holds, g is said to be in the upper (respectively, lower) class, written $g \in \mathscr{U}$ (respectively, $g \in \mathscr{D}$). By means of an appropriate transformation, Motoo used his result to give simple proofs of iterated logarithm type results for the radial part of d-dimensional Brownian motion ($d \geqq 1$). In simplified form, Motoo's argument can be found in Itô and McKean (1965), Section 4.12.

It is our purpose here to show that a refinement of Motoo's method yields an asymptotic expression for

$$(1.3) \qquad P_x\{X(t) \geqq g(t) \text{ for some } t \geqq t_0\}$$

(as $t_0 \uparrow \infty$) for certain upper class functions g. Roughly speaking, our result is

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that if the tails of the speed measure m decrease sufficiently rapidly, and if $g \in \mathscr{U}$ increases sufficiently slowly, then (1.3) is asymptotic to

$$\frac{1}{m(I)} \int_{t_0}^{\infty} \frac{dt}{g(t)}$$

as $t_0 \uparrow \infty$). It is to be noted that, in accordance with Freedman's convention, our speed measure is twice the usual choice (confer Lemmas 4.2 and 4.3 below).

Any diffusion Y can be transformed to one in natural scale by means of a monotonic transformation. It may be of some help to review how this is accomplished in the following common situation. Suppose that Y has infinitesimal generator Γ_Y satisfying

$$\Gamma_Y f(y) = \mu_Y(y) f'(y) + \sigma_Y^2(y) f''(y)/2$$

where μ_Y and $\sigma_Y > 0$ are continuous functions; roughly speaking, starting from state y, Y behaves instantaneously like a Brownian motion with drift coefficient $\mu_Y(y)$ and scale (or diffusion) coefficient $\sigma_Y^2(y)$. Take for S any increasing solution to the equation $\Gamma S = 0$, and put $X(t) = S(Y(t))$. Then $X = (X(t))_{t \geqq 0}$ is a diffusion whose infinitesimal generator Γ_X is of the same form as Γ_Y, with drift coefficients $\mu_X(x) = 0$ and scale coefficients given by

$$\sigma_X^2(x) = \sigma_Y^2(S^-(x))(S'(S^-(x)))^2$$

(here S^- denotes the inverse of S). As X is driftless, it is in natural scale; moreover, its speed measure m is given by

$$m(dx) = 2\sigma_X^{-2}(x)\, dx .$$

See Breiman (1968) pages 386–387 for details. In view of the monotoneity of S, X crosses g if and only if Y crosses $S^- g$, so our results can easily be interpreted in terms of Y.

Here is the main result:

THEOREM 1.1. *Let $(X(t))_{t \geqq 0}$ be a diffusion in natural scale, with state space $I = (-\infty, \infty)$ and speed measure m satisfying*

$$m\{\xi : |\xi| \geqq x\} = O(1/x) \tag{1.4}$$

as $x \uparrow \infty$. Suppose that $g : [0, \infty) \to (0, \infty)$ satisfies

$$g(t) \uparrow \infty \qquad \text{as } t \uparrow \infty \tag{1.5}$$

$$G(t) \equiv \int_t^{\infty} du/g(u) \downarrow 0 \qquad \text{as } t \uparrow \infty \tag{1.6}$$

$$G(t(1 + t^{-c})) \sim G(t) \qquad \text{as } t \uparrow \infty , \tag{1.7}$$

for some $c < \frac{1}{3}$. Let ρ be an initial distribution satisfying

$$\rho\{\xi : |\xi| \geqq x\} = O(1/x^{\gamma}) \qquad \text{as } x \uparrow \infty \tag{1.8}$$

for some $\gamma > 0$. Then

$$P_\rho\{X(t) \geqq g(t) \text{ for some } t \geqq t_0\} \sim G(t_0)/m(I) \qquad \text{as } t_0 \uparrow \infty . \tag{1.9}$$

Condition (1.7) implies (confer Freedman (1971), Theorem 74, page 213) that the convex function G must satisfy

$$G(t) \geqq \alpha e^{-\beta t^c} \tag{1.10}$$

for all large t, for some appropriate positive numbers α and β. The function g does not need to be smooth, nor to satisfy a "smooth growth" condition analogous to (1.7). Indeed, given any increasing function f for which $f(t) \geqq t$ for all t, there exists a step function g such that (1.5), (1.6), and (1.7) hold, while $\limsup_t g(t)/f(t) = \infty$; to see this, determine t_n, $n \geqq 1$, inductively so that $t_{n+1} - t_n = f(t_n)$, and set $g(t) = 2^n f(t_n)$ for $t \in [t_n, t_{n+1})$, $n \geqq 1$. It is undoubtedly true that (1.9) holds under a condition that is less artifical than (1.7); however, (1.7) seems to be about the weakest condition under which our method of proof can yield (1.9). By (1.4), condition (1.8) is satisfied when one takes $m^* = m/m(I)$ for ρ; in this connection it is of interest to note that, under our general assumptions, m^* serves as an invariant measure for X (confer Maruyama and Tanaka (1951) page 139–140). Another situation in which (1.8) is trivially satisfied arises when ρ is degenerate; thus (1.4)—(1.7) imply

$$P_x\{X(t) \geqq g(t) \text{ for some } t \geqq t_0\} \sim G(t_0)/m(I) \tag{1.11}$$

for all states x. However, (1.9) does not follows from this because there is no guarantee that the $1 + o(1)$ terms implicit in (1.11) are bounded. Along the same lines, it should be noted that (1.9) does not in general hold for arbitrary ρ. To see this, suppose X has the property that

$$\lim_{y \uparrow \infty} P_y\{X \text{ reaches } x \text{ by time } t\} = 0$$

for each x and t; this is to say that $+\infty$ is a natural boundary and is equivalent to the condition $\int^\infty xm(dx) = \infty$ (confer Breiman (1968) pages 366–367). Take any g satisfying (1.5) and choose $x(t)$'s increasing to ∞ with t in such a way that $\lim_{t \uparrow \infty} P_{x(t)}\{X \text{ reaches } g(t) \text{ by time } t\} = 0$. Then the left-hand side in (1.9) is minorized by $(1 + o(1))\rho([x(t_0), \infty))$, which can be made to tend to 0 as slowly as desired by choosing ρ appropriately.

Theorem 1.1 has several variants, of which we mention three.

(A) Suppose that in (1.4) one strengthens the condition on the right hand tail of m by demanding that

$$\int^\infty xm(dx) < \infty ,$$

i.e., that $+\infty$ be an entrance boundary (confer Breiman (1968) page 366). Then in (1.8) the condition on the right-hand tail of ρ can be dropped entirely; one can even allow ρ to give positive mass to $+\infty$. If both $\pm\infty$ are entrance boundaries, then (1.9) holds for every ρ.

(B) The conclusion (1.11) remains in force if (1.4) is weakened to

$$m\{\xi : |\xi| \geqq x\} = O(1/x^\delta) \qquad \text{for all } \delta < 1$$

and if (1.7) is strengthened to

$$\lim_{\varepsilon \downarrow 0} \liminf_{t \uparrow \infty} G(t(1 + \varepsilon))/G(t) = 1 .$$

(C) Under (1.5)—(1.7), (1.9) is valid for a diffusion X in natural scale with state space $I = [0, \infty)$, for which 0 is an instantaneous state (confer Freedman (1971) page 107) and for which one-sided versions of (1.4) and (1.8) hold; moreover, (1.9) holds for any ρ if $+\infty$ is an entrance boundary.

Here are some consequences of Theorem 1.1. The first is a sharpening of Motoo's (1959) upper class criterion for the Uhlenbeck process, corresponding to the infinitesimal generator

$$\Gamma f(x) = f''(x)/2 - \alpha x f'(x) \tag{1.12}$$

$(\alpha > 0)$.

COROLLARY 1.1. *Let $(U(t))_{t \geq 0}$ be the Uhlenbeck diffusion process with generator (1.12). Let $\psi: [0, \infty) \to (0, \infty)$ be a function such that*

$$\psi(t) \uparrow \infty \qquad \textit{as } t \uparrow \infty \tag{1.13}$$

$$\Psi(t) \equiv \alpha^{\frac{1}{2}} \pi^{-\frac{1}{2}} \int_t^\infty \psi(u) e^{-\alpha\psi^2(u)}\, du < \infty \qquad \textit{for large } t \tag{1.14}$$

$$\Psi(t(1 + t^{-c})) \sim \Psi(t) \qquad \textit{as } t \uparrow \infty\,, \tag{1.15}$$

for some $c < \frac{1}{3}$. Then for each initial distribution ρ satisfying

$$\rho\{\xi : |\xi| \geqq x\} = O(e^{-\gamma x^2}) \qquad \textit{as } x \uparrow \infty \tag{1.16}$$

for some $\gamma > 0$, one has

$$P_\rho\{U(t) \geqq \psi(t) \ \textit{for some} \ t \geqq t_0\} \sim \Psi(t_0) \qquad \textit{as } t_0 \uparrow \infty\,.$$

For example,

$$P_\rho\{U(t) \geqq ((\log(t) + (\tfrac{3}{2} + \delta)\log_2 t)/\alpha)^{\frac{1}{2}} \ \text{for some} \ t \geqq t_0\} \sim \alpha/(\pi^{\frac{1}{2}}\delta \log^\delta t_0)$$

for $\delta > 0$.

Next, we give a sharpening of Motoo's (1959) upper-class criterion for d-dimensional Brownian motion (confer Lévy (1954) page 55):

COROLLARY 1.2. *Let $(R(t))_{t \geq 0}$ be the radial part of a normalized d-dimensional Brownian motion process ($d \geqq 1$). Let $\psi: [0, \infty) \to (0, \infty)$ be a function satisfying* (1.13) *and*

$$\Psi(t) \equiv (\Gamma(d/2)2^{d/2})^{-1} \int_t^\infty \psi^d(u) e^{-(\frac{1}{2})\psi^2(u)}\, du/u < \infty \qquad \textit{for large } t \tag{1.17}$$

$$\Psi(e^t e^{t^{1-c}}) \sim \Psi(e^t) \qquad \textit{as } t \uparrow \infty \tag{1.18}$$

for some $c < \frac{1}{3}$. Then for each initial distribution ρ on $[0, \infty)$ satisfying

$$\rho([x, \infty)) = O(e^{-\gamma x^2}) \qquad \textit{as } x \uparrow \infty \tag{1.19}$$

for some $\gamma > 0$, one has

$$P_\rho\{R(u) \geqq (u+1)^{\frac{1}{2}}\psi(u) \ \textit{for some} \ u \geqq u_0\} \sim \Psi(u_0) \tag{1.20}$$

as $u_0 \uparrow \infty$.

As special cases, one has for $\alpha < \frac{1}{3}$

$$(1.21)\qquad P_\rho\{R(u) \geqq (2u \log^\alpha u)^{\frac{1}{2}} \text{ for some } u \geqq u_0\} \sim (\alpha\Gamma(d/2))^{-1}(\log u_0)^{1+(d/2-1)\alpha}e^{-\log^\alpha u_0}$$

as $u_0 \uparrow \infty$; for $c > 1$,

$$(1.22)\qquad P_\rho\{R(u) \geqq (2cu \log_2 u)^{\frac{1}{2}} \text{ for some } u \geqq u_0\} \sim \frac{c^{d/2} \log_2{}^{d/2} u_0}{\Gamma(d/2)(c-1) \log^{c-1} u_0}$$

as $u_0 \uparrow \infty$ ($\log_2 u$ denotes $\log(\log(u))$); and for $\delta > 0$, $k \geqq 3$

$$(1.23)\qquad \begin{aligned} &P_\rho\{R(u) \geqq (2u(\log_2 u + (d/2)\log_3 u + (\log_3 u + \cdots \\ &\qquad + \log_{k-1} u + (1+\delta)\log_k u)))^{\frac{1}{2}} \text{ for some } u \geqq u_0\} \\ &\qquad\sim 1/(\Gamma(d/2)\delta \log_{k-1}^{\delta} u_0)\,. \end{aligned}$$

An analogous result holds in regards to the Feller-Erdös-Kolmogorov-Petrovski criterion for univariate Brownian motion $(B(t))_{t\geqq 0}$; namely if ψ satisfies (1.13), (1.17), and (1.18) with $d = 1$, and if ρ satisfies (1.16), then

$$(1.24)\qquad P_\rho\{B(u) \geqq (u+1)^{\frac{1}{2}}\psi(u) \text{ for some } u > u_0\} \sim \Psi(u_0)/2$$

with Ψ defined by (1.17).

The "lower class" result of Spitzer (1958) is refined to

COROLLARY 1.3. *Let $(R(t))_{t\geqq 0}$ be the radial part of a normalized 2-dimensional Brownian motion process. Let $\psi : [0, \infty) \to (0, \infty)$ be a function such that*

$$(1.25)\qquad \psi(u) \downarrow 0 \qquad \textit{as } u \uparrow \infty$$

$$(1.26)\qquad \Psi(u) \equiv \int_u^\infty (2v|\log \psi(v)|)^{-1}\, dv < \infty \qquad \textit{for large } u$$

$$(1.27)\qquad \Psi(e^t e^{t^{1-c}}) \sim \Psi(e^t) \qquad \textit{as } t \uparrow \infty$$

for some $c > \frac{1}{3}$. Then for each initial distribution ρ satisfying (1.19), *one has*

$$(1.28)\qquad P_\rho\{R(u) \leqq (1+u)^{\frac{1}{2}}\psi(u) \textit{ for some } u \geqq u_0\} \sim \Psi(u_0)$$

as $u_0 \uparrow \infty$.

In particular, for $\delta > 0$ and

$$\begin{aligned} \psi_{k,\delta}(u) &= u^{-(\log_k u)^\delta}, &&\textit{if } k = 1 \\ &= u^{-(\log_2 u)\cdots(\log_{k-1} u)(\log_k u)^{1+\delta}}, &&\textit{if } k \geqq 2, \end{aligned}$$

one has

$$(1.29)\qquad P_\rho\{R(u) \leqq u^{\frac{1}{2}}\psi_{k,\delta}(u) \text{ for some } u \geqq u_0\} \sim (2\delta \log_k{}^\delta u_0)^{-1}$$

as $u_0 \uparrow \infty$.

Finally, here is a sharpening of the corresponding result in higher dimensions (confer Dvoretzky and Erdös (1951)):

COROLLARY 1.4. *Let $d \geqq 3$, and let $(R(u))_{u\geqq 0}$ be the radial part of a normalized d-dimensional Brownian motion process. Let $\psi : [0, \infty) \to (0, \infty)$ be a function such* *that*

$$\psi(u) \downarrow 0 \qquad as\ u \uparrow \infty \tag{1.30}$$

$$\Psi(u) \equiv (\Gamma(d/2 - 1)2^{d/2-1})^{-1} \int_u^\infty \psi^{d-2}(v)v^{-1}\,dv < \infty \qquad for\ large\ u \tag{1.31}$$

$$\Psi(e^t e^{t^{1-c}}) \sim \Psi(e^t) \qquad as\ t \uparrow \infty \tag{1.32}$$

for some $c < \frac{1}{3}$. *Then for each initial distribution* ρ *satisfying* (1.19), *one has*

$$P_\rho\{R(u) \leqq (1 + u)^{\frac{1}{2}}\psi(u) \ \text{for some}\ u \geqq u_0\} \sim \Psi(u_0) \tag{1.33}$$

as $u_0 \uparrow \infty$.

In particular for $k \geqq 1$, $\delta > 0$,

$$P_\rho\{R(u) \leqq u^{\frac{1}{2}}((\log u) \cdots (\log_{k-1} u)(\log_k u)^{1+\delta})^{-1/(d-2)} \ \text{for some}\ u \geqq u_0\} \sim (\Gamma(d/2 - 1)2^{d/2-1}\delta \log_k{}^\delta u_0)^{-1}\,. \tag{1.34}$$

For univariate Brownian motion, Strassen (1967) has a result which is much superior to (1.24), in that it applies to a broad class of "smooth" upper class functions, and is formulated in terms of the density of the time, T, of last crossing, instead of the tail probabilities of T. It is to be suspected that there is a Strassen type version of Theorem 1.1. Strassen gives a summary of the relevant literature prior to 1967. Among recent related results, the work of Robbins and Siegmund (1970) merits special attention. In this paper martingale techniques are used to find the exact value of (1.3) when X is univariate Brownian motion and g belongs to a certain class of curves. These exact results are a major improvement over the asymptotic ones. Unfortunately the Robbins–Siegmund curves are defined through a rather complicated procedure. One consequence of this is that, aside from some important special cases, most of these curves do not have closed form expressions, and can at best be evaluated asymptotically. Another consequence is that is difficult to tell whether a given curve is in the Robbins–Siegmund class. In contrast, it is generally easy to check whether a given curve satisfies Strassen's conditions, or those we have used here. Neither Strassen nor Robbins–Siegmund impose a condition like (1.18) (which is (1.7) in disguise), so their methods apply to curves which are much farther from the "boundary" between the upper and lower classes. In this connection we note that the restriction on α in (1.21) is due entirely to (1.18); in view of Strassen's results, (1.21) is undoubtedly true for all α.

2. Statement of results: maxima and minima of processes. The methods used to establish Theorem 1.1 and its corollaries can be used in connection with other stochastic processes. We present here some results concerning the partial maxima and minima of certain processes.

To begin with, fix an integer $k \geqq 1$, and for each $t \geqq 0$, let $V_k(t)$ be the kth-smallest ordinate of points with abscissa $\leqq t$ in a homogeneous Poisson process, N, in $[0, \infty)^2$. The process $V_k = (V_k(t))_{t \geqq 0}$ is a Markov process whose sample paths are right-continuous, non-increasing step functions. For $s < t$ and $y > x$, one has

$$P\{V_k(s) \geqq y,\ V_k(t) \geqq x\}$$
$$= \sum_{0 \leqq i \leqq k-1} e^{-sx}(sx)^i(i!)^{-1}G(k-i, s; y-x)G(k-i, t-s; x)\ ,$$

where $G(r, \lambda; \xi) = \lambda^r \int_\xi^\infty z^{r-1}e^{-\lambda z}\,dz/\Gamma(r)$ is the probability assigned to the interval $[\xi, \infty)$ by the Gamma $G(r, \lambda)$ distribution. It follows that $V_k(s)$ has a Gamma $G(k, s)$ distribution, and that

$$P\{V_k(t) \geqq x \mid V_k(s) = y\}$$
$$= \sum_{0 \leqq i \leqq k-1} B(k-1, x/y; i)G(k-i, t-s; x)\ , \qquad \text{if} \quad x \leqq y$$
$$= 0\ , \qquad \text{if} \quad x > y\ ,$$

where $B(n, p; j) = \binom{n}{j}p^j(1-p)^{n-j}$; in particular, V_k has stationary transition probabilities. Given that $V_k(s) = y$, the process remains in state y for a length of time which is exponentially distributed with parameter y (i.e., $G(1, y)$), and then jumps down to a state x which is distributed according to the Beta $(k, 1)$ distribution on $(0, y)$ (the density is kx^{k-1}/y^k, $0 < x < y$). The following theorem gives some information on how fast $V_k(t) \downarrow 0$ as $t \uparrow \infty$.

THEOREM 2.1. *Let* $\psi : [0, \infty) \to (0, \infty)$ *be a function such that*

$$\psi(u) \uparrow \infty \qquad as \quad u \uparrow \infty \tag{2.1}$$

$$\Psi(u) \equiv \left(\int_u^\infty \psi^k(v)e^{-\psi(v)}dv/v\right)/\Gamma(k) < \infty \qquad for\ large\ \ u \tag{2.2}$$

$$\Psi(e^t e^{t^{1-c}}) \sim \Psi(e^t) \qquad as \quad t \uparrow \infty \tag{2.3}$$

for some $c < \frac{1}{3}$. *Let* ρ *be an initial distribution on* $(0, \infty]$ *such that*

$$\rho((0, v]) = O(v^\gamma) \qquad as \quad v \downarrow 0 \tag{2.4}$$

for some $\gamma > 0$. *Then*

$$P_\rho\{V_k(u) \geqq \psi(u)/u \ \ for\ some \ \ u \geqq u_0\} \sim \Psi(u_0) \tag{2.5}$$

as $u_0 \uparrow \infty$.

The natural choice for ρ places unit mass at $+\infty$; we shall, however, use other degenerate ρ's in the corollary below. This result sharpens the upper half of the following integral test, which can be proved by the same methods: if ψ satisfies (2.1) and if ρ is an arbitrary initial distribution on $(0, \infty]$, then

$$\int \psi^k(v)e^{-\psi(v)}v^{-1}\,dv \begin{Bmatrix} < \infty \\ = \infty \end{Bmatrix} \quad \text{implies} \quad P_\rho\{V_k(u) \geqq \psi(u)/u \ \ \text{i.o.}\} \begin{Bmatrix} = 0 \\ = 1\ . \end{Bmatrix}$$

Now let $(U_n)_{n \geqq 1}$ be a sequence of independent random variables, each uniformly distributed over (0, 1). Fix $k \geqq 1$, and for $n \geqq 1$, let $m_k(n)$ be the kth smallest of $U_1, U_2, \cdots, U_{k-1+n}$. The process $(m_k(n))_{n \geqq 1}$ can be represented using the $(V_k(t))_{t \geqq 0}$ process introduced above, in the following manner. Put $T_0 = 0$ and let $T_1 < T_2 < \cdots < T_n < \cdots$ be the successive abscissae of points of N (the Poisson process from which V_k was derived) in $(0, \infty) \times (0, 1)$. The $(T_n - T_{n-1})$'s are i.i.d. $G(1, 1)$ random variables; moreover, the process $(m_k(n))_{n\geq 1}$ has the same distribution as $(V_k(T_n))_{n\geq 1}$, conditional on $V_k(0) = 1$. Theorem 2.1 then leads to

COROLLARY 2.1. *Let* $(m_k(n))_{n\geq 1}$ *be as defined above. Let* $\psi: [0, \infty) \to (0, \infty)$ *be a function satisfying* (2.1), (2.2), (2.3), *and, in addition*

$$\psi(v) = o(v^c) \tag{2.6}$$

for some $c < \frac{1}{3}$. *Then*

$$P\{m_k(n) \geq \psi(n)/n \text{ for some } n \geq \nu\} \sim \textstyle\int_\nu^\infty \psi^k(t)e^{-\psi(t)}t^{-1}\,dt/\Gamma(k) \tag{2.7}$$

as $\nu \to \infty$.

For example, when $\psi(u) = \gamma \log_2 u$, the right-hand side of (2.7) is asymptotic to

$$\gamma^k \log_2{}^k \nu/(\Gamma(k)(\gamma - 1)\log^{\gamma-1}\nu)$$

$(\gamma > 1)$. Corollary 2.1 extends easily to the case of non-uniform U_n's by means of probability integral transformations.

A result analogous to Corollary 2.1 holds for the partial maxima of certain recurrent diffusion processes:

COROLLARY 2.2. *Let* $(X(t))_{t\geq 0}$ *be a diffusion in natural scale, with state space* $I = (-\infty, \infty)$ *and with speed measure* m *satisfying*

$$m\{\xi : |\xi| \geq x\} = O(1/x) \tag{2.8}$$

as $x \uparrow \infty$. *Let* ψ *be a function satisfying* (2.1), (2.6), (2.2) (*with* $k = 1$), *and* (2.3). *Then for each state* x

$$P_x\{\sup_{\tau\leq t} X(t) \leq t/(m(I)\psi(t)) \text{ for some } t \geq t_0\} \sim \Psi(t_0) \tag{2.9}$$

(*with* Ψ *defined by* (2.2) *with* $k = 1$).

Via appropriate transformations, one can get results similar to those of Corollary 1.2 of Section 1. As special cases we mention the following: for a normalized univariate Brownian motion process,

$$\begin{aligned} P_0\{\max_{r\leq s} B(r)/(r+1)^{\frac{1}{2}} &\leq (2\log_2(1+s) + \log_3(1+s) \\ &\quad - 2\log_4(1+s) - \log(c^2 4\pi))^{\frac{1}{2}} \text{ for some } s \geq s_0\} \\ &\sim c\log_3 s_0/((c-1)\log_2{}^{c-1}s_0) \end{aligned} \tag{2.10}$$

$(c > 1, s_0 \uparrow \infty)$, and for the radial part of a d-dimensional Brownian Motion

$$\begin{aligned} P_0\{\max_{r\leq s} R(r)/(r+1)^{\frac{1}{2}} &\leq (2\log_2(1+s) + d\log_3(1+s) \\ &\quad - 2\log_4(1+s) - \log(c^2 4\Gamma^2(d/2)))^{\frac{1}{2}} \text{ for some } s \geq s_0\} \\ &\sim 2c\log_3 s_0/((2c-1)\log_2{}^{2c-1}s_0) \end{aligned} \tag{2.11}$$

$(c > \frac{1}{2}, s_0 \uparrow \infty)$.

For the process $(m_1(n))_{n\geq 1}$, the integral test corresponding to (2.5) was in effect established by Barndorff-Nielsen (1961) (see also (1963)). Robbins and Siegmund (1972) reestablished this result, and used their martingale methods to get exact expressions for the tail probabilities for the time of last crossing of certain functions. Pursuing a line of investigation initiated by Darling and Erdös (1956), they also established the integral test corresponding to (2.10).

3. Statement of results: non-decreasing stable processes. We present here a refinement of Breiman's (1968) law for non-decreasing stable processes. Let $(X(t))_{t\geqq 0}$ be such a process, having exponent $\alpha \in (0, 1)$, and intensity m; thus for each t

$$\log(E(e^{-\theta X(t)})) = mt \int_0^\infty (e^{-\theta x} - 1)x^{-(1+\alpha)}\,dx = -tm\Gamma(1-\alpha)\theta^\alpha/\alpha$$

$(\theta \geqq 0)$. Set

$$\lambda = \alpha/(1-\alpha)\,, \qquad \mu = (\Gamma(1-\alpha)m)^{1/(1-\alpha)}(1-\alpha)/\alpha\,.$$

Breiman showed that for any function $\psi\colon [0, \infty) \to (0, \infty)$ satisfying

$$\psi(s) \uparrow \infty \qquad \text{as } s \uparrow \infty\,, \tag{3.1}$$

one has

$$\int^\infty (\psi(s))^{\frac{1}{2}} e^{-\psi(s)}\,ds/s \begin{Bmatrix} < \infty \\ = \infty \end{Bmatrix} \text{ if and only if } P\{X(s) \leqq s^{1/\alpha}\mu^{1/\lambda}/\psi^{1/\lambda}(s) \text{ i.o. as } s \uparrow \infty\} \begin{cases} = 0 \\ = 1\,. \end{cases}$$

Actually, this integral test was in effect first established by Lipshutz (1956) in the context of partial sums of i.i.d. random variables in the domain of attraction of the law of $X(1)$. Our methods yield

THEOREM 3.1. *Let X be as above. Suppose that ψ satisfies* (3.1), *and that for some $c < \frac{1}{3}$, one has*

$$\Psi(e^t e^{t^{1-c}}) \sim \Psi(e^t) \qquad \textit{as } t \uparrow \infty\,, \tag{3.2}$$

where

$$\Psi(s) = (2\pi\alpha)^{-\frac{1}{2}} \int_s^\infty \psi^{\frac{1}{2}}(s) e^{-\psi(s)}\,ds/s < \infty\,. \tag{3.3}$$

Let ρ be an initial distribution on $[0, \infty)$ *such that*

$$\rho([x, \infty)) = O(x^{-\gamma}) \qquad \textit{as } x \uparrow \infty$$

for some $\gamma > 0$. Then

$$P_\rho\{X(s) \leqq s^{1/\alpha}\mu^{1/\lambda}/\psi^{1/\lambda}(s) \textit{ for some } s \geqq s_0\} \sim \Psi(s_0) \qquad \textit{as } s_0 \uparrow \infty\,. \tag{3.4}$$

In particular, for $\psi(s) = c\log_2 s$ $(c > 1)$, the right-hand side of (3.4) is asymptotic to $(c \log_2 s_0)^{\frac{1}{2}}/((2\pi\alpha)^{\frac{1}{2}}(c-1)\log^{c-1} s_0)$. For a result related to Theorem 3.1, see Robbins and Siegmund (1970) page 1427.

4. Proof of Theorem 1.1 and its corollaries.

(A) *Preliminaries.* The proof of Theorem 1.1 is based on the principle that rare independent events are almost disjoint (confer, e.g., Chung (1968) page 60).

LEMMA 4.1. *Let A_n, $n \geqq 1$, be (pairwise) independent events in some probability* *space, and suppose that* $\sum_n P(A_n) < \infty$. *Then*

$$P(\bigcup_{n\geq k} A_n) \sim \sum_{n\geq k} P(A_n)$$

as $k \uparrow \infty$.

PROOF. This follows from the more precise relation

$$\gamma_k(1-\gamma_k) \leqq \gamma_k - \sum_{n\neq p;n,p\geq k} P(A_n \cap A_p) \leqq P(\bigcup_{n\geq k} A_n) \leqq \gamma_k$$

where $\gamma_k = \sum_{n\geq k} P(A_n)$. □

The next three lemmas are all to be considered in the following context. Let $(X(t))_{t\geq 0}$ be a diffusion in natural scale, with state space $I = (-\infty, \infty)$, and speed measure m having total mass $m(I) < \infty$. For $a \in I$, let τ_a denote the first hitting time of a. For $x \in I$ and $n \geqq 0$, set

$$e_a^{(n)}(x) = E_x(\tau_a^{\,n}) . \tag{4.1}$$

Note $e_a^{(0)}(x) \equiv 1$; we shall write $e_a(x)$ for $e_a^{(1)}(x)$.

The moments (4.1) are related to the speed measure through the following recursion formulas, which can be found in Mandl (1968) page 112 (for $n = 1$, confer also Freedman (1971), Formula 67b page 128).

LEMMA 4.2. *For* $n \geqq 1$ *and* x *to the right of* a, *one has*

$$e_a^{(n)}(x) = n \int_a^\infty G_a(x; y) e_a^{(n-1)}(y) m(dy) \tag{4.2}$$

$$= n \int_a^x \left(\int_u^\infty e_a^{(n-1)}(y) m(dy)\right) du \tag{4.3}$$

where

$$\begin{aligned} G_a(x; y) &= y - a , &\quad \text{if} \quad a \leqq y < x \\ &= x - a , &\quad \text{if} \quad x \leqq y < \infty . \end{aligned}$$

A dual result holds for x to the left of a. Adding the "left" and "right" versions of (4.2) for $n = 1$ gives the well known

LEMMA 4.3. *For any two states* x, y *in* I,

$$e_x(y) + e_y(x) = |y - x| m(I) . \tag{4.4}$$

From Lemma 4.2, it is clear that the less mass their is in the tails of m, the more moments τ_a will have. The following lemma makes this more precise:

LEMMA 4.4 *If the speed measure* m *satisfies*

$$m([x, \infty)) = O(1/x^\delta) \qquad \text{as } x \uparrow \infty \tag{4.5}$$

for some δ *in* $((n-1)/n, 1)$, *then* $e_a^{(n)}(x) < \infty$ *for each* a *and* x *in* I *with* $a < x$. *If* m *satisfies* (4.5) *with* $\delta = 1$ *then for each* a *in* I *one has*

$$E_\rho(e^{\theta\tau_a}) < \infty \qquad \text{for } |\theta| \text{ sufficiently small} \tag{4.6}$$

for any initial distribution ρ *on* $[a, \infty)$ *satisfying* $\rho([x, \infty)) = O(1/x^\gamma)$ *as* $x \uparrow \infty$ *for some* $\gamma > 0$. *Finally if* m *satisfies* $\int^\infty x m(dx) < \infty$, *then* (4.6) *holds for any initial distribution* ρ *on* $[a, \infty)$.

PROOF. Suppose first that (4.5) holds. Fix a in I. If m^* is a speed measure such that $m([x, \infty)) \leqq m^*([x, \infty))$ for all $x \geqq a$, then $e_a^{(n)}(x) \leqq (e_a^{(n)}(x))^*$ for all n and all $x \geqq a$; this follows from (4.3) and induction on n. Thus we may without loss of generality assume that m has a continuous density over $[a, \infty)$ satisfying

$$m(dx)/dx \leqq \kappa(a)/(x - a + 1)^{1+\delta} \tag{4.7}$$

for some $\kappa(a) < \infty$.

Suppose now $\delta = 1$. Choose and fix ζ in (0, 1) with $\zeta < \gamma$. Then (4.7) holds with δ replaced by $1 - \zeta$, and (4.3) implies

$$e_a^{(1)}(x) \leqq (\kappa(a)/(\zeta(1 - \zeta)))(x - a + 1)^{\zeta}$$

for $x \geqq a$. Use (4.3), (4.7) with δ replaced by 1, and induction on n to get

$$e_a^{(n)}(x) \leqq n!\, (\kappa(a)/(\zeta(1 - \zeta)))^n(x - a + 1)^{\zeta}$$

for $x \geqq a$ and $n \geqq 1$. Thus relative to the initial distribution ρ, the moment generating function of τ_a is finite at least in the interval $\{\theta : |\theta| < \kappa(a)/(\zeta(1 - \zeta))\}$. Next, suppose $(n - 1)/n < \delta < 1$. Use (4.3), (4.7), and induction to get

$$e_a^{(n)}(x) \leqq n!\, \kappa(a)^n(x - a + 1)^{n - n\delta}/d_n(\delta)$$

where

$$d_n(\delta) = [\delta(1 - \delta)][(2\delta - 1)(2 - 2\delta)] \cdots [(n\delta - (n - 1))(n - n\delta)]\,.$$

To finish up, suppose that $\int^\infty xm(dx) < \infty$. Then by (4.2) $\sup_{x \geqq a} e_a(x) < \infty$, and so (confer Freedman (1971) pages 111–112)

$$\sup_{x \geqq a} E_x(e^{\theta\tau_a}) < \infty$$

for all $|\theta|$ sufficiently small. □

REMARK. Lemmas 4.3 and 4.4 are valid also in the case that $I = [0, \infty)$ with 0 an instantaneous state (confer Freedman (1970) pages 135–136).

We shall make use of the following simple estimate of the rate of convergence in the strong law of large numbers:

LEMMA 4.5. *Let $V_1, V_2, \cdots$ be independent random variables whose moment generating functions are finite in a neighborhood of* 0, *and suppose that $V_2, V_3, \cdots$ are identically distributed and have mean zero. Put $S_n = V_1 + \cdots + V_n$. Let $0 < c < \frac{1}{2}$. Then there exists a number $\theta > 0$ such that*

$$P\{|S_n/n| \geqq 1/m^c \text{ for some } n \geqq m\} \leqq P\{|S_n/n| \geqq 1/n^c \text{ for some } n \geqq m\} \leqq e^{-\theta m^{1-2c}} \tag{4.8}$$

for all large m.

PROOF. Observe that

$$P\{S_n \geqq n^{1-c} \text{ for some } n \geqq m\} \leqq \sum_{n \geqq m} \inf_{t>0} (\psi^*(t)\psi^n(t)/e^{tn^{1-c}})$$

where ψ^* (respectively ψ) is the moment generating function of V_1 (respectively, of V_2). Using the fact that

$$\psi(t) = 1 + O(t^2)$$

for t near 0, it is easy to deduce that the nth term in the sum above is majorized by

$$e^{-\zeta n^{1-2c}}$$

for a suitable $\zeta > 0$, and thus that (4.8) holds, with θ chosen $< \zeta$. □

(B) *Proof of Theorem* 1.1. We turn now to the proof of Theorem 1.1. Using (1.7) choose and fix $c < \frac{1}{3}$ such that

$$G(t(1 \pm 5t^{-c})) \sim G(t) \tag{4.9}$$

as $t \uparrow \infty$. Define Markov times T_n, $n \geqq 0$, and S_n, $n \geqq 1$, inductively as follows:

$$T_0 = \tau_0$$

and for $n \geqq 1$

$$\begin{aligned} S_n &= \inf\{t > T_{n-1} : X(t) = 1\} \\ T_n &= \inf\{t > S_n : X(t) = 0\} . \end{aligned} \tag{4.10}$$

In view of (1.4), Lemmas 4.4 and 4.3, and the strong Markov property, relative to ρ the random variables T_0, $(T_n - T_{n-1})$, $n \geqq 1$, are independent and have moment generating functions finite near 0, and the $(T_n - T_{n-1})$'s are identically distributed with mean

$$\mu = (1 - 0)m(I) = m(I) .$$

For $n \geqq 1$, put

$$M_n \equiv \sup\{X(t) : T_{n-1} \leqq t \leqq T_n\} = \sup\{X(t) : S_n \leqq t \leqq T_n\} . \tag{4.11}$$

For each $t_0 > 0$, define $k \equiv k(t_0)$ to be the largest multiple of μ not exceeding t_0; thus $t_0 \sim k\mu$ as $t_0 \uparrow \infty$. Also, set $\varepsilon \equiv \varepsilon(t_0) = 1/t_0^c$ (confer (4.9)). Set

$$\begin{aligned} L_{t_0} &= \{M_n \geqq g(n\mu(1+\varepsilon)) \text{ for some } n \geqq k(1+\varepsilon)^2\} \\ U_{t_0} &= \{M_n \geqq g(n\mu(1-\varepsilon)^2) \text{ for some } n \geqq k(1-\varepsilon)\} \\ C_{t_0} &= \{X(t) \geqq g(t) \text{ for some } t \geqq t_0\} \\ A_{t_0} &= \{|T_n/n - \mu| \geqq \varepsilon\mu \text{ for some } n \geqq k(1+\varepsilon)\} \\ B_{t_0} &= \{|T_n/n - \mu| \geqq \varepsilon\mu \text{ for some } n \geqq k(1-\varepsilon)\} . \end{aligned}$$

Using $k\varepsilon^2 \uparrow \infty$ and the monotoneity of g, one finds that for t_0 large

$$\begin{aligned} L_{t_0} - A_{t_0} &\subset \{M_n \geqq g(T_n) \text{ for some } n \geqq k(1+\varepsilon)^2\} - A_{t_0} \subset C_{t_0} \\ &\subset \{M_n \geqq g(T_{n-1}) \text{ for some } n \geqq k(1-\varepsilon) + 1\} \cup B_{t_0} \subset U_{t_0} \cup B_{t_0} , \end{aligned}$$

and so

$$P_\rho(L_{t_0}) - P_\rho(A_{t_0}) \leqq P_\rho(C_{t_0}) \leqq P_\rho(U_{t_0}) + P_\rho(B_{t_0}) . \tag{4.12}$$

By the strong Markov property, the M_n's are i.i.d. random variables, and because $(X(t))_{t \geqq 0}$ is in natural scale, we have

$$P\{M_n \geqq \gamma\} = P\{X \text{ hits } \gamma \text{ before } 0 \mid X(0) = 1\} = 1/\gamma \tag{4.13}$$

for $\gamma \geqq 1$. Lemma 4.1 now implies that as $t_0 \uparrow \infty$

$$P_\rho(L_{t_0}) = (1 + o(1))(\mu(1+\varepsilon))^{-1} \sum_{n \geqq k(1+\varepsilon)^2} \mu(1+\varepsilon)/g(n\mu(1+\varepsilon)) \tag{4.14}$$
$$\geqq (1 + o(1))\mu^{-1} \int_{t_0(1+\varepsilon)^4}^{\infty} (g(t))^{-1}\, dt = (1 + o(1))G(t_0)/\mu\,.$$

by (4.9). A similar argument shows that as $t_0 \uparrow \infty$

$$P_\rho(U_{t_0}) \leqq (1 + o(1))G(t_0)/\mu\,. \tag{4.15}$$

Moreover, Lemma 4.5 implies

$$P_\rho(A_{t_0}) \leqq P_\rho(B_{t_0}) \leqq e^{-\# t_0^{1-2c}} \tag{4.16}$$

for some number $\# > 0$ and all large t_0. Since $c < 1 - 2c$, (4.16) and (1.10) imply $P_\rho(B_{t_0}) = o(G(t_0))$, and combining this with (4.12), (4.14), and (4.15) we get

$$P_\rho(C_{t_0}) \sim G(t_0)/\mu$$

as $t_0 \uparrow \infty$.

(C) *Proof of Corollary* 1.1. The scale function for the process U is (confer Breiman (1968) page 386)

$$S(y) = \int_0^y e^{\alpha z^2}\, dz\,. \tag{4.17}$$

Note

$$S(y) = (2\alpha y)^{-1} e^{\alpha y^2}(1 + O(1/y^2))$$

as $y \uparrow \infty$, and

$$S^-(x) = (\alpha^{-1}[\log(x) + 2^{-1}\log_2 x + \log(2\alpha^{\frac{1}{2}}) + o(1)])^{\frac{1}{2}}$$

as $x \uparrow \infty$ (S^- denotes the inverse of the function S). The process $(X(t))_{t\geqq 0} = (S(U(t)))_{t \geqq 0}$ is in natural scale, and its speed measure m has density

$$m(dx)/dx = 2/(S'(S^-(x)))^2$$

with respect to Lebesgue measure on $I = (-\infty, \infty)$. In particular,

$$m(I) = 2 \int 1/S'(y)\, dy = 2(\pi/\alpha)^{\frac{1}{2}}\,; \tag{4.18}$$

moreover as $|x| \uparrow \infty$,

$$m(dx)/dx = (1 + o(1))/(2\alpha x^2 \log|x|)\,.$$

Corollary 1.1 now follows directly from Theorem 1.1.

(D) *Proof of Corollary* 1.2. We give the proof for $d \geqq 2$; the proof for $d = 1$ is similar, but requires a few changes stemming from the fact that $S(0)$ below is finite for $d = 1$ (see (4.19)). Put $Y(t) = e^{-t}R^2(e^t - 1)$, $t \geqq 0$. $(Y(t))_{t\geqq 0}$ is a diffusion process with infinitesimal generator

$$\Gamma f(y) = 4yf''(y)/2 + (d - y)f'(y)$$

(confer Itô and McKean (1965), page 163), and thus scale

$$S(y) = \int_1^y z^{-d/2} e^{z/2}\, dz\,. \tag{4.19}$$

Note that

$$(4.20)\qquad S(y) = 2y^{-d/2}e^{y/2}(1 + O(1/y)) \qquad as\ y \uparrow \infty$$

$$(4.21)\qquad S^{-}(x) = 2\log(x) + d\log_2 x + (d-2)\log(2) + o(1) \qquad as\ x \uparrow \infty$$

while

$$(4.22)\qquad \begin{aligned} S(y) &= \log(y) + O(1)\,, && \text{if } d = 2 \\ &= -2(1 + o(1))/((d-2)y^{d/2-1})\,, && \text{if } d \geqq 3 \qquad \text{as } y \downarrow 0 \end{aligned}$$

$$(4.23)\qquad \begin{aligned} S^{-}(x) &= O(1)e^{-|x|}\,, && \text{if } d = 2 \\ &= (2(1 + o(1))/((d-2)|x|))^{2/(d-2)}\,, && \text{if } d \geqq 3 \qquad as\ x \downarrow -\infty\,. \end{aligned}$$

Put $(X(t))_{t \geqq 0} = (S(Y(t)))_{t \geqq 0}$. Then X is a diffusion in natural scale whose speed measure m has density

$$m(dx)/dx = 2/(4S^{-}(x)(S'(S^{-}(x)))^2)$$

with respect to Lebesgue measure on $I = (-\infty, \infty)$. In particular

$$(4.24)\qquad m(I) = 2\int_0^\infty y^{d/2}e^{-y/2}/(4y)\,dy = \Gamma(d/2)2^{d/2-1}\,.$$

Moreover, from (4.21) and (4.23), it is clear that (1.4) holds and that $-\infty$ is an entrance boundary. Since

$$\begin{aligned} P_x\{R(u) &\geqq (u+1)^{\frac{1}{2}}\psi(u) \text{ for some } u \geqq u_0\} \\ &= P_{S(x^2)}\{X(t) \geqq S(\psi^2(e^t - 1)) \text{ for some } t \geqq t_0\} \end{aligned}$$

($t_0 = \log(u_0 + 1)$), Corollary 1.2 follows from Theorem 1.1.

(E) *Proof of Corollaries* 1.3 *and* 1.4. We use the same notation as in part (D) above. Here one has

$$\begin{aligned} P_x\{R(u) &\leqq (u+1)^{\frac{1}{2}}\psi(u) \text{ for some } u \geqq u_0\} \\ &= P_{-S(x^2)}\{X^*(t) \geqq -S(\psi^2(e^t - 1)) \text{ for some } t \geqq t_0\} \end{aligned}$$

where $(X^*(t))_{t \geqq 0} = (-X(t))_{t \geqq 0}$. Corollaries 1.3 and 1.4 follow from this, Theorem 1.1, and (4.22) and (4.24).

5. Proof of Theorem 2.1 and its corollaries.

(A) *Proof of Theorem* 2.1. We begin by transforming V_k in much the same way that the process R was transformed in part (D) of Section 4. For each $t \geqq 0$, put $X(t) = e^t V(e^t)$. $X \equiv (X(t))_{t \geqq 0}$ is a Markov process with

$$\begin{aligned} P\{X(\tau) \geqq y \mid X(t) = x\} &= \textstyle\sum_{0 \leqq i \leqq k-1} B(k-1, e^{-(\tau-t)}y/x; i) \\ &\qquad \times G(k-i, 1 - e^{-(\tau-t)}; y)\,, \qquad \text{if } y \leqq e^{\tau-t}x \\ &= 0\,, \qquad \text{if } y > e^{\tau-t}x \end{aligned}$$

($t < \tau$). Thus X has stationary transition probabilities, and because $X(t)$ has a Gamma $G(k, 1)$ distribution for each t, X is stationary. In this connection, we note that as $t \uparrow \infty$

$$(5.1)\qquad L_{X(t)|X(0)=x} \to G(k, 1)$$

(the left-hand side denotes the conditional distribution of $X(t)$, given $X(0) = x$). We shall have need of the infinitesimal generator of X. To get this, observe that given $X(0) = x$, for small t, the process moves (along an "exponential" trajectory) to $e^t x$ with probability $1 - tx + o(t)$ and with probability $tx + o(t)$ to a state y distributed according to the density $ky^{k-1}/(e^t x)^k$, $0 < y < xe^t$. Thus for any continuous bounded function f with a continuous derivative, one has

$$(5.2) \qquad \lim_{t \downarrow 0} (E_x f(X(t)) - f(x))/t = kx^{1-k} \int_0^x f(\xi)\xi^{k-1}\, d\xi - xf(x) + xf'(x) \equiv \Gamma f(x)$$

$(x > 0)$.

Notice that

$$P_\rho\{V_k(u) \geqq \psi(u)/u \ \text{ for some } \ u \geqq u_0\} = P_{\rho^*}\{X(t) \geqq \psi(e^t) \ \text{ for some } \ t \geqq t_0\}$$

where $t_0 = \log(u_0)$, and where ρ^* is the law of $V_k(1)$ under the initial distribution ρ. It is easily checked that (2.4) holds with ρ replaced by ρ^*. So to prove Theorem 2.1 it suffices to show that when (2.1) holds

$$(5.3) \qquad P_{\rho^*}\{X(t) \geqq \psi(t) \ \text{ for some } \ t \geqq t_0\} \sim \left(\int_{t_0}^\infty \psi^k(t)e^{-\psi(t)}\, dt\right)/\Gamma(k) \equiv \Psi^*(t_0)$$

provided

$$\Psi^*(t(1 + t^{-c})) \sim \Psi^*(t)$$

for some $c < \frac{1}{3}$. The argument used to prove (5.3) is similar to that used to prove Theorem 1.1, with the successive returns to state 1 playing the role of the T_n's (see (4.10)). The only details we shall give here concern the mean recurrence time of state 1, and the distribution of the maximum of X between successive visits to state 1. As an analogue to Lemmas 4.3 and 4.4 one has

LEMMA 5.1. *For each state $x > 0$, let τ_x be the time of first return to x. Then τ_x is a strictly positive random variable for which*

$$(5.4) \qquad \mu(x) \equiv E_x(\tau_x) = \Gamma(k)e^x/x^k$$

and for which

$$(5.5) \qquad E_\rho(e^{\theta\tau_x}) < \infty \qquad \textit{for all } |\theta| \textit{ sufficiently small}$$

whenever ρ is an initial distribution for X such that

$$\rho((0, x]) = O(x^\gamma) \qquad \textit{as } x \downarrow 0$$

for some $\gamma > 0$.

PROOF. We evaluate $\mu(x)$ using the renewal theorem (confer Breiman (1968) page 219) which implies that

$$(5.6) \qquad \lim_{h \downarrow 0} \lim_{t \uparrow \infty} P_x\{\text{visit } x \text{ during } [t, t + h)\}/h = 1/\mu(x)\,.$$

Write

$$P_x\{\text{visit } x \text{ during } [t, t + h)\} = \sum_{1 \leq i \leq 4} \int_{J_i} P_y\{\text{visit } x \text{ during } [0, h)\} P_x\{X(t) \in dy\}$$

where

$$J_1 = [0, e^{-h}x)\,, \qquad J_2 = [e^{-h}x, x)\,, \qquad J_3 = [x, h^{-\frac{1}{3}})\,, \qquad J_4 = [h^{-\frac{1}{3}}, \infty)\,.$$

Using the instantaneous behavior of X and (5.1), one easily finds that (as $t \uparrow \infty$ and then $h \downarrow 0$) the term involving J_1 is 0; that involving J_2 is $(1 - 0(h))P_x\{X(t) \in J_2\}$; that involving J_3 is $O(h^{\frac{4}{3}})$; and that involving J_4 is $o(h)$. It follows that the left-hand side of (5.6) is x times the density of the invariant distribution, $G(k, 1)$, for X evaluated at x, i.e. $x^k e^{-x}/\Gamma(k)$. This proves (5.4). Standard arguments utilizing geometic waiting times suffice to establish (5.5); we omit the details. □

Since the sample paths of X move upwards continuously and downwards in jumps, one has, for $x < z$

$$P_x\{\sup_{t \leqq \tau_x} X(t) \geqq z\} = P_x\{X(\sigma_z) = z\}$$

where σ_z is the first exit time from the interval $[x, z)$. As an analogue to (4.13), one has

LEMMA 5.2. *$P_x\{X(\sigma_z) = z\} \sim e^x x^{-k}/(e^z z^{-k})$ as $z \uparrow \infty$.*

PROOF. Suppose we can find a function f: $(0, \infty) \to [0,1]$ such that the restriction of f to $[x, \infty)$ is continuous, $f = 0$ over $(0, x)$, $f = 1$ over $[z, \infty)$, and

$$\Gamma f(y) = 0 \tag{5.7}$$

for all y in (x, z). Then standard results in the theory of Markov processes (confer Dynkin (1965) pages 141–3) imply that $f(y) = P_y\{X(\sigma_z) \geqq z\}$ for all y; we want $f(x)$.

Using (5.2), (5.7) becomes

$$k \textstyle\int_x^y \xi^{k-1} f(\xi)\, d\xi - y^k f(y) + y^k f'(y) = 0\,. \tag{5.8}$$

Differentiate this to get

$$f''(y) - (1 - k/y)f'(y) = 0\,;$$

thus over (x, z), f must have the form

$$f(y) = \alpha(\textstyle\int_x^y e^t/t^k\, dt + \beta) \equiv h(y)\,.$$

Choosing α so that $h(z) = 1$ and then β so that h satisfies (5.8) at $y = x$, one finds

$$f(y) = (\textstyle\int_x^y e^t/t^k\, dt + e^x/x^k)/(\int_x^z e^t/t^k\, dt + e^x/x^k)\,,$$

and so

$$f(x) = (e^x/x^k)/(\textstyle\int_x^z e^t/t^k\, dt + e^x/x^k) \sim (e^x/x^k)/(e^z/z^k)$$

as $z \uparrow \infty$. □

(B) *Proof of Corollary* 2.1. Choose $c < \frac{1}{3}$ so that (2.3) and (2.6) hold. For $t > 0$, set $\varepsilon_t = 1/t^c$ and write $h(t)$ for $\psi(t)/t$. Put $J_n = (1/(1 + \varepsilon_n), 1 + \varepsilon_n)$. Making use of the T_n's introduced prior to the statement of the corollary, we have

$$P\{m_k(n) \geqq h(n) \text{ for some } n \geqq \nu\}$$
$$= P\{V_k(T_n) \geqq h(n) \text{ for some } n \geqq \nu \mid V_k(0) = 1\}$$
$$\leqq P\{T_n/n \notin J_n \text{ for some } n \geqq \nu\} \tag{5.9}$$
$$+ P\{V_k(n/(1+\varepsilon_n)) \geqq h(n) \text{ for some } n \geqq \nu \mid V_k(0) = 1\}$$
$$\leqq P\{T_n/n \notin J_n \text{ for some } n \geqq \nu\}$$
$$+ P\{V_k(u) \geqq h(u(1+\delta_u)) \text{ for some } u \geqq t(\nu) \mid V_k(0) = 1\}$$

where $t(\nu) = \nu/(1 + \varepsilon_\nu)$ and δ_u is ε_τ for $\tau = t^-(u)$ (t^- denotes the inverse of the function t). The substitution $v = u(1 + \delta_u)$ and (2.6) and (2.3) imply that

$$\int_{t(\nu)}^\infty (uh(u(1+\delta_u)))^k \exp[-uh(u(1+\delta_u))]u^{-1}\,du$$
$$\sim \int_{t(\nu)(1+\delta_{t(\nu)})}^\infty (vh(v))^k \exp[-vh(v)/(1+\delta_u)]v^{-1}\,dv$$
$$\sim \int_\nu^\infty \psi^k(v)e^{-\psi(v)}v^{-1}\,dv\,.$$

It follows from Theorem 2.1 that the second term of (5.9) is asymptotic to $\Psi(\nu)$, with Ψ defined by (2.2). But (2.3) implies $\Psi(\nu) \geqq \# e^{-\#\nu^c}$ (confer (1.10)), and Lemma 4.5 implies that the first term of (5.9) is $\leqq e^{-\#\nu^{1-2c}}$. Thus

$$P\{m_k(n) \geqq h(n) \text{ for some } n \geqq \nu\} \leqq (1 + o(1))\Psi(\nu)$$

as $\nu \uparrow \infty$. The analogous lower bound is established by applying the same sort of reasoning to the following relation, wherein $(1+\varepsilon_\tau)^2\tau f(\tau) \equiv \sup_{t\leqq\tau}(\psi(t)(1+\varepsilon_t)^3)$:

$$P\{V_k(T_n) \geqq h(n) \text{ for some } n \geqq \nu \mid V_k(0) = 1\}$$
$$\geqq P\{V_k(\tau(1+\varepsilon_\tau)^2) \geqq f(\tau) \text{ for some } \tau \geqq \nu \mid V_k(0) = 1\}$$
$$- P\{T_n \notin J_n \text{ for some } n \geqq \nu\}\,.$$

(C) *Proof of Corollary* 2.2. Choose $c < \frac{1}{3}$ so that (2.3) and (2.6) hold, and set $\varepsilon_t = 1/t^c$. Define T_n, $n \geqq 0$, as in part (B) of Section 4 (see (4.10)), but with 0, 1 replaced by x, $x + 1$ if $x > 0$, and define M_n by (4.11). Write μ for $m(I)$, $g(t)$ for $t/(\mu\psi(t))$, and determine $k = k(t_0)$ as in Section 4 (B). Over $\{X(0) = x\}$, one has

$$\{|T_n/n - \mu| < \varepsilon_n \text{ for all } n \geqq k(1 + 2\varepsilon_k/\mu)\}$$
$$\cap \{\max_{j\leqq n} M_j \leqq g(n(\mu - \varepsilon_n)) \text{ for some } n \geqq k(1+2\varepsilon_k/\mu)\}$$
$$\subset \{\max_{\tau\leqq t} X(\tau) \leqq g(t) \text{ for some } t \geqq t_0\}$$
$$\subset \{|T_{n+1}/n - \mu| \geqq \varepsilon_n \text{ for some } n \geqq k(1 - 2\varepsilon_k/\mu)\}$$
$$\cup \{\max_{j\leqq n} M_j \leqq g(n(\mu + \varepsilon_n)) \text{ for some } n \geqq k(1 - 2\varepsilon_k/\mu)\}\,.$$

We will omit the rest of the proof, which is similar to that used in Section 4 (B) and in part (B) above. In this connection, we note that, e.g. when $x \leqq 0$, $1/M_j$ is uniformly distributed on (0, 1) for each j (confer (4.13)).

(D) *Proof of* (2.10) *and* (2.11). First consider (2.10). Put $U(t) = e^{-t/2}B(e^t - 1)$, $t \geqq 0$; $(U(t))_{t\geqq 0}$ is the Ornstein–Uhlenbeck process of Section 1 (confer Corollary

1.1), with $\alpha = \frac{1}{2}$. For suitable functions f, Corollary 2.2 implies

$$
\begin{aligned}
&P_0\{\max_{r \leqq s} B(r)/(r+1)^{\frac{1}{2}} \leqq f(\log(1+s)) \text{ for some } s \geqq s_0\} \\
(5.10) \qquad &= P_0\{\max_{\tau \leqq t} U(\tau) \leqq f(t) \text{ for some } t \geqq t_0 \equiv \log(1+s_0)\} \\
&\sim \gamma^{-1} \int_{t_0}^{\infty} (Sf(t))^{-1} \exp[-t/(\gamma S f(t))]\, dt ,
\end{aligned}
$$

where $\gamma = 2(2\pi)^{\frac{1}{2}}$ (confer (4.18)) and S is given by (4.17). For

$$f(t) = (2\log(t) + \log_2 t - 2\log_3 t - \log(c^2 4\pi))^{\frac{1}{2}} ,$$

it is easily checked that (5.10) is asymptotically equivalent to the right-hand side of (2.10).

The proof of (2.11) is carried out in much the same way, utilizing the transformation $Y(t) = e^{-t}R^2(e^t - 1)$ introduced in Section 4, part (D). One finds that for suitable functions f,

$$
\begin{aligned}
&P_0\{\max_{r \leqq s} R(r)/(r+1)^{\frac{1}{2}} \leqq f(\log(1+s)) \text{ for some } s \geqq s_0\} \\
&\qquad \sim \delta^{-1} \int_{t_0} (S(f^2(t))^{-1} \exp[-t/(\delta S(f^2(t)))]\, dt
\end{aligned}
$$

where $t_0 = \log(1+s_0)$, δ is given by (4.24) and S by (4.19).

6. Proof of Theorem 3.1. The proof of Theorem 3.1 draws upon Breiman's arguments and the methods utilized in Sections 4 and 5 above. We will omit the details, except for the analogues of Lemmas 5.1 and 5.2. For $t \geqq 0$, put $Z(t) = X(e^t)/e^{t/\alpha}$; as Breiman shows, Z is a stationary Markov process with stationary transition probabilities.

LEMMA 6.1. *For each state x, let τ_x be the time of first return to x for the Z process. Then (with respect to P_x) τ_x is strictly positive with probability one, and*

$$E_x(\tau_x) = \alpha/(xD(x)) , \tag{6.1}$$

where D denotes the density of the stationary distribution for Z, i.e., the law of $X(1)$. Moreover for any initial distribution ρ on $[0, \infty)$ satisfying

$$\rho([x, \infty)) = O(1/x^{\gamma}) \qquad \textit{as } x \uparrow \infty$$

for some $\gamma > 0$, one has

$$E_\rho(e^{\theta\tau_x}) < \infty \qquad \textit{for all } |\theta| \textit{ sufficiently small.} \tag{6.2}$$

The proof is similar to that of Lemma 5.1 and will be omitted.

Now let $0 < z < x$ be two states, and let σ_z denote the first exit time of the Z process from the interval $(z, x]$, and put

$$u_x(z) = P_x\{Z(\sigma_z) = z\} .$$

LEMMA 6.2. *For each x,*

$$u_x(z) \sim (\alpha\mu/2\pi)^{\frac{1}{2}}/(z^{\lambda/2}e^{\mu z^{-\lambda}}xD(x))$$

as $z \downarrow 0$.

PROOF. This is basically Breiman's Theorem 3, with closer attention being given to the constants of proportionality. Tracing through Breiman's argument (and making a few minor corrections), one finds

$$u_x(z) = 1/(1 + \int_z^x \theta(\xi)\, d\xi)$$

where

$$\int_z^x \theta(\xi)\, d\xi = (1 + o(1))R(z)m \int_0^x D(\xi)/(x - \xi)^\alpha\, d\xi \tag{6.3}$$

with

$$R(z) = \int_{z^\delta}^\infty \exp[z^{-\lambda}(-\xi + \beta\xi^\alpha)]\xi^{-1}\, d\xi$$

($\delta = 1/(1 - \alpha)$ and $\beta = m\Gamma(1 - \alpha)/\alpha$). The Laplace transform of $x \to \int_0^x D(\xi)(x - \xi)^{-\alpha}\, d\xi$ is

$$s \to m\Gamma(1 - \alpha)s^{\alpha-1}e^{-\beta s^\alpha} = -(d/ds)e^{-\beta s^\alpha},$$

so the right-hand side of (6.3) is in fact $(1 + o(1))R(z)xD(x)$. Laplace's method can be used to get the asymptotic expansion for $R(z)$, with the result that

$$R(z) \sim cz^{\lambda/2}e^{\mu z^{-\lambda}}$$

where

$$c = (2\pi/|w''(t_0)|)^{\frac{1}{2}}/t_0 = (2\pi/\alpha\mu)^{\frac{1}{2}}$$

with $t_0 = (\alpha\beta)^\delta$ being the point that maximizes $w : t \to -t + \beta t^\alpha$.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS 60637