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EXCURSIONS OF STATIONARY GAUSSIAN PROCESSES ABOVE HIGH MOVING BARRIERS[1]

BY SIMEON M. BERMAN

New York University

Let $X(t)$ be a real stationary Gaussian process with covariance function $r(t)$; and let $f(t)$, $t \geqq 0$, be a nonnegative continuous function which vanishes only at $t = 0$. Under certain conditions on $r(t)$ and $f(t)$, we find, for fixed $T > 0$ and for $u \to \infty$ (i) the asymptotic form of the probability that $X(t)$ exceeds $u + f(t)$ for some $t \in [0, T]$; and (ii) the conditional limiting distribution of the time spent by $X(t)$ above $u + f(t)$, $0 \leqq t \leqq T$, given that the time is positive.

0. Statement of the main result. Let $X(t)$, $t \geqq 0$, be a stationary Gaussian process with mean 0, variance 1, continuous covariance function $r(t)$, and continuous sample functions on every finite interval. For $T > 0$, let $1 - r(t)$ be strictly positive on $(0, T]$, and regularly varying with exponent α for $t \downarrow 0$:

$$(0.1) \qquad \lim_{x \downarrow 0} \frac{1 - r(xt)}{1 - r(x)} = t^\alpha , \qquad t \geqq 0 ,$$

for some α, $0 < \alpha \leqq 2$. Let $f(t)$ be a continuous function on $[0, T]$, strictly positive on $(0, T]$, and regularly varying at $t = 0$, and vanishing with exponent β, $\beta \geqq \alpha/2$, for $t \downarrow 0$:

$$(0.2) \qquad \lim_{x \downarrow 0} \frac{f(xt)}{f(x)} = t^\beta , \qquad t \geqq 0 .$$

Let I_A be the indicator of the event A; then

$$(0.3) \qquad \xi_T = \int_0^T I_{[X(s) - f(s) > u]} \, ds$$

represents the time spent by $X(s)$ above the curve $u + f(s)$ on the interval $[0, T]$. For $u > 0$, let v be the largest solution of the equation

$$(0.4) \qquad u^2[1 - r(1/v)] = 1 ,$$

and w the largest solution of

$$(0.5) \qquad uf(1/w) = 1 ;$$

under the conditions on r and f, these solutions exist for all sufficiently large positive u. Put

$$(0.6) \qquad \phi(u) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}u^2} .$$

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The main results are: If

$$p = \lim_{t \downarrow 0} \frac{f(t)}{(1 - r(t))^{\frac{1}{2}}} \tag{0.7}$$

exists and is finite, then there exists a bounded concave non-decreasing function $F(x)$, $x \geqq 0$, such that

$$\lim_{u \to \infty} \frac{P\{\max_{[0,T]} (X(t) - f(t)) > u\}}{(v/w)\phi(u)/u} = F'(0) , \tag{0.8}$$

and such that

$$\lim_{u \to \infty} \frac{P\{v\xi_T > x\}}{P\{v\xi_T > 0\}} = \frac{F'(x)}{F'(0)} . \tag{0.9}$$

F is given explicitly in terms of α, β and p. (See (6.11) and (6.12).)

1. Discussion of the results. This is a continuation of the author's work on the amount of time spent by a stationary Gaussian process above a high level [1], [2], [3], and by a Gaussian process with stationary increments [4]. The present study is different from the others: the high level barrier u is replaced by a moving barrier in the form of a function $u + f(t)$, where u is large and f is fixed. Previous work on the crossing of a moving barrier has been done by Leadbetter [10], [11], Leadbetter and Cryer [12], Chover [5], and Fieger [8]. The only asymptotic result on the high moving barrier is a bound for the probability of crossing a linear barrier in the case of a stationary process with a finite second spectral moment, obtained by Miroshin [13].

One of the features of this work is the use of a simple but very general integral equation for the distribution of the occupation time of an arbitrary stochastic process. While integral equations have been useful in studying occupation times of Markov processes, or results are obtained for non-Markovian processes. The integral equation method replaces the method of moments used in the previous work: all the earlier results can be obtained by the method used here. One of the important lemmas in [2] has an error so that this paper also serves to correct that result.

In addition to the more powerful method used here, the conditions on the covariance function are more general than those in [2]; they are comparable to those in [15].

The condition given on f is that it have a unique minimum at $t = 0$, where it has the value 0. The theorem holds also in a modified form for any function f with a unique minimum at some point in the interval from 0 to T, and with a minimum value not necessarily equal to 0. This modification is indicated in Section 9.

Our result on the moving barrier is not a generalization of the previous one for the fixed barrier, but is different: The *moving* barrier is likely to be crossed just at the low point of the barrier, but the *fixed* barrier can be crossed at any point. The two limiting distributions of the time spent above the barrier are different. Finally we remark that the case $0 < \beta < \alpha/2$ and other extensions have been considered by the author, and will appear in a future publication.

2. An integral equation for the excursion distribution of an arbitrary stochastic process.

LEMMA 2.1. *Let* $X(t)$, $0 \leqq t \leqq T$, *be a measurable, separable stochastic process with Borel sample functions; let* $b(t)$, $0 \leqq t \leqq T$, *be a fixed Borel function; and put*

$$\xi_t = \int_0^t I_{[X(s)>b(s)]}\, ds\,, \qquad 0 \leqq t \leqq T\,. \tag{2.1}$$

Then for $0 < A < B$:

$$\int_A^B P\{\xi_T > x\}\, dx = \int_0^T P\{A < \xi_t \leqq B,\, X(t) > b(t)\}\, dt\,. \tag{2.2}$$

PROOF. Let $G(x)$ be a bounded, measurable function; then

$$E\{\int_0^T G(X(s))\, ds\}^m = \int_0^T \cdots \int_0^T E\{G(X(s_1)) \cdots G(X(s_m))\}\, ds_1 \cdots ds_m\,.$$

By the symmetry of the integrand, the latter integral may be written as

$$m \int_0^T E\{G(X(s_m)) \int_0^{s_m} \cdots \int_0^{s_m} G(X(s_1)) \cdots G(X(s_{m-1}))\}\, ds_1 \cdots ds_m\,,$$

which is identical with

$$m \int_0^T E\{G(X(t))(\int_0^t G(X(s))\, ds)^{m-1}\}\, dt\,.$$

Now replace $X(t)$ by $X(t) - b(t)$, and let $G(x)$ be the indicator function of the positive real axis; then the relations displayed above imply that

$$E\xi_T{}^m = m \int_0^T E(I_{[X(t)>b(t)]}\xi_t)^{m-1}\, dt\,, \qquad m \geqq 2\,. \tag{2.3}$$

Now we get another expression for $E\xi_T{}^m$ by simple integration by parts:

$$E\xi_T{}^m = m \int_0^\infty x^{m-1} P\{\xi_T > x\}\, dx\,, \qquad m \geqq 1\,;$$

hence, by substitution in (2.3), we obtain

$$\int_0^\infty x^m P\{\xi_T > x\}\, dx = \int_0^T E(I_{[X(t)>b(t)]}\xi_t)^m\, dt\,, \qquad m \geqq 1\,. \tag{2.4}$$

The left-hand side of (2.4) is the mth moment of a bounded monotone function on the positive axis whose derivative is

$$\begin{array}{ll} P\{\xi_T > x\}\,, & \text{for } 0 < x \leqq T \\ 0\,, & \text{elsewhere.} \end{array}$$

The right-hand side of (2.4) is the mth moment of the distribution of $I_{[X(t)>b(t)]}\xi_t$, mixed over t with respect to the uniform Lebesgue measure on 0, T. It follows from the uniqueness of the moment sequences that the distributions coincide on the positive axis:

$$\int_A^B P\{\xi_T > x\}\, dx = \int_0^T P\{A < I_{[X(t)>b(t)]}\xi_t \leqq B\}\, dt\,, \qquad 0 < A < B\,. \tag{2.5}$$

This is equivalent to (2.2) because the indicator assumes only the values 0 and 1. The proof is complete.

When X is stationary the integrand on the right-hand side of (2.2) may be transformed by means of the equation

$$(2.6) \qquad P\{A < \xi_t \leqq B, X(t) > b(t)\} \\ = P\{A < \textstyle\int_0^t I_{[X(s)>b(t-s)]}\,ds \leqq B, X(0) > b(t)\}\,.$$

The proof is as follows: The left-hand side of (2.6) is, by definition,

$$P\{A < \textstyle\int_0^t I_{[X(s)>b(s)]}\,ds \leqq B, X(t) > b(t)\}\,.$$

Since the process is invariant under time shifts and time reversals, we may replace the time index s in the process by $t - s$, and then change the variable of integration from s to $t - s$; this yields the right-hand side of (2.6).

3. Functions of regular variation. We recall some of the properties of functions of regular variation. Let $g(t)$, $t \geqq 0$, be a continuous function of regular variation with exponent α (that is, $g = 1 - r$ in (0.1)). By Karamata's theorem [9] there exist functions $a(y)$ and $c(y) \geqq 0$ such that

$$\lim_{x\downarrow 0} c(x) = c_0\,, \qquad (0 < c_0 < \infty)$$
$$\lim_{x\downarrow 0} a(x) = \alpha\,,$$

and such that for all sufficiently small t—for example, $0 \leqq t \leqq 1$—the function $g(t)$ has the form

$$(3.1) \qquad g(t) = c(t)\exp\left(-\int_t^1 \frac{a(y)}{y}\,dy\right).$$

From this representation we deduce an important estimate:

LEMMA 3.1. *For every h, $0 < h < \alpha$, there exists $T > 0$ such that*

$$(3.2) \qquad \tfrac{1}{2}t^\alpha \exp\{-h|\log t|\} \leqq g(xt)/g(x) \leqq 2t^\alpha \exp\{h|\log t|\}$$

for all x and t such that $0 \leqq xt \leqq T$, $0 < x \leqq T$.

PROOF. Pick T so small that

$$\tfrac{1}{2} \leqq c(xt)/c(x) \leqq 2\,, \qquad |a(x) - \alpha| < h\,,$$

for the indicated values of x and t; then write the integrand in the exponent in (3.1) as

$$\frac{a(y) - \alpha}{y} + \frac{\alpha}{y}\,.$$

We deduce several more properties of such functions.

LEMMA 3.2. *The convergence of $g(xt)/g(x)$ to t^α is uniform on every closed bounded interval.*

PROOF. This is known for intervals bounded away from 0 [15]; here we extend it to intervals $[0, T]$. Without loss of generality suppose that $0 < T \leqq 1$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that $g(xt)/g(x) < \varepsilon/2$ (apply (3.2) with $h = \alpha/2$) and $t^\alpha < \varepsilon/2$, for all $0 \leqq t \leqq \delta$, and all $0 \leqq xt \leqq T$, $0 < x \leqq T$. Hence, for such x and t we have $|g(xt)/g(x) - t^{\alpha}| < \varepsilon$. Since ε is arbitrary, the uniformity of the convergence for all $0 \leqq t \leqq T$ now follows from the known result about the uniform convergence for the interval $[\delta, T]$.

Now we consider the relation between the variations of $1 - r(t)$ and $f(t)$, and the relative order of v and w, which are defined by equations (0.4) and (0.5). Since $1 - r$ and f tend to 0 at the origin, v and w tend to ∞ as u does.

LEMMA 3.3. *Let p be the limit in* (0.7); *then*

$$\lim_{u\to\infty} (w/v) = p^{1/\beta} = q \,. \tag{3.3}$$

In particular, if $\beta > \alpha/2$, this holds with $p = q = 0$.

PROOF. By the definitions of p, v and w, we have

$$f(1/v) \sim p(1 - r(1/v))^{\frac{1}{2}} = pf(1/w) \,.$$

From this relation and the representation (3.1) for f, it follows that

$$\begin{aligned} \lim_{u\to\infty} \log (w/v) &= (1/\beta) \log p \,, && \text{if } p > 0 \,, \\ &= -\infty \,, && \text{if } p = 0 \,. \end{aligned}$$

LEMMA 3.4. *For every $h > 0$:*

$$\lim_{u\to\infty} (v^{\alpha-h}/u^2) = \lim_{u\to\infty} (w^{\beta-h}/u) = 0 \,.$$

PROOF. This is a direct consequence of the definitions of v and w and of the representation (3.1).

4. A weak convergence theorem and an inequality for the distribution of the maximum. In this section we present certain preliminary results on Gaussian processes which will be used in the proofs of our main theorems. Let $X(t)$, $t \geqq 0$, be a stationary Gaussian process satisfying the conditions leading to and including (0.1). For $u > 0$ and arbitrary z form the process

$$X_u(t) = u[X(t/v) - u] - u^2[r(t/v) - 1] - zr(t/v) \,, \tag{4.1}$$

conditioned by $X_u(0) = 0$. (The finite dimensional distributions of the conditioned process are obtained from those of the original process (4.1) by replacing the joint Gaussian distributions by the conditional Gaussian distributions, given $X_u(0)$.) This process naturally arises in the analysis of the excursions of X above u [2]. By elementary computations, we find the conditional first and second order moments:

$$E\{X_u(t) \,|\, X_u(0) = 0\} = 0 \,, \tag{4.2}$$

$$\begin{aligned} &\operatorname{Var} \{X_u(t) - X_u(s) \,|\, X_u(0) = 0\} \\ &\qquad = 2u^2\left[1 - r\left(\frac{t - s}{v}\right)\right] - u^2[r(s/v) - r(t/v)]^2 \,. \end{aligned}$$

Let $U(t)$, $t \geqq 0$, be a Gaussian process with stationary increments and with the specific moment functions

$$E(U(t)) = 0\,, \qquad E(U(0))^2 = 0\,, \qquad E(U(t) - U(s))^2 = 2|t - s|^\alpha\,,$$

where α is the exponent of variation of $1 - r$.

LEMMA 4.1. *Let* **J** *be a closed finite subinterval of the nonnegative t-axis. Then the process* $X_u(t)$, $t \in \mathbf{J}$, *conditioned by* $X_u(0) = 0$, *converges for* $u \to \infty$ *weakly over* $C(\mathbf{J})$ *to the process* $U(t)$, $t \in \mathbf{J}$.

PROOF. The proof of the convergence in distribution of the conditioned process is exactly the same as in [2]. There the weak compactness of the family of distributions was proved under the additional assumption that $1 - r(t) \sim g(t)^\alpha$, where $g(t)$ is slowly varying and increasing. The proof of weak compactness under our more general assumption (0.1) is the same as the earlier one except for one point: we need a suitable bound on the conditional variance in (4.2). Such a bound is furnished by Lemma 3.1:

$$\begin{aligned}\operatorname{Var}\{X_u(t) - X_u(s) \mid X_u(0) = 0\} &\leqq 2u^2\left[1 - r\left(\frac{t-s}{v}\right)\right] \\ &= 2\,\frac{1 - r\left(\frac{t-s}{v}\right)}{1 - r\left(\frac{1}{v}\right)} \\ &\leqq 4|t-s|^\alpha \exp\{h|\log|t-s||\}\,, \qquad 0 < h < \alpha\,.\end{aligned}$$

Now the higher central moments of a normally distributed random variable are multiples of powers of the standard deviation; hence,

$$\begin{aligned}&E\{(X_u(t) - X_u(s))^{2m} \mid X_u(0) = 0\} \\ &\qquad \leqq K_m|t-s|^{m\alpha} \exp\{mh|\log|t-s||\}\,, \qquad s, t \in \mathbf{J},\ m \geqq 1\,.\end{aligned}$$

Weak compactness of the family of measures now follows by the well-known moment criterion [14]; we take m so large that $m(\alpha - h) > 1$.

Several other forms of Lemma 4.1 follow from it:

COROLLARY 4.1. *The process* $u[X(t/v) - u]$, $t \in \mathbf{J}$, *conditioned by* $u[X(0) - u] = z$, *converges weakly to* $U(t) - t^\alpha + z$. *Similarly the conditioned process* $u[X(t/v) - u] - uf(t/w)$ *converges weakly to* $U(t) - t^\alpha - t^\beta + z$.

PROOF. Write $u[X(t/v) - u]$ as $X_u(t) + u^2[r(t/v) - 1] + zr(t/v)$. The condition $u[X(0) - u] = z$ is equivalent to $X_u(0) = 0$. Now $u^2[r(t/v) - 1] \to -t^\alpha$ and $uf(t/w) \to t^\beta$ uniformly on **J** (Lemma 3.2), and $zr(t/v) \to z$ uniformly on **J**. It follows from the linearity of weak convergence over C that the conditioned processes $X_u(t) + u^2[r(t/v) - 1] + zr(t/v)$ and $X_u(t) + u^2[r(t/v) - 1] + zr(t/v) - uf(t/w)$ converge weakly to the indicated limiting processes.

Lemma 4.1 and its corollary are used to estimate the distribution of the time spent above a high barrier and the probability that the process crosses the barrier.

The latter probability is small if u is large; however the *conditional* probability given $X(0) = u + z/u$ has a limit for fixed z. We express the unconditional probability as the integral of the conditional probability, apply Lemma 4.1, and then pass to the limit.

As a preliminary to the proof of the following theorem, we recall the symbol $\phi(u)$ in (0.6), and note the simple identity

$$\phi(u + z/u) = \phi(u)e^{-z}e^{-z^2/2u^2} . \tag{4.3}$$

THEOREM 4.1. *For given z, let P_z be the measure on the function space $C(\mathbf{J})$ induced by the Gaussian process $U(t) + z$, $t \in \mathbf{J}$. Let $V(f), f \in C(\mathbf{J})$, be a functional which is continuous at all points of $C(\mathbf{J})$ except for those in a set of P_z-measure 0, for all $-\infty < z < \infty$. Then for $-\infty < c < d \leqq \infty$:*

$$\lim_{u\to\infty} \frac{1}{\phi(u)/u} P\{V(u[X(t/v) - u]) \leqq x, c \leqq u[X(0) - u] \leqq d\} = \int_c^d P\{V(U(t) - t^\alpha + z) \leqq x\}e^{-z}\, dz , \tag{4.4}$$

for all x in the continuity set of the latter function. If $c = -\infty$ then the relation (4.4) holds if

$$P\{V(u[X(t/v) - u]) \leqq x \mid u[X(0) - u] = z\}e^{-z}$$

is uniformly (in u) dominated over the negative axis by an integrable function of z.

PROOF. Write the joint probability on the left-hand side of (4.4) as the integral of the conditional probability given $u[X(0) - u] = z$, and then divide by $\phi(u)/u$:

$$\frac{\int_c^d P\{V(u[X(t/v) - u]) \leqq x \mid X(0) = u + z/u\}\phi(u + z/u)\, dz/u}{\phi(u)/u} .$$

Apply (4.3) and let $u \to \infty$; then, by Corollary 4.1 and dominated convergence, the expression above converges to the limit in (4.4).

We shall use Theorem 4.1 for the particular functionals

$$\max_J f \qquad \text{and} \qquad V(f) = \int_{\mathbf{J}} I_{[f(s)>x]}\, ds , \qquad \text{for fixed } x.$$

The first of these is continuous in the uniform topology, but the second is not; however, in our applications the points of discontinuity of V will form a set of measure 0:

LEMMA 4.2. *Let $Y(t)$, $t \in \mathbf{J}$, be a Gaussian process with continuous sample functions and such that the set*

$$\{t : \text{Var } Y(t) = 0, t \in \mathbf{J}\}$$

is of Lebesgue measure 0. Then the points of discontinuity of $V(f), f \in C(\mathbf{J})$, form a set of measure 0 under the probability measure induced by the process Y.

PROOF. It follows from the hypothesis that the measure on the line

$$m(A) = \int_{\mathbf{J}} P\{Y(t) \in A\}\, dt , \quad A = \text{Lebesgue measurable set,}$$

is absolutely continuous with respect to Lebesgue measure. For arbitrary $\varepsilon > 0$, let $G(y)$ be a continuous distribution function such that

$$G(y) = 0\,, \quad \text{for} \quad y \leqq x$$
$$= 1\,, \quad \text{for} \quad y \geqq x + \varepsilon\,;$$

and let $H(y)$ be a continuous distribution function such that

$$H(y) = 0\,, \quad \text{for} \quad y \leqq x - \varepsilon$$
$$= 1\,, \quad \text{for} \quad y \geqq x\,;$$

then

$$\int_J G(f(t))\,dt \leqq V(f) \leqq \int_J H(f(t))\,dt\,, \qquad f \in \mathbf{J}\,.$$

The extreme members of this inequality are continuous functionals on $C(\mathbf{J})$; furthermore,

$$E\left|\int_J H(Y(t))\,dt - \int_J G(Y(t))\,dt\right| \leqq \int_J P\{|Y(t) - x| \leqq \varepsilon\}\,dt\,.$$

By the absolute continuity of the measure $m(A)$, the latter integral tends to 0 with ε. Donsker's criterion [6] now implies that V is continuous almost everywhere.

Now we derive a bound on the probability that $X(0)$ is below a certain level, but the maximum on an interval to the right of the origin is above another level. It is based on Fernique's inequality [7]. This inequality states that if $Y(t)$, $t \in \mathbf{J}$, is a Gaussian process with mean 0, and if the second moment of $Y(t)$ is bounded on J, and if there is a continuous, monotonically non-decreasing function ψ such that $E(Y(t) - Y(s))^2 \leqq \psi^2(|t - s|)$, and $\int_1^\infty \psi(e^{-x^2})\,dx < \infty$, then there exist positive constants K, L, and M such that

$$P\{\max_J Y > u\} \leqq K \int_{u/L}^\infty \phi(x)\,dx\,, \qquad \text{for} \quad u \geqq M\,. \tag{4.5}$$

By the well-known inequality for the tail of the normal distribution,

$$\int_u^\infty \phi(x)\,dx \leqq \phi(u)/u\,, \qquad \text{for} \quad u \geqq 0\,, \tag{4.6}$$

it follows that the right-hand side of the inequality (4.5) is at most

$$\frac{K\phi(u/L)}{u/L} \qquad \text{for} \quad u \geqq M\,. \tag{4.7}$$

It follows from the form of the moments in (4.2) and from the estimates of the variance in the proof of Lemma 4.1 that the conditioned process $X_u(t)$ satisfies Fernique's inequality. Now we prove a more special form of the inequality for the process $X_u(t)$.

LEMMA 4.3. *Given an arbitrary closed bounded interval* $\mathbf{J}$ *on the nonnegative axis, there exist positive numbers* K, L, *and* M *such that for all sufficiently large* u, *the following inequality holds for all* x *and* y *satisfying* $x \geqq y + M$, $x \geqq M$:

$$P\{\max_{t \in \mathbf{J}} X(t/v) > u + x/u,\ X(0) \leqq u + y/u\} \leqq K(\phi(u)/u)\left(e^{-2x} + \int_{(x-y)/L}^\infty \phi(z)\,dz\right)\,. \tag{4.8}$$

PROOF. Write the joint probability in (4.8) as the integral of the conditional probability, and use the notation of (4.1):

$$(4.9)\qquad \int_{-\infty}^{y} P\{\max_{t\in \mathbf{J}}[X_u(t)+u^2[r(t/v)-1]+zr(t/v)]>x \mid X_u(0)=0\}\phi(u+z/u)\,dz/u\ .$$

Let u (and v) be so large that

$$(4.10)\qquad \tfrac{1}{2}\leqq r(t/v)\leqq 1\ ,\qquad \text{for } t\in\mathbf{J}\ ;$$

then, by (4.3), the expression (4.9) is not more than

$$(4.11)\qquad \frac{\phi(u)}{u}\int_{-\infty}^{0} P\left\{\max_{\mathbf{J}} X_u(t)>\frac{2x-z}{2}\,\middle|\, X_u(0)=0\right\}e^{-z}\,dz + \frac{\phi(u)}{u}\int_0^y P\{\max_{\mathbf{J}}\{X_u(t)+zr(t/v)\}>x\mid X_u(0)=0\}e^{-z}\,dz\ .$$

By changing the variable of integration in the first term in (4.11) from z to $z-2x$, we see that it is dominated by

$$\frac{\phi(u)}{u}e^{-2x}\int_{-\infty}^{\infty} P\{\max_{\mathbf{J}} X_u(t)>-\tfrac{1}{2}z\mid X_u(0)=0\}e^{-z}\,dz\ .$$

Now apply the inequality (4.5) to the conditioned process X_u for large negative z; then, by the estimate (4.7), the integral above is finite. This accounts for the first part of the right-hand side of (4.8).

The second term in (4.11) has the same sign as y. If y is negative, it may be discarded, and the inequality (4.8) certainly holds. If y is positive, then the second term in (4.11) is at most equal to

$$\frac{\phi(u)}{u}P\{\max_J X_u(t)>x-y\mid X_u(0)=0\}\ ,$$

which, by (4.5) and (4.7), yields the second part of the bound (4.8).

5. Preliminary estimate of the probability of crossing the barrier. In this section we establish the order of magnitude of the probability that $X(t)$ exceeds the barrier $u+f(t)$ for some $0\leqq t\leqq T$ and large u:

THEOREM 5.1. *For any $T>0$:*

$$\limsup_{u\to\infty}\frac{P\{\max_{[0,T]}[X(t)-f(t)]>u\}}{(v/w)\phi(u)/u}<\infty\ .$$

(Later we show that the ratio above actually approaches a limit.)

The proof is accomplished in a series of lemmas. The main idea of the proof is that if X crosses the barrier $u+f$ at some point in the interval, then the crossing is likely to take place near the beginning of the interval, where the movable part of the barrier, the function $f(t)$, has not yet grown too large. The probability of crossing over this small interval is then estimated by means of the limit result of Theorem 4.1, with V as the maximum functional.

First we show that the maximum over a fixed interval is asymptotically equivalent to the maximum over a fixed subinterval of arbitrarily small length.

LEMMA 5.1. *For any* d, $0 < d < T$:

$$\lim_{u\to\infty} \frac{P\{\max_{[d,T]}[X(t) - f(t)] > u\}}{(v/w)\phi(u)/u} = 0 .$$

PROOF. Since $f(t) = 0$ only for $t = 0$, f is bounded away from 0 on $[d, T]$; hence, it suffices to prove that for every interval $\mathbf{J}$ and every $\varepsilon > 0$,

$$\lim_{u\to\infty} \frac{P\{\max_{\mathbf{J}} X(t) > u + \varepsilon\}}{(v/w)\phi(u)/u} = 0 .$$

The interval J is divisible into approximately v intervals of equal length; hence, by Boole's inequality and the stationarity of the process X, the ratio displayed above is not more than

$$\frac{wP\{\max_{t\in \mathbf{J}} X(t/v) > u + \varepsilon\}}{\phi(u)/u} .$$

For arbitrary $M > 1$, write this as

$$(5.1) \qquad \frac{wP\{\max_{t\in J} X(t/v) > u + \varepsilon,\, X(0) > u + (M \log w)/u\}}{\phi(u)/u} + \frac{wP\{\max_{t\in J} X(t/v) > u + \varepsilon,\, X(0) \leqq u + (M \log w)/u\}}{\phi(u)/u} .$$

The first term in (5.1) is at most

$$\frac{wP\{X(0) > u + (M \log w)/u\}}{\phi(u)/u} ,$$

which, by (4.6) and (4.3), is at most w^{1-M}, which tends to 0 as $u \to \infty$.

By Lemma 3.4, $u\varepsilon - M \log w \to \infty$ with u; thus by Lemma 4.3, the second term in (5.1) is dominated by

$$Kw(e^{-2\varepsilon u} + \textstyle\int_{(u\varepsilon - M\log w)/L}^{\infty} \phi(z)\, dz) \to 0 .$$

In the next lemma we show that maximum of $X - f$ over an interval of fixed length is asymptotically equivalent to the maximum over an initial segment of the interval of length of order $1/w$. As part of the proof we need this estimate: If X has a standard normal distribution, then, for every $M > 1$ and every positive integer k:

$$(5.2) \qquad \textstyle\sum_{j\geqq k} P\{X > u + (M \log j)/u\} \leqq \frac{\phi(u)}{u} \sum_{j\geqq k} j^{-M} .$$

This follows from (4.6) and (4.3).

LEMMA 5.2. *For every* $T > 0$:

$$\lim_{D\to\infty} \limsup_{u\to\infty} \frac{P\{\max_{[D/w,T]}(X(t) - f(t)) > u\}}{(v/w)\phi(u)/u} = 0 .$$

PROOF. By virtue of Lemma 5.1, we may assume that T is arbitrarily small —for example, so small that (by Lemma 3.1)

$$f(t/w)/f(1/w) \geqq \tfrac{1}{2}t^{\beta}\exp\{-h|\log t|\}\,, \qquad \text{for } 0 \leqq t \leqq T,\ 0 \leqq t \leqq wT\,, \tag{5.3}$$

where $0 < h < \beta$. Cut the interval $[D/w, T]$ into approximately $Tw - D$ intervals of length $1/w$. By Boole's inequality and the stationarity of X we have

$$P\{\max_{[D/w,T]}(X(t) - f(t)) > u\} \leqq \textstyle\sum_{D\leqq j<Tw} P\{\max_{[0,1/w]} X(t) > u + \min_{t\in[j,j+1]} f(t/w)\}\,.$$

By (5.3) and (0.5) the sum above is at most

$$\textstyle\sum_{D\leqq j<Tw} P\{\max_{[0,1/w]} X(t) > u + j^{\beta-h}/2u\}\,. \tag{5.4}$$

In estimating (5.4) there are two cases to consider: $p > 0$ and $p = 0$ in (0.7).

Case $p > 0$. Let $\mathscr{E}_j$ be the event

$$\max_{[0,1/w]} X(t) > u + j^{\beta-h}/2u$$

and $\mathscr{D}_j$ the event

$$X(0) \leqq u + (M\log j)/u\,.$$

From the elementary inequality

$$\textstyle\sum_j P(\mathscr{E}_j) \leqq \sum_j P(\mathscr{E}_j \cap \mathscr{D}_j) + \sum_j (1 - P(\mathscr{D}_j))\,,$$

and the estimate (5.2), it follows that the sum (5.4) is at most

$$\textstyle\sum_{D\leqq j<Tw} P\{\max_{[0,1/w]} X(t) > u + j^{\beta-h}/2u,\ X(0) \leqq u + (M\log j)/u\} + (\phi(u)/u)\sum_{j\geqq D} j^{-M}\,. \tag{5.5}$$

Divide each term in (5.5) by $(v/w)\phi(u)/u$, and let $u \to \infty$. By Lemma 3.3 w is of the same order as v; we may apply Lemma 4.3 to the probabilities in (5.5) with $x = \frac{1}{2}j^{\beta-h}$ and $y = M\log j$. Then the first sum in (5.5), after division by $(v/w)\phi(u)/u$, is at most

$$K\textstyle\sum_{j\geqq D}\{e^{-j^{\beta-h}} + \int_{(j^{\beta-h}-M\log j)/2L}^{\infty}\phi(z)\,dz\}\,,$$

which tends to 0 as $D \to \infty$. Upon division by $(v/w)\phi(u)/u$, the second term in (5.5) converges to

$$q\textstyle\sum_{j\leqq D}' j^{-M}\,,$$

which tends to 0 as $D \to \infty$.

Case $p = 0$. By Lemma 3.3, v/w tends to ∞. For large u, the interval from 0 to $1/w$ may be broken up into approximately v/w intervals of equal length $1/v$. By Boole's inequality and the stationarity of X, the sum (5.4) is dominated by

$$(v/w)\textstyle\sum_{D\leqq j<Tw} P\{\max_{[0,1/v]} X(t) > u + j^{\beta-h}/2u\}\,.$$

When divided by $(v/w)\phi(u)/u$ it becomes the same as the sum considered above in the case $p > 0$; the proof is completed as in that case.

Finally we estimate the tail of the distribution of $\max X - f$ over an interval of length of order $1/w$:

LEMMA 5.3. *For every $D > 0$:*

$$\limsup_{u\to\infty} \frac{P\{\max_{[0,D/w]}(X(t) - f(t)) > u\}}{(v/w)\phi(u)/u} < \infty .$$

PROOF. *Case $p > 0$.* By Lemma 3.3 we may replace w by v; thus it suffices to show that

$$\frac{P\{\max_{[0,D]}(X(t/v) - f(t/v)) > u\}}{\phi(u)/u}$$

is bounded for $u \to \infty$. Write this as

$$\frac{1}{\phi(u)/u} P\{\max_{[0,D]}(u[X(t/v) - u] - uf(t/v)) > 0,\ -\infty < u[X(0) - u] < \infty\} ;$$

then the latter converges to

$$\int_{-\infty}^{\infty} P\{\max_{[0,D]}[U(t) - t^\alpha - (qt)^\beta + y] > 0\} e^{-y}\, dy .$$

This follows from Theorem 4.1 and from the reasoning used in the proof of Corollary 4.1; the hypothesis of the theorem for the case $c = -\infty$ is fulfilled by virtue of Fernique's inequality (4.5) and the estimate (4.7).

Case $p = 0$. As in the proof of Lemma 5.2 we break the interval $[0, D/w]$ into approximately Dv/w subintervals of length $1/v$; then, by Boole's inequality and the stationarity of X, we obtain

$$(5.6)\qquad \begin{aligned} &P\{\max_{[0,D/w]}[X(t) - f(t)] > u\} \\ &\quad \leqq \sum_{0\leqq j<Dv/w} P\{\max_{t\in[0,1]} X(t/v) > u + \min_{t\in[j,j+1]} f(t/v)\} . \end{aligned}$$

Let N be an arbitrary fixed positive integer. If u is sufficiently large, then Dv/w exceeds some multiple of N. Suppose for simplicity, that it is *equal* to a multiple of N; then the sum in (5.6) is at most

$$\frac{Dv/w}{N} \sum_{1\leqq k\leqq N} P(\max_{t\in[0,1]} X(t/v) > u + \min_{t\in[(k-1)D/N,kD/N]} f(t/w)\} .$$

Divide this by $(v/w)\phi(u)/u$, and let $u \to \infty$: By Theorem 4.1 and the uniform convergence of $uf(t/w)$ to t^β (Lemma 3.2), the expression above converges to

$$(5.7)\qquad (D/N) \sum_{k=1}^{N} \int_{-\infty}^{\infty} P\left\{\max_{t\in[0,1]}[U(t) - t^\alpha + y] > \left[\frac{(k-1)D}{N}\right]^\beta\right\} e^{-y}\, dy .$$

(The same argument holds for all large Dv/w—even when it is not a multiple of N.) The average (5.7) is a Riemann sum which approximates the integral

$$\int_0^D \int_{-\infty}^{\infty} P\{\max_{s\in[0,1]} U(s) - s^\alpha + y > t^\beta\} e^{-y}\, dy\, dt .$$

The latter integral is finite; this follows by application of Fernique's inequality (4.5) to the process U.

Theorem 5.1 is a direct consequence of Lemmas 5.1, 5.2, and 5.3: By Boole's inequality the tail of the distribution of the maximum over $[0, T]$ is dominated by those over the three subintervals $[0, D/w]$, $[D/w, d]$, and $[d, T]$, respectively.

6. A limit theorem for ξ_T. Now we study the asymptotic form of the distribution of the random variable $v\xi_T$ for $u \to \infty$, where ξ_T is defined by (0.3) as the time spent by the process above the barrier $u + f(t)$. We apply Lemma 2.1 with $b(t) = u + f(t)$; then, by a change of variable of integration, we get

$$\int_A^B P\{v\xi_T > x\}\, dx = v \int_0^T P\{A < v\xi_t \leqq B, X(t) > u + f(t)\}\, dt .$$

Divide both members by $(v/w)\phi(u)/u$:

$$\int_A^B \frac{P\{v\xi_T > x\}\, dx}{(v/w)\phi(u)/u} = w \int_0^T \frac{P\{A < v\xi_t \leqq B, X(t) > u + f(t)\}\, dt}{\phi(u)/u} . \tag{6.1}$$

We let $u \to \infty$ on both sides of this equation. The right-hand side will be shown to converge to $F(B) - F(A)$, where F is bounded and monotonic. Then we pass to the limit under the integral sign on the left-hand side of (6.1) to obtain

$$\int_A^B \lim_{u\to\infty} \frac{P\{v\xi_T > x\}}{(v/w)\phi(u)/u}\, dx = F(B) - F(A) .$$

From this we shall conclude that the limit of the integrand exists and is equal to $F'(x)$ almost everywhere.

First we transform the right-hand side of (6.1). By the relation (2.6) the probability in the integrand is equal to

$$P\{A < v \int_0^t I_{[X(s)>u+f(t-s),X(0)>u+f(t)]}\, ds \leqq B\} .$$

Insert this in the integrand on the right-hand side of (6.1), change the variable of integration from s to vs, and then the other variable of integration from t to wt; then the right-hand side of (6.1) becomes

$$\frac{\int_0^{Tw} P\{A < \int_0^{tv/w} I_{[X(s/v)>u+f(t/w-s/v),X(0)>u+f(t/v)]}\, ds \leqq B\}\, dt}{\phi(u)/u} . \tag{6.2}$$

Our task is to show that the expression (6.2) converges for $u \to \infty$.

We need the following simple estimate of the double tail of the bivariate normal distribution:

LEMMA 6.1. *Let X and Y have a joint normal distribution, and let c be an arbitrary positive number. Then for every $x \geqq 0$ and every y,*

$$P\{X > x, Y > y\} \leqq \exp\{-cx + \tfrac{1}{2}c^2 \operatorname{Var}(X \mid Y)\} E\{e^{cE(X\mid Y)} I_{[Y>y]}\} .$$

PROOF. If X is a normally distributed random variable, then (cf. [2])

$$P\{X > x\} \leqq \exp\{-cx + cEX + \tfrac{1}{2}c^2 \operatorname{Var}(X)\} , \qquad \text{for } x \geqq 0 .$$

Write $P\{X > x, Y > y\}$ as

$$E\{I_{[Y>y]}\, P\{X > x \mid Y\}\} ,$$

and apply the inequality to the conditional probability that $X > x$.

Our plan in estimating (6.2) is to pass to the limit under the sign of integration; so now we show that the integrand is uniformly dominated by an integrable function.

LEMMA 6.2. *There exist positive numbers* a, b, *and* c *such that the integrand in* (6.2) *is uniformly in large* t *dominated by* $c \cdot \exp(-bt^a)$ *for all sufficiently large* u.

PROOF. By the Chebyshev inequality and the nonnegativity of f, the integrand is at most

$$\frac{1}{A\phi(u)/u} E\{\textstyle\int_0^{tv/w} I_{[X(s/v)>u, X(0)>u+f(t/w)]}\, ds\},$$

which, by Fubini's theorem, is

$$\frac{1}{A\phi(u)/u} \int_0^{tv/w} P\{u[X(s/v)-u] > 0,\, u[X(0)-u] > uf(t/w)\}\, ds\,. \tag{6.3}$$

Now apply Lemma 6.1, with $0 < c < 1$, $x = 0$, and $y = uf(t/w)$, and note that by (4.2)

$$E\{u[X(s/v)-u] \mid u[X(0)-u] = z\} = u^2[r(s/v)-1] + zr(s/v)\,,$$
$$\operatorname{Var}\{u[X(s/v)-u] \mid u[X(0)-u] = z\} = u^2[1-r^2(s/v)]\,;$$

then (6.3) is at most

$$\frac{1}{A\phi(u)/u} \int_0^{tv/w} \exp\{\tfrac{1}{2}c^2u^2[1-r^2(s/v)]\}$$
$$\times \int_{uf(t/w)}^{\infty} \exp\{cu^2[r(s/v)-1] + czr(s/v)\}\phi(u+z/u)\, dz\, ds\,.$$

By the relation (4.3), the nonnegativity of z and of $1-r$, and the nonnegativity of $r(1/w)$ for large w, the expression displayed above is at most

$$\frac{1}{A(1-c)} \int_0^{tv/w} \exp\{-c(1-c)u^2[1-r(s/v)] - (1-c)uf(t/w)\}\, ds\,.$$

By Lemma 3.1 this is uniformly dominated, for all sufficiently large u by

$$\frac{1}{A(1-c)} \exp\{-\tfrac{1}{2}(1-c)t^\beta e^{-h|\log t|}\} \int_0^\infty \exp\{-\tfrac{1}{2}c(1-c)s^\alpha e^{-h|\log s|}\}\, ds\,. \tag{6.4}$$

The first exponential factor is at most $\exp(-\frac{1}{2}(1-c)t^{\beta-h})$ for $t \geqq 1$; and the integral converges.

By Lemma 6.2 we may take the limit under the integral sign in (6.2); therefore, it suffices to determine the limit of

$$\frac{P\{A < Y_t \leqq B\}}{\phi(u)/u}\,, \tag{6.5}$$

where

$$Y_t = \int_0^{tv/w} I_{[X(s/v)>u+f(t/w-s/v),\, X(0)>u+f(t/w)]}\, ds\,.$$

If tv/w remains bounded, as in the case $p > 0$, then we can find the limit of (6.5) by means of the weak convergence theorems of Section 4; however, this is not directly possible when $v/w \to \infty$, as when $p = 0$. For this reason we introduce a convergence factor in the integral Y_t which puts the weight of the integral on a bounded set (cf. [3]). For arbitrary $\lambda > 0$, define

$$Y_{t,\lambda} = \int_0^{tv/w} e^{-\lambda s} I_{[X(s/v)>u+f(t/w-s/v),X(0)>u+f(t/w)]}\,ds\,.$$

We shall show that if λ is small then $Y_{t,\lambda}$ may be substituted for Y_t in (6.5) with only a small alteration of the limit:

LEMMA 6.3. *The ratio*

$$\frac{E|Y_t - Y_{t,\lambda}|}{\phi(u)/u} \tag{6.6}$$

tends to 0 *with* λ *uniformly in* $t \geqq 0$ *and all sufficiently large* u.

PROOF. By the definitions of Y_t and $Y_{t,\lambda}$, the ratio (6.6) is equal to

$$\int_0^{tv/w} \frac{(1 - e^{-\lambda s})}{\phi(u)/u} P\left\{X(s/v) > u + f\left(\frac{t}{w} - \frac{s}{v}\right),\ X(0) > u + f(t/w)\right\} ds\,,$$

which is dominated by

$$\int_0^{tv/w} \frac{(1 - e^{-\lambda s})}{\phi(u)/u} P\{X(s/v) > u,\ X(0) > u + f(t/w)\}\,ds\,.$$

This is proportional to the expression (6.3), except for a factor $1 - e^{-\lambda s}$ in the integrand. By the same reasoning as in the proof of Lemma 6.2, the latter integral is at most equal to A times the expression (6.4), with the alteration of the integral by a factor $1 - e^{-\lambda s}$. The integral tends to 0 monotonically with λ; and the convergence can be shown to be uniform in $t \geqq 0$.

LEMMA 6.4. *For arbitrary* $\lambda > 0$, *the ratio*

$$\frac{P\{A < Y_{t,\lambda} \leqq B\}}{\phi(u)/u} \tag{6.7}$$

converges to

$$\int_{t^\beta}^{\infty} P\{A < \int_0^{t/q} e^{-\lambda s} I_{[U(s)-s^\alpha-(t-qs)^\beta>-y]}\,ds \leqq B\}e^{-y}\,dy\,, \qquad \textit{for}\ \ p > 0\,, \tag{6.8}$$

and to

$$e^{-t^\beta} \int_0^{\infty} P\{A < \int_0^{\infty} e^{-\lambda s} I_{[U(s)-s^\alpha>-y]}\,ds \leqq B\}e^{-y}\,dy\,, \qquad \textit{for}\ \ p = 0\,. \tag{6.9}$$

PROOF. By the definition of $Y_{t,\lambda}$, the ratio (6.7) is equal to

$$\frac{1}{\phi(u)/u} P\{A < \int_0^{tv/w} e^{-\lambda s} I_{[u[(X(s/v)-u]-uf(t/w-s/v)>0]}\,ds \leqq B,\quad u[X(0) - u] > uf(t/w)\}\,. \tag{6.10}$$

Case $p > 0$. By Lemmas 3.2 and 3.3,

$$uf\left(\frac{t}{w} - \frac{s}{v}\right) \to (t - qs)^\beta$$

uniformly in s-values satisfying $0 \leqq s \leqq tv/w$, and

$$uf(t/w) \to t^\beta ;$$

therefore, by Theorem 4.1 (with $c = t^\beta$ and $d = \infty$) the expression (6.10) converges to (6.8).

Case $p = 0$. By Lemma 3.3 the upper limit of integration, tv/w, in the excursion integral in (6.10) tends to ∞. For arbitrary $R > 0$, express the integral as the sum of an integral over the domain $[0, R]$, and an integral over $[R, tv/w]$. The second integral is at most $e^{-\lambda R}/\lambda$; hence, for large R, it hardly affects the limit of (6.10). The first integral, over the domain $[0, R]$, behaves in the same way as the integral in the case $p > 0$, except that

$$uf\left(\frac{t}{w} - \frac{s}{v}\right) \to t^\beta .$$

We find the limit of (6.10) with R in place of tv/w:

$$\int_{t^\beta}^\infty P\{A < \int_0^R e^{-\lambda s} I_{[U(s) - s^\alpha - t^\beta > -y]}\, ds \leqq B\} e^{-y}\, dy .$$

Then we let $R \to \infty$, and change the variable of integration from y to $y - t^\beta$, to get (6.9).

For $x > 0$, define $F(x)$ as

$$\int_0^\infty \int_{t^\beta}^\infty P\{0 < \int_0^{t/q} I_{[U(s) - s^\alpha - (t - qs)^\beta > -y]}\, ds \leqq x\} e^{-y}\, dy\, dt , \quad \text{for } p > 0 \text{ (and } \beta = \alpha/2) , \tag{6.11}$$

and as

$$\int_0^\infty e^{-t^\beta}\, dt \cdot \int_0^\infty P\{0 < \int_0^\infty I_{[U(s) - s^\alpha > -y]}\, ds \leqq x\} e^{-y}\, dy , \qquad \text{for } p = 0 . \tag{6.12}$$

LEMMA 6.5. *The integral* (6.2) *converges to* $F(B) - F(A)$.

PROOF. According to the remarks following Lemma 6.2, it suffices to evaluate the limit of (6.5) and then integrate over t. By Lemma 6.3 we may first find the corresponding limit for $Y_{t,\lambda}$, and then let $\lambda \to 0$. According to Lemma 6.4, we get this limit by setting $\lambda = 0$ in (6.8) and (6.9), and then obtain $F(B) - F(A)$ by integration over t.

Here is the main result of this section:

THEOREM 6.1. *Let* $F'(x)$ *be the derivative of the monotone function* $F(x)$, *defined by* (6.11) *and* (6.12); F' *is defined almost everywhere. Then*

$$\lim_{u \to \infty} \frac{P\{v\xi_T > x\}}{(v/w)\phi(u)/u} = F'(x)$$

for almost all $x > 0$.

PROOF. By (6.1) and Lemma 6.5:

$$\lim_{u \to \infty} \int_A^B \frac{P\{v\xi_T > x\}}{(v/w)\phi(u)/u}\, dx = F(B) - F(A) , \tag{6.13}$$

for all $0 < A < B$. The integrand in (6.13) is uniformly (in u) bounded in x on $[A, \infty)$, for every $A > 0$. To prove this, note that the integrand cannot but increase when x is replaced by A, and then apply Chebyshev's inequality: the integrand is at most

$$\frac{wE\xi_T}{A\phi(u)/u}, \qquad \text{or} \qquad \int_0^{Tw} \frac{P\{X(s/w) > u + f(s/w)\}}{\phi(u)/u}\, ds\,.$$

$X(s/w)$ has the standard normal distribution; hence, by (4.6) and (4.3), the integral above is dominated by

$$\int_0^{Tw} \exp(-uf(s/w))\, ds\,.$$

This converges to $\int_0^\infty \exp(-s^\beta)\, ds$; the proof is based on the inequality in Lemma 3.1 and the fact that $f(t)$ is bounded away from 0 when t is.

We have just shown that the integrand in (6.13) is uniformly bounded. It is also monotonically non-increasing. Therefore, it has a weak limit $W(x)$ over some sequence of u-values. It follows from (6.13) by bounded convergence that

$$\int_A^B W(x)\, dx = F(B) - F(A)\,; \tag{6.14}$$

hence, $W(x) = F'(x)$ for almost all $x > 0$. F' appears as the unique weak limit of the integrand in (6.13); therefore, the integrand necessarily converges over every sequence of u-values to the same limit $W(x)$.

As the indefinite integral of a non-increasing function, $F(x)$ is necessarily concave.

7. Properties of the function F.

LEMMA 7.1. *F is of positive variation on the positive axis.*

PROOF. It follows from the definition of $F(x)$ that

$$F(0+) = 0\,; \tag{7.1}$$

hence, it suffices to prove that $F(\infty) > 0$.

Case $p > 0$. Here $F(\infty)$ is equal to

$$\int_0^\infty \int_{t^\beta}^\infty P\{\int_0^{t/q} I_{[U(s)-s^\alpha-(t-qs)^\beta > -y]}\, ds > 0\} e^{-y}\, dy\, dt\,. \tag{7.2}$$

We shall show that this is positive by assuming the contrary and deducing a contradiction. If the integral (7.2) is equal to 0, then so is the integrand, almost everywhere:

$$P\{\int_0^{t/q} I_{[U(s)-s^\alpha-(t-qs)^\beta > -y]}\, ds > 0\} = 0$$

for almost all y and t such that $0 < t^\beta < y < \infty$. This implies that

$$P\{\max_{s \in [0,t/q]} U(s) - s^\alpha - (t - qs)^\beta \leqq -y\} = 1$$

for all such y and t; in particular, letting $y \to \infty$, we find that

$$P\{\max_{s \in [0,t/q]} U(s) - s^\alpha - (t - qs)^\beta = -\infty\} = 1\,.$$

But this cannot be true because $U(0) = 0$, almost surely.

Case $p = 0$. Here $F(\infty)$ is

$$\int_0^\infty e^{-t^\beta}\,dt \cdot \int_0^\infty P\{\int_0^\infty I_{[U(s)-s^\alpha>-y]}\,ds > 0\}e^{-y}\,dy\,;$$

the proof of positivity is the same as in the previous case.

Next we show that F has a positive (finite or infinite) right-handed derivative at the origin; later, it will be shown to be finite.

LEMMA 7.2. *The limit*

$$F'(0) = \lim_{h\downarrow 0} F(h)/h$$

exists and is greater than 0.

PROOF. By (6.14):

$$F(h)/h = (1/h)\int_0^h W(x)\,dx\,.$$

Since $W(x)$ is non-increasing, the limit $F'(0)$ of the expression above exists for $h \to 0$, and satisfies $F'(0) \geqq W(x)$, for all $x \geqq 0$. If $F'(0) = 0$, then $W(x) = 0$ for all $x > 0$, and so, by (6.14), $F(x) = 0$ for all $x > 0$. But this would contradict the fact that F has positive variation.

8. An exact asymptotic formula for the probability of crossing the barrier. The convergence in Theorem 6.1 holds for almost all $x > 0$. We want to extend this to hold at $x = 0$:

$$\frac{P\{v\xi_T > 0\}}{(v/w)\phi(u)/u} \to F'(0)\,.$$

Since the sample functions of the process are continuous, the probability in the numerator above is $P\{\max_{[0,T]} X(t) - f(t) > u\}$. Then the limit relation above would imply

$$P\{\max_{[0,T]} X(t) - f(t) > u\} \sim F'(0)(v/w)\phi(u)/u\,, \qquad u \to \infty\,.$$

Theorem 5.1 would then imply that $F'(0) < \infty$. These results are established in the following theorem:

THEOREM 8.1

$$\lim_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u\}}{(v/w)\phi(u)/u} = F'(0) < \infty\,. \tag{8.1}$$

PROOF. First we show that

$$F'(0) \leqq \liminf_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u\}}{(v/w)\phi(u)/u}\,. \tag{8.2}$$

If $X(t) - f(t)$ spends positive time above u, then its maximum certainly exceeds u: for any $x > 0$,

$$P\{v\xi_T > x\} \leqq P\{\max_{[0,T]} X(t) - f(t) > u\}\,.$$

Average this inequality over x from $A > 0$ to B, and then divide by $(v/w)\phi(u)/u$:

$$\frac{1}{B-A}\int_A^B \frac{P\{v\xi_T > x\}}{(v/w)\phi(u)/u}\,dx \leqq \frac{P\{\max_{[0,T]} X(t) - f(t) > u\}}{(v/w)\phi(u)/u}\,.$$

Pass to the limit and apply (6.13):

$$\frac{F(B) - F(A)}{B - A} \leqq \liminf_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u\}}{(v/w)\phi(u)/u}.$$

Let $A \to 0$ and then $B \to 0$; this proves (8.2). From this and Theorem 5.1 we also infer that $F'(0) < \infty$.

Now we establish the reverse inequality corresponding to (8.2), with *lim sup* in place of *lim inf*. The proof of this part is much more complicated. The main idea is the converse of that used in the first part: if the process spends very little time above the barrier, then the process is hardly likely to go too far over the barrier. For arbitrary $x > 0$, we decompose the event $\{\max_{[0,T]} X(t) - f(t) > u\}$ into its intersections with $\{v\xi_T > x\}$ and $v\xi_T \leqq x\}$, and then apply the argument following (8.2):

$$P\{\max_{[0,T]} X(t) - f(t) > u\}$$
$$= P\{v\xi_T > x\} + P\{\max_{[0,T]} X(t) - f(t) > u,\, v\xi_T \leqq x\}\,.$$

Average this relation over $[A, B]$, and note that the second term on the right-hand side is non-decreasing in x:

$$P\{\max_{[0,T]} X(t) - f(t) > u\}$$
$$\leqq \frac{1}{B - A} \int_A^B P\{v\xi_T > x\}\, dx + P\{\max_{[0,T]} X(t) - f(t) > u,\, v\xi_T \leqq B\}\,.$$

Divide both sides of this inequality by $(v/w)\phi(u)/u$, and let $u \to \infty$:

$$\limsup_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u\}}{(v/w)\phi(u)/u}$$
$$\leqq \frac{F(B) - F(A)}{B - A} + \limsup_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u,\, v\xi_T \leqq B\}}{(v/w)\phi(u)/u}\,.$$

Then let $A \to 0$ and then $B \to 0$:

(8.3)
$$\limsup_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u\}}{(v/w)\phi(u)/u}$$
$$\leqq F'(0) + \lim_{B\to 0} \limsup_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u,\, v\xi_T \leqq B\}}{(v/w)\phi(u)/u}\,.$$

Then (8.1) will follow from (8.2) and (8.3) as soon as we prove

(8.4)
$$\lim_{B\to 0} \limsup_{u\to\infty} \frac{P\{\max_{[0,T]} X(t) - f(t) > u,\, v\xi_T \leqq B\}}{(v/w)\phi(u)/u} = 0\,.$$

According to Lemmas 5.1 and 5.2 we may replace the interval $[0, T]$ in (8.4) by an interval $[0, T/w]$, where T is large but fixed. Therefore, it suffices to replace the ratio in (8.4) by

(8.5)
$$\frac{P\{\max_{[0,T/w]} X(t) - f(t) > u,\, v\xi_{T/w} \leqq B\}}{(v/w)\phi(u)/u}\,.$$

We shall show that this converges to 0 if we take the limit over $u \to \infty$ and then $B \to 0$.

Case $p = 0$. By Lemma 3.3, we may replace w by qv in (8.5); the ratio may be written as

$$(8.6) \qquad \frac{P\{\max_{t\in[0,T/q]} X(t/v) - f(t/v) > u,\ \int_0^{T/q} I_{[X(s/v)-f(s/v)>u]}\, ds \leqq B\}}{(1/q)\phi(u)/u}\,.$$

Now condition by $X(0) = u + z/u$. The conditioned process $u[X(t/v) - u] - uf(t/v)$ converges weakly to $U(t) - t^\alpha - (qt)^\beta + z$ (Corollary 4.1). It follows that the conditional joint distribution of the two functionals in (8.6), the maximum functional and the occupation time functional, converges to the joint distribution under the limiting process, indeed, the maximum functional is continuous in the uniform topology, and the occupation time functional is continuous almost everywhere under the limiting probability measure (Lemma 4.2). It follows, as in Theorem 4.1, that the ratio (8.6) converges to

$$(8.7) \qquad q \int_{-\infty}^{\infty} P\{\max_{[0,T/q]} U(t) - t^\alpha - (qt)^\beta > -z,\ \int_0^{T/q} I_{[U(s)-s^\alpha-(qs)^\beta>-z]}\, ds \leqq B\} e^{-z}\, dz\,.$$

(The special hypothesis of the theorem for the case $c = -\infty$ is fulfilled by virtue of Fernique's inequality (4.5) and the estimates (4.6) and (4.7); the reasoning is similar to that for the case $p > 0$ in the proof of Lemma 5.3.)

Now let $B \to 0$ in the integral (8.7): it tends to 0; indeed, the limit of the probability in the integrand is the probability that $U(t) - t^\alpha - (qt)^\beta$ exceeds $-z$ at some point on $[0, T/q]$ but spends no time above $-z$; and the latter probability is equal to 0 because the sample function is continuous.

Case $p = 0$. We estimate (8.5) by the method of proof of Lemma 5.3: as in that proof, the ratio (8.5) is dominated by

$$\frac{1}{(v/w)\phi(u)/u} \sum_{0\leqq j<Tv/w} P\{\max_{t\in[j,j+1]} X(t/v) > u + \min_{t\in[j,j+1]} f(t/v),\ \int_j^{j+1} I_{[X(s/v)>u+\max_{t\in[j,j+1]} f(t/v)]}\, ds \leqq B\}\,.$$

By stationarity, this is equal to

$$\frac{1}{(v/w)\phi(u)/u} \sum_{0\leqq j<Tv/w} P\{\max_{0\leqq t\leqq 1} X(t/v) > u + \min_{t\in[j,j+1]} f(t/v),\ \int_0^{1} I_{[X(s/v)>u+\max_{t\in[j,j+1]} f(t/v)]}\, ds \leqq B\}\,;$$

the latter converges to

$$\int_0^T \int_{-\infty}^{\infty} P\{\max_{s\in[0,1]} U(s) - s^\alpha + y > t^\beta,\ \int_0^1 I_{[U(s)-s^\alpha+y>t^\beta]}\, ds \leqq B\} e^{-y}\, dy\, dt\,.$$

As in the case $p > 0$, this integral converges to 0 with B. This completes the proof of the theorem.

Combining Theorems 6.1 and 8.1, we obtain the following form of the theorem on the time spent above the moving barrier:

THEOREM 8.2. *For every $T > 0$, and almost every $x > 0$:*

$$\lim_{u\to\infty} \frac{P\{v\xi_T > x\}}{P\{v\xi_T > 0\}} = \frac{F'(x)}{F'(0)} .$$

In other words, $1 - F'(x)/F'(0)$ is the limit of the conditional distribution function of $v\xi_T$, given that it is positive.

PROOF. The denominator on the left-hand side is equivalent to $P\{\max_{[0,T]} X(t) - f(t) > u\}$. Divide the numerator and denominator by $(v/w)\phi(u)/u$, and let $u \to \infty$; and then apply Theorems 6.1 and 8.1.

9. Extensions and applications. As stated in Section 1, the essence of the condition on the function f is that it have a unique minimum t_0 on the interval $[0, T]$, and that f be of regular variation at t_0:

$$\lim_{x\to 0} \frac{f(xt + t_0) - f(t_0)}{x} = t^\beta .$$

If t_0 is in the interior of the interval then the same limit theorems hold except that the process $U(s) - s^\alpha - (sq)^\beta$ over $s \geqq 0$ is extended to $U(s) - |s|^\alpha - |sq|^\beta$ over $-\infty < s < \infty$. This is a generalization of the two-sided excursion limit theorem considered in [3], Section 2.

This result can be applied to finding the limiting distribution of $\max_{[0,t]} Y(s)$, for $t \to \infty$, where Y is a Gaussian process with a stationary covariance function, but a periodic mean value function. Such a process is of the form $X(s) - f(s)$, where X is a stationary Gaussian process with mean 0 and f is a periodic function. Suppose that f has period $1/T$, a unique minimum value t_0 on the fundamental interval $[0, T]$, and is of regular variation at t_0. For simplicity, suppose also that $f(t_0) = 0$. For $u > 0$, the probability that $\max_{[0,T]} Y \leqq u$ is the probability that the stationary process X does not overshoot the moving barrier $u + f(t)$. For large $t > 0$, $\max_{[0,t]} X(s) - f(s)$ can be expressed as the maximum of approximately t/T partial maxima of the process over subintervals of equal length T. As in [2], it can be shown that in the calculation of the limiting distribution of the maximum for $t \to \infty$, the partial maxima may be assumed to be mutually independent random variables. It follows that the limiting distribution of the maximum over $[0, t]$ is equivalent to that of the maximum of a sequence of independent random vriables with the common distribution function

$$G(x) = P\{\max_{[0,T]} X(s) - f(s) \leqq x\} .$$

The normalizing coefficients for the limiting distribution are obtained from the formula of Theorem 8.1 for the tail of G. The limiting distribution function is the usual double exponential, $\exp(-e^{-x})$.

10. Correction and extension of previous results. In [2] we obtained the conditional limiting distribution of $v\xi_T$ given $\xi_T > 0$, where

$$\xi_T = \textstyle\int_0^T I_{[X(s)>u]}\, ds ;$$

this is the time spent above the level barrier u. Using the integral-equation method of the present work, we can get the same results as in [2]; furthermore, the condition stated here for $1 - r$ is weaker than the one imposed in [2], where it was required that $1 - r(t) \sim g(t)t^{\alpha}$, $t \to 0$, where $g(t)$ is slowly varying and increasing. More important than the relaxation of the condition on r is the correction of an error in the proof of the main result in [2]. There is a slip in the statement of Lemma 2.2: the exponent $\alpha/2$ in formula (2.16) should be replaced by $2/\alpha$. This does not disturb the proofs of Theorems 2.2 and 2.3 when $1 < \alpha \leqq 2$; however, when $0 < \alpha < 1$, the conclusion about the finiteness of the moment generating function is in doubt. (When $\alpha = 1$, the moment generating function has a positive radius of convergence; this is demonstrated in [3].) This error invalidates the method of moments used in [2]; however, the results about the existence of the limiting distribution are still correct, but the distribution has to be described without reference to moments. By the method used in this paper we can prove that

$$\lim_{u\to\infty} \frac{P\{v\xi_T > x\}}{Tv\phi(u)/u}$$

exists for $x > 0$, and is equal almost everywhere to $(d/dx)F_{\alpha}(x)$, where

$$F_{\alpha}(x) = \int_0^{\infty} P\{\int_0^{\infty} I_{[U(s)-s^{\alpha}>-y]}\, ds \leqq x\}e^{-y}\, dy\, .$$

The proof of the existence and finiteness and positivity of

$$\lim_{u\to\infty} \frac{P\{\max_{[0,T]} X(s) > u\}}{Tv\phi(u)/u}$$

is practically the same as in [2] because the moments are not directly involved. Combining the two limit relations above we get the same result as in Theorem 8.2 for the level barrier u.

In [4] we extended the results of [2] on the limiting distribution of ξ_T to a process with stationary Gaussian increments. Let $X(t)$ be such a process, with mean 0 and with $X(0) = 0$ almost surely; and define $Y(t) = X(t)/(EX^2(t))^{\frac{1}{2}}$, $t > 0$. Let B be a closed bounded subinterval of the positive axis, bounded away from 0. Under certain conditions on $EX^2(t)$ it is shown that the tail of the distribution of $\max_B Y$ is of the same form as in the stationary case, namely,

$$P\{\max_B Y > u\} \sim \text{constant} \quad v\phi(u)/u\,, \qquad \text{for } u \to \infty\,.$$

It is then shown that the conditional limiting distribution of the time spent above u by $Y(t)$ is a scale mixture of the distribution F_{α} defined above:

$$\lim_{u\to\infty} \frac{P\{v \int_B I_{[Y(t)>u]}\, ds > x\}}{P\{v \int_B I_{[Y(s)>u]}\, ds > 0\}} = \frac{\int_B \frac{d}{dx} F\left(\frac{x}{s}\right) dH(s)}{\int_B \frac{d}{dx} F\left(\frac{x}{s}\right)\Big|_{x=0} dH(s)}\,,$$

where $H(s)$ is a distribution function on B. The given proof is based on the method of moments; therefore, it is valid only for $1 \leqq \alpha \leqq 2$ because of the error. However, the proof can be modified and corrected by the method of the present paper. Start with the relation (6.1), with $f \equiv 0$ and $w = 1$. Then, as in Lemma 6.3 we introduce the convergence factor $e^{-\lambda s}$ in the excursion integral; then the integral becomes a bounded random variable, and the method of moments may be applied to it. The conclusion about the limiting distribution is then obtained by letting $\lambda \to 0$, as justified by Lemma 6.3.

Note added in proof. The referee suggested this better proof of (2.2): Simply take expectations on both sides of the identity

$$\int_A^B I_{[\xi_T > x]}\, dx = \int_0^T I_{[A < \xi_t \leqq B, X(t) > b(t)]}\, dt .$$

Professor Roy Davies showed me how to verify this, but only after the paper was sent for publication.

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Department of Mathematics
New York University
New York, New York 10012