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LOCAL ASYMPTOTIC LAWS FOR BROWNIAN MOTION[1]

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Upper and lower functions are defined for the large values of $|X_d(t+u) - X_d(t-v)|$ as $(u+v) \downarrow 0$ where X_d is a standard Brownian motion in R^d, and it is shown that the integral test for two-sided growth in R^d is the same as that for one-sided growth in R^{d+2}. It is also shown that, for $d \geqq 4$, the lower asymptotic growth rate of $|X_d(t+u) - X_d(t-v)|$ for small $(u+v) = h$ is the same as the lower growth rate of $|X_{d-2}(t+h) - X_{d-2}(t)|$. Integral tests are also obtained for local asymptotic growth rates of the associated processes $P_d(a) = \inf_{t \geqq 0} \{t : |X(t)| \geqq a\}$ and $M_d(t) = \sup_{0 \leqq s \leqq t} |X_d(t)|$.

1. Introduction. We consider a standard Brownian motion process $X_d(t)$ taking values in R^d. The celebrated "law of iterated logarithm" states

$$P\left\{\limsup_{h \downarrow 0} \frac{|X_d(t+h) - X_d(t)|}{\{2h \log|\log h|\}^{\frac{1}{2}}} = 1\right\} = 1 \tag{1.1}$$

and it is clear that, for $t > 0$, we also have

$$P\left\{\limsup_{h \downarrow 0} \frac{|X_d(t) - X_d(t-h)|}{\{2h \log|\log h|\}^{\frac{1}{2}}} = 1\right\} = 1 . \tag{1.2}$$

From (1.1) and (1.2) it is easy to deduce that

$$P\left\{1 \leqq \limsup_{u+v=h \to 0,\, u \geqq 0,\, v \geqq 0} \frac{|X_d(t+u) - X_d(t-v)|}{\{2h \log|\log h|\}^{\frac{1}{2}}} \leqq 2^{\frac{1}{2}}\right\} = 1$$

but this is not good enough to complete the argument used in [12] to obtain the strong variation. In fact it was shown there that

$$P\left\{\limsup_{u+v=h \to 0,\, u \geqq 0,\, v \geqq 0} \frac{|X_d(t+u) - X_d(t-v)|}{\{2h \log|\log h|\}^{\frac{1}{2}}} = 1\right\} = 1 . \tag{1.3}$$

In order to get further information about the large values of $|X_d(t+h) - X_d(t)|$ as $h \downarrow 0$, Lévy divided growth functions φ into an upper class $\mathscr{U}_d$ and a lower class $\mathscr{L}_d$ (see Section 3 for precise definitions), and there is an integral test to determine whether φ is in $\mathscr{U}_d$ or $\mathscr{L}_d$. The first objective of this paper is to obtain the corresponding criterion for two-sided growth. In Section 3 we show that the two-sided growth rate in R^d is precisely the same as the one-sided rate in R^{d+2}.

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As a further example of two-sided growth problem we consider "rates of escape" in Section 4. We formulate a local two-sided escape rate in terms of the small values of $|X_d(t+u) - X_d(t-v)|$ as $(u+v) \downarrow 0$ and obtain an integral test which for $d = 4$ is the same as that obtained by Spitzer [10] for one-sided escape in the plane; and for $d \geqq 5$ is the same as that obtained by Dvoretzky and Erdös [4] for one-sided escape in R^{d-2}. We have no explanation for the observed connection between two-sided growth conditions for the process in R^d and the corresponding one-sided criteria in $R^{d\pm 2}$.

In Section 5 we obtain the division into upper and lower classes for the first passage time

$$P_d(a) = \inf\{t > 0; |X_d(t)| > a\} \tag{1.4}$$

considered as a random function of a. The law of iterated logarithm for $P_d(a)$ was obtained in [3]. In Section 6 we exploit the connection between $P_d(a)$ and

$$M_d(t) = \sup_{0 \leqq s \leqq t} |X_d(t)| \tag{1.5}$$

to obtain the lower envelope as $t \downarrow 0$ for $M_d(t)$. The results of Chung [1] lead immediately to a criterion for the lower envelope of $M_d(t)$ as $t \to \infty$, and his methods could be adapted to give our result for $M_d(t)$. However, the independence difficulties in the proof seem to be more easily overcome using $P_d(a)$. Other related processes are also considered in Section 6.

Our results are all proved for local behaviour of X_d near a fixed time $t > 0$. They all have analogues as $t \to \infty$—some of these are formulated and others are left to the reader. Our methods are standard—using a version of the Borel–Cantelli lemma. In order to exhibit the common features of the proofs we have collected, in Section 2, the general lemmas which are used in many different situations. Some of these may be of independent interest.

2. Preliminaries. If $a(h)$, $b(h)$ are parameters defined for small positive h we say

$$a(h) \approx b(h) \qquad \text{as} \quad h \to 0\,,$$

if there are positive constants c, c', δ such that

$$ca(h) \leqq b(h) \leqq c'a(h) \qquad \text{for} \quad 0 < h < \delta\,.$$

We also use the standard notations

$$a(h) \sim b(h) \qquad \text{and} \qquad a(h) = o(b(h))$$

to mean

$$\lim_{h\to 0} \frac{a(h)}{b(h)} = 1, 0 \quad \text{respectively.}$$

We use c, c', c'' to denote positive constants whose values may differ from line to line. In each situation all the random variables considered are defined on some fixed probability space $(\Omega, \mathscr{F}, P)$, but we usually suppress the dependence on ω. In applying the Markov property we use $P^x\{\cdot\}$ for the probability given that the process starts at x. For x in R^d, Euclidean space of d dimensions, $|x|$ denotes the length of the vector x, and $S(x, r)$ denotes the closed ball of radius r centered at x.

For a process $X(t)$, $t \geqq 0$, $\sigma\{X(s), s \leqq t\}$ denotes the σ-algebra generated by $X(s)$, $s \leqq t$. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two sub-σ-algebras of $\mathscr{F}$, then $\mathscr{F}_1 \vee \mathscr{F}_2$ denotes the σ-algebra generated by $\mathscr{F}_1$ and $\mathscr{F}_2$. Φ_ε denotes the class of functions φ from $(0, \varepsilon)$ to $[0, \infty)$ such that $\varphi(u) \uparrow \infty$ as $u \downarrow 0$.

The following lemmas are standard, see e.g. [4]; we state them here for ready reference.

LEMMA 2.1. *Let $S(x, r)$ denote the ball with center x and radius r in R^d. Let $X_d(t)$ be standard Brownian motion in R^d, $d \geqq 3$. Then*

$$(2.1) \qquad P\{X_d(t) \in S(x, r) \text{ for some } t > 0\} = \min[1, (r/|x|)^{d-2}]\,.$$

LEMMA 2.2. *Let $X_d(t)$ be standard Brownian motion R^d, $d \geqq 3$. Then for $h > 0$, $0 \leqq r \leqq h^{\frac{1}{2}}$ we have*

$$(2.2) \qquad P\{|X_d(t)| \leqq r \text{ for some } t > h\} \approx (rh^{-\frac{1}{2}})^{d-2}\,.$$

LEMMA 2.3. *Let $X^d(t)$ be standard Brownian motion in R^d, $d \geqq 3$. Then for $h > 0$, $0 \leqq r \leqq h^{\frac{1}{2}}$, we have*

$$(2.3) \qquad P\{|X_d(t)| \leqq r \text{ for some } h \leqq t \leqq 4h\} \approx (rh^{-\frac{1}{2}})^{d-2}\,.$$

Note that the constants involved in (2.2) and (2.3) are independent of both r and h. The following lemma is a simple corollary of Lemma 2.2.

LEMMA 2.4. *Let $X_d(t)$ and $Y_d(t)$ be independent, standard Brownian motions in R^d, $d \geqq 3$. Then for $h_1 \geqq 0$, $h_2 > 0$, $0 \leqq r \leqq h_1^{\frac{1}{2}}$, we have*

$$(2.4) \qquad P\{|X_d(t) - Y_d(h_2)| \leqq r \text{ for some } t > h_1\} \approx (r(h_1 + h_2)^{-\frac{1}{2}})^{d-2}\,.$$

We will also need the following two lemmas, whose proofs are obvious modifications of the proofs given in [4] of Lemmas 2.2 and 2.3 respectively.

LEMMA 2.5. *Let $X_d(t)$ be standard Brownian motion in R^d, $d \geqq 3$. Let $S(x, r) \subset S(0, \frac{1}{2}h^{\frac{1}{2}})$. Then*

$$(2.5) \qquad P\{X_d(t) \in S(x, r) \text{ for some } t \geqq h\} \approx (rh^{-\frac{1}{2}})^{d-2}\,.$$

LEMMA 2.6. *Let $X_d(t)$, $Y_d(t)$ be standard independent Brownian motions, $d \geqq 3$. Then there exists $c > 0$, depending only on d, such that if $0 \leqq r \leqq ch^{\frac{1}{2}}$, $h \leqq h_1 \leqq 4h$, then*

$$(2.6) \qquad P\{|X_d(t) - Y_d(h_1)| \leqq r \text{ for some } h \leqq t \leqq 4h\} \approx (rh^{-\frac{1}{2}})^{d-2}\,.$$

The constants involved in (2.6) are independent of r, h and h_1.

In the following lemma the exact tail estimate for $d = 1$ is given in [6]. For $d > 1$ see [12].

LEMMA 2.7. *Let $X_d(t)$ be standard Brownian motion in R^d and let* $R(X; s, t) = \sup_{s \leq u \leq v \leq t} |X(u) - X(v)|$. *Let* $R(h) = R(X; 0, h)$. *Then*

$$P\{R(h) > \lambda h^{\frac{1}{2}}\} \approx \lambda^{d-2} e^{-\lambda^2/2} . \tag{2.7}$$

The estimates in the following lemmas are well known for $d = 1$, the generalization for $d > 1$ is done in [9].

LEMMA 2.8. *Let U be a standard Gaussian random variable in R^d (with identity matrix as the covariance matrix). Then*

$$P\{|U| \geqq \lambda\} \sim c_d \lambda^{d-2} e^{-\lambda^2/2} . \tag{2.8}$$

LEMMA 2.9. *Let U and V be standard Gaussian random variables in R^d with $E(U_i V_j) = \rho\delta_{ij}$, $1 \leqq i, j \leqq d$, where ρ is a constant and δ_{ij} is the usual Kronecker symbol. If $|\rho| < (ab)^{-1}$, then*

$$P\{|U| > a, |V| > b\} \leqq cP\{|U| > a\}P\{|V| > b\} . \tag{2.9}$$

LEMMA 2.10. *Let U, V be standard Gaussian random variables in R^d with $E(U_i V_j) = \rho\delta_{ij}$, $1 \leqq i, j \leqq d$. Then*

$$P|U| > a, |V| > a\} \leqq ce^{-(1-\rho^2)a^2/8} P\{|U| > a\} . \tag{2.10}$$

The following lemma will be used in making some Fubini arguments. Its proof is similar to the usual proof of the "strong" independent increments property for a process with *stationary* and independent increments.

LEMMA 2.11. *Let $X_d(t)$ be standard Brownian motion in R^d. Let $\mathscr{F}_1$ be a σ-algebra which is independent of this process. Let $T \geqq 0$ be a random variable such that $\{T < t\} \in \sigma\{X_s, s \leqq t\} \vee \mathscr{F}_1$. Then $X_d(T + t) - X_d(T)$, $t \geqq 0$, is independent of $\mathscr{F}_1$ and $X_d(t)$ up to time T.*

The next lemma establishes the equivalence of the convergence of a certain series and an integral. The proof consists of elementary computations. We write $\log_m$ for the m-iterated logarithm to base e.

LEMMA 2.12. *Let $\varphi_i \in \Phi_\varepsilon$, $i = 1, 2$, $\varepsilon > 0$, such that for some $0 < a < b$,*

$$a(\log_2 u^{-1})^{i/2} \leqq \varphi_i(u) \leqq b(\log_2 u^{-1})^{i/2} . \tag{2.11}$$

If $a_j = e^{-j/\log j}$, $j \geqq 1$, then for any real number m, we have

$$\sum_j \{\varphi_i(a_j)\}^m \exp[-\{\varphi_i(a_j)\}^{2/i}] < \infty \tag{2.12}$$

$$\Leftrightarrow \int_{0+} \frac{\{\varphi_i(u)\}^{m+2/i}}{u} \exp[-\{\varphi_i(u)\}^{2/i}]\, du < \infty .$$

The next lemma allows us to obtain the equivalence of integral tests for functions and their inverses.

LEMMA 2.13. *Let $\psi \in \Phi_\varepsilon$. Let $\alpha > 0$, $\theta > 0$. Let $u(t)$ denote the strictly monotone function*

$$u = \alpha t^\theta \psi(t)^{-1} . \tag{2.13}$$

If $t(u)$ is the inverse function given by (2.13), *define* $\varphi(u)$ *by*

$$\varphi(u) = \{\psi(t(u))\}^{1/\theta} . \tag{2.14}$$

Then $\varphi \in \Phi_{\varepsilon'}$ *for some* $\varepsilon' > 0$, *and for any real* $d, m > 0$, *we have*

$$\int_{0+} \frac{\{\varphi(u)\}^d}{u} \exp[-\{\varphi(u)\}^m]\, du < \infty \tag{2.15}$$

$$\Leftrightarrow \int_{0+} \frac{\{\psi(u)\}^{d/\theta}}{u} \exp[-\{\psi(u)\}^{m/\theta}]\, du < \infty .$$

PROOF. It is clear that $\varphi \in \Phi_{\varepsilon'}$ for some $\varepsilon' > 0$. Let $u_j = 2^{-j}$ and let t_j be the unique solution of (2.13) for $u = u_j$. Note that $t_j \downarrow 0$. Let $\psi_j = \psi(t_j)$. Then we have

$$\int_{0+} \frac{\{\varphi(u)\}^d}{u} \exp[-\{\varphi(u)\}^m]\, du < \infty \tag{2.16}$$

$$\Leftrightarrow \sum_j \{\varphi(u_j)\}^d \exp[-\{\varphi(u_j)\}^m] < \infty$$
$$\Leftrightarrow \sum_j \psi_j^{d/\theta} \exp[-\psi_j^{m/\theta}] < \infty .$$

From (2.13) we get

$$t_j = \beta u_j^{1/\theta} \psi_j^{1/\theta} , \qquad \text{where } \beta = \alpha^{-1/\theta} . \tag{2.17}$$

Since $\psi_j \nearrow$, we have

$$(t_j - t_{j+1}) t_{j+1}^{-1} = (u_j u_{j+1}^{-1} \psi_j \psi_{j+1}^{-1})^{1/\theta} - 1 \leqq 2^{1/\theta} - 1 . \tag{2.18}$$

Hence

$$\int_{0+} \frac{\{\psi(t)\}^{d/\theta}}{t} \exp[-\{\psi(t)\}^{m/\theta}]\, dt \tag{2.19}$$

$$= \sum_j \int_{t_{j+1}}^{t_j} \frac{\{\psi(t)\}^{d/\theta}}{t} \exp[-\{\psi(t)\}^{m/\theta}]\, dt$$

$$\leqq c \sum_j \psi_j^{d/\theta} \exp[-\psi_j^{m/\theta}] \frac{t_j - t_{j+1}}{t_{j+1}} .$$

By (2.16), (2.18) and (2.19) we get

$$\int_{0+} \frac{\{\varphi(u)\}^d}{u} \exp[-\{\varphi(u)\}^m]\, du < \infty \tag{2.20}$$

$$\Rightarrow \int_{0+} \frac{\{\psi(t)\}^{d/\theta}}{t} \exp[-\{\psi(t)\}^{m/\theta}]\, dt < \infty .$$

To get the reverse implication in (2.16) note that

$$\int_{0+} \frac{\{\psi(t)\}^{d/\theta}}{t} \exp[-\{\psi(t)\}^{m/\theta}]\, dt \geqq \sum_j \psi_{j+1}^{d/\theta} \frac{(t_j - t_{j+1})}{t} \exp[-\psi_j^{m/\theta}] \tag{2.21}$$

$$= \sum_j \gamma_{j+1}[1 - \psi_{j+1}^{1/\theta}(2\psi_j)^{-1/\theta}] ,$$

where for convenience we write γ_j for $\psi_j^{d/\theta} \exp[-\psi_j^{d/\theta}]$. We have

$$\sum_j \gamma_{j+1}[1 - \psi_{j+1}^{1/\theta}(2\psi_j)^{-1/\theta}] \geqq (1 - (\tfrac{3}{4})^{1/\theta}) \sum_{\psi_{j+1} \leqq \frac{3}{2}\psi_j} \psi_{j+1} . \tag{2.22}$$

Since $\sum' \gamma_j$, where $\sum'$ denotes the sum over those j for which $\psi_{j+1} > (\frac{3}{2})\psi_j$, converges, we conclude that the divergence of $\sum_j \gamma_j$ implies the divergence of the right-hand side in (2.22). The reverse implication in (2.20) now follows from (2.16) and (2.21), and the lemma is proved.

Next we will give a general version of an argument used in [2]. This allows us, in proving integral tests, to consider only functions which are bounded above and below by two standard functions.

LEMMA 2.14. *Let g be an eventually monotone decreasing function from $[0, \infty)$ to $[0, \infty)$. Let h from $(0, \varepsilon)$ to $[0, \infty)$ be a measurable function. For $\psi \in \Phi_\varepsilon$, define*

$$F(\psi) = \int_0^\varepsilon g(\psi(t))h(t)\, dt\,, \tag{2.23}$$

which may be finite or infinite. Assume the following conditions hold:

(2.24) *For each $\psi \in \Phi_\varepsilon$,*

$$0 < s < \varepsilon,\ \int_s^\varepsilon g(\psi(t))h(t)\, dt < \infty\,.$$

(2.25) *There exist $\psi_1, \psi_2 \in \Phi_\varepsilon$, $\psi_1 \leqq \psi_2$, such that $F(\psi_1) = \infty$, $F(\psi_2) < \infty$, and* $\lim_{s\to 0} g(\psi_1(s)) \int_s^\varepsilon h(t)\, dt = \infty$.

For $\psi \in \Phi_\varepsilon$, let $\hat{\psi}$ denote

$$\hat{\psi} = \min\,[\max\,(\psi, \psi_1), \psi_2]\,. \tag{2.26}$$

Then for any $\psi \in \Phi_\varepsilon$ we have

$$F(\psi) < \infty \Rightarrow \hat{\psi} \leqq \psi \quad \textit{near}\ 0 \ \textit{ and }\ F(\hat{\psi}) < \infty\,, \tag{2.27}$$

$$F(\psi) = \infty \Rightarrow F(\hat{\psi}) = \infty\,. \tag{2.28}$$

PROOF. In view of (2.24) there is no loss of generality in assuming g to be decreasing over its entire domain. Then $\psi_1 \leqq \psi_2 \Rightarrow F(\psi_1) \geqq F(\psi_2)$. Let $F(\psi) < \infty$. If there exists a sequence $t_n \downarrow 0$ such that $\psi(t_n) < \hat{\psi}(t_n)$, then by (2.26) we have $\psi(t_n) < \psi_1(t_n)$. Since g is eventually decreasing, we have for all sufficiently large n,

$$\int_{t_n}^\varepsilon g(\psi(t))h(t)\, dt \geqq g(\psi(t_n)) \int_{t_n}^\varepsilon h(t)\, dt \geqq g(\psi_1(t_n)) \int_{t_n}^\varepsilon h(t)\, dt\,. \tag{2.29}$$

By assumption (2.25) this implies $F(\psi) = \infty$, a contradiction. Hence $\hat{\psi} \leqq \psi$ near 0. This implies $\psi_1 \leqq \psi$ near 0. Hence for some $\delta > 0$,

$$\begin{aligned} \int_0^\delta g(\hat{\psi}(t))h(t)\, dt &= \int_{[\psi_1 \leqq \psi \leqq \psi_2] \cap (0,\delta)} + \int_{[\psi \geqq \psi_2] \cap (0,\delta)} \\ &\leqq F(\psi) + F(\psi_2) < \infty\,. \end{aligned} \tag{2.30}$$

This together with (2.24) proves (2.27). Now let $F(\psi) = \infty$. If $\hat{\psi} \leqq \psi$ near 0 then $F(\hat{\psi}) = \infty$. On the other hand, if there exists a sequence $t_n \downarrow 0$ such that $\psi(t_n) < \hat{\psi}(t_n)$, then $\hat{\psi}(t_n) = \psi_1(t_n)$, and we have

$$\int_{t_n}^\varepsilon g(\hat{\psi}(t))h(t)\, dt \geqq g(\hat{\psi}(t_n)) \int_{t_n}^\varepsilon h(t)\, dt \geqq g(\psi_1(t_n)) \int_{t_n}^\varepsilon h(t)\, dt\,. \tag{2.31}$$

It follows from assumption (2.25) again that $F(\hat{\psi}) = \infty$.

Finally we state an extension of the Borel–Cantelli lemma due to Kochen and Stone [7].

LEMMA 2.15. *Let* $\{E_n\}$ *be a sequence of events. Then*

(i) $$\sum_n P(E_n) < \infty \Rightarrow P\{E_n \text{ occur i.o.}\} = 0.$$

(ii) $$\liminf_n \Big[\sum_{j=1}^n \sum_{k=1}^n P(E_j \cap E_k)\Big]\Big[\sum_{j=1}^n \sum_{k=1}^n P(E_j)P(E_k)\Big]^{-1} \leqq c \Rightarrow P\{E_n \text{ occur i.o.}\} \geqq c^{-1}.$$

3. Two-sided upper growth conditions. If $X_d(t)$ is a standard Brownian motion in R^d, it is well-known that

$$\limsup_{t\downarrow 0} \frac{|X_d(t)|}{(2t\log|\log t|)^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{3.1}$$

Recently it was observed [12] that this can be strengthened to a two-sided result

$$\limsup_{u\geqq 0, v\geqq 0, 0<u+v<\delta, \delta\to 0} \frac{|X_d(t+v) - X_d(t-u)|}{\{2(u+v)\log|\log(u+v)|\}^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{3.2}$$

Our object is to obtain the integral test corresponding to (3.2) and compare it with the test corresponding to (3.1).

Let Φ_ε' denote the class of functions φ from $(0, \varepsilon)$ to $[0, \infty)$ such that $\varphi(t) \uparrow \infty$ as $t \downarrow 0$ and $t^{\frac{1}{2}}\varphi(t) \downarrow 0$ as $t \downarrow 0$. We can divide the functions of Φ_ε' into upper and lower classes corresponding to (3.1) and (3.2).

DEFINITION 3.1. A function $\varphi \in \Phi_\varepsilon'$ belongs to $\mathscr{U}_1^{(d)}$ if, for each ω, $\exists\delta > 0$ such that $|X_d(t)| < (2t)^{\frac{1}{2}}\varphi(t)$ for $0 < t < \delta$. $\mathscr{L}_1^{(d)}$ is the complement of $\mathscr{U}_1^{(d)}$ in Φ_ε'. $\mathscr{U}_2^{(d)}$ is defined to consist of those $\varphi \in \Phi_\varepsilon'$ for which, given $t > 0$, for almost every ω there exists $\delta > 0$ such that if $u \geqq 0$, $v \geqq 0$, $0 < u + v < \delta$, then $|X_d(t+v) - X_d(t-u)| < 2^{\frac{1}{2}}(u+v)^{\frac{1}{2}}\varphi(u+v)$. $\mathscr{L}_2^{(d)}$ is the complement of $\mathscr{U}_2^{(d)}$ in Φ_ε'.

Lévy claims ([8] page 244) that Kolmogorov proved (in case $d = 1$) that if $\varphi \in \Phi_\varepsilon'$ then,

$$\varphi \in \mathscr{U}_1^{(d)} \Leftrightarrow \int_0 \frac{\varphi(x)^d}{x} e^{-\varphi(x)^2}\, dx < \infty\,. \tag{3.3}$$

We now obtain the corresponding result for $\mathscr{U}_2$.

THEOREM 3.1. *Let* $\varphi \in \Phi_\varepsilon'$. *Then*

$$\varphi \in \mathscr{U}_2^{(d)} \Leftrightarrow \int_{0+} \frac{\varphi(x)^{d+2}}{x} e^{-\varphi(x)^2}\, dx < \infty\,. \tag{3.4}$$

REMARK 3.1. Though the iterated logarithm result (3.1) does not depend on the dimension d, (3.3) shows that the fine division into upper and lower functions depends (very marginally) on d. However, the result of our theorem shows that the class $\mathscr{U}_2^{(d)}$ is precisely the same as the class $\mathscr{U}_1^{(d+2)}$. One feels that there ought to be some way of proving this directly, but we have not succeeded in finding it.

REMARK 3.2. It is well known that there is a result corresponding to (3.3) for the large values of $|X_d(t)|$ as $t \to \infty$. Similar arguments show that $\mathscr{U}_2^{(d)}$ is also the class of $\varphi \in \Phi_\varepsilon'$ such that, if $X_d(t)$ and $Y_d(t)$ are independent Brownian motions in R^d, then given ω, there exists K with the property that $u \geqq 0$, $v \geqq 0$, $u + v > K$ imply

$$|X_d(u) - Y_d(v)| < 2^{\frac{1}{2}}(u+v)^{\frac{1}{2}}\varphi\left(\frac{1}{u+v}\right).$$

Before starting the main proof it is convenient to obtain a trapping lemma which follows easily from lemma 2.14.

LEMMA 3.1. *It is sufficient to consider* $\varphi \in \Phi_\varepsilon'$ *satisfying*

$$\left(\log\log\frac{1}{x}\right)^{\frac{1}{2}} \leqq \varphi(x) \leqq 2\left(\log\log\frac{1}{x}\right)^{\frac{1}{2}}. \tag{3.5}$$

PROOF. In Lemma 2.14 let $\psi_1(x) = (\log\log x^{-1})^{\frac{1}{2}}$ and $\psi_2(x) = 2\psi_1(x)$; let $g(x) = x^{d+2}e^{-x^2}$ and $h(x) = x^{-1}$. Note that $\Phi_\varepsilon' \subset \Phi_\varepsilon$. Then $F(\varphi)$ in Lemma 2.14 is exactly the integral in (3.4). All the conditions of Lemma 2.14 are satisfied. Writing $\hat{\varphi} = \min[\max(\varphi, \psi_1), \psi_2]$, we will show that (3.4) is true for φ if it is true for $\hat{\varphi}$. If $F(\varphi) < \infty$, then by Lemma 2.10 we have $F(\hat{\varphi}) < \infty$ and $\hat{\varphi} \leqq \varphi$ near 0. Since we assume (3.4) to be true for $\hat{\varphi}$, it follows that $\hat{\varphi} \in \mathscr{U}_2^{(2)}$. Since $\hat{\varphi} \leqq \varphi$ near 0 we conclude that $\varphi \in \mathscr{U}_2^{(d)}$. On the other hand, if $F(\varphi) = \infty$, then by Lemma 2.14 we have $F(\hat{\varphi}) = \infty$, and since (3.4) is assumed true for $\hat{\varphi}$, we must have $\hat{\varphi} \in \mathscr{L}_2^{(d)}$. This means for almost every ω there exist $u_n \downarrow 0$, $v_n \downarrow 0$ such that $|X_d(t + v_n) - X_d(t - u_n)| \geqq 2^{\frac{1}{2}}(u_n + v_n)^{\frac{1}{2}}\hat{\varphi}(u_n + v_n)$. Since $\psi_2 \in \mathscr{U}_2^{(d)}$, we must have $\hat{\varphi}(u_n + v_n) < \psi_2(u_n + v_n)$ for all sufficiently large n. But this strict inequality implies that $\varphi(u_n + v_n) \leqq \hat{\varphi}(u_n + v_n)$. Hence $\varphi \in \mathscr{L}_2^{(d)}$. This proves the lemma.

REMARK 3.3. Let $\varphi \in \Phi_\varepsilon'$. If we pick ψ_1, ψ_2, g and h in Lemma 2.14 as in the proof of Lemma 3.1, we see that $F(\psi) < \infty \Leftrightarrow F(\hat{\psi}) < \infty$. Now using the fact that $\psi_1 \leqq \hat{\psi} \leqq \psi_2$, we see that

$$F(\hat{\psi}) < \infty \Leftrightarrow \int_{0+} u^{-1}(\log|\log u|)^{(d+2)/2}\exp[-\{\hat{\varphi}(u)\}^2]\,du < \infty.$$

Now pick $h(u) = u^{-1}(\log\log u^{-1})^{(d+2)/2}$, $g(u) = e^{-u^2}$, ψ_1, ψ_2 as before, in Lemma 2.14. Then we conclude by Lemma 2.14 that

$$\int_{0+}\frac{(\log|\log u|)^{(d+2)/2}}{u}\exp[-\{\hat{\varphi}(u)\}^2]\,du < \infty \Leftrightarrow$$

$$\int_{0+}\frac{(\log|\log u|)^{(d+2)/2}}{u}\exp[-\{\varphi(u)\}^2]\,du < \infty.$$

Hence this last integral might as well be used in (3.4). A similar argument would allow us to reformulate all the integral tests of this paper in a form involving φ only in the exponential term.

PROOF OF THEOREM. We do the case $d = 1$; general d follows by the same argument since all the relevant estimates are available for general d. φ will be assumed to satisfy (3.5). Let $R(X; s, t) = \sup_{s \leqq t_1 < t_2 \leqq t} |X(t_1) - X(t_2)|$, $R(h) = R(X; 0, h)$. The following estimates are available from Lemmas 2.7 and 2.8 for $d = 1$

(3.6) $$P\{|X(h)| > \lambda h^{\frac{1}{2}}\} \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{\lambda} e^{-\lambda^2/2}$$

(3.7) $$P\{|R(h)| > \lambda h^{\frac{1}{2}}\} \approx \frac{1}{\lambda} e^{-\lambda^2/2}\,.$$

Consider the sequence

(3.8) $$a_m = e^{-m/\log m}\,, \qquad m \geqq 2\,.$$

Then

(3.9) $$a_n/a_{n+1} \sim 1 + (\log n)^{-1} \qquad \text{as } n \to \infty\,.$$

Let

(3.10) $$u_{n,i} = \frac{i}{\log n} a_n;\ v_{n,i} = \left(1 - \frac{i}{\log n}\right) a_n\,, \qquad 0 \leqq i \leqq \log n\,.$$

Note that for each i, $u_{n,i} + v_{n,i} = a_n$. Let the integral in (3.4) converge for a given φ. In order to show that $\varphi \in \mathscr{U}_2(= \mathscr{U}_2^{(1)})$ it is enough to show that a.s. for all i and n sufficiently large,

(3.11) $$R(X; t - u_{n,i+1}, t + v_{n,i-1}) \leqq 2^{\frac{1}{2}} a_{n+1}^{\frac{1}{2}} \varphi(a_{n+1})\,.$$

For if for some u, v

$$|X(t + v) - X(t - u)| > 2^{\frac{1}{2}}(u + v)^{\frac{1}{2}} \varphi(u + v)\,,$$

then there is an n such that $a_{n+1} \leqq u + v < a_n$ and $|X(t + v) - X(t - u)| > 2^{\frac{1}{2}} a_{n+1}^{\frac{1}{2}} \varphi(a_{n+1})$. This implies for a suitable $i \leqq \log n$ that $u_{n,i} \leqq u \leqq u_{n,i+1}$, $v \leqq v_{n,i-1}$ and

(3.12) $$R(X; t - u_{n,i+1}, t + v_{n,i-1}) > 2^{\frac{1}{2}} a_{n+1}^{\frac{1}{2}} \varphi(a_{n+1})\,.$$

Let $E_{n,i}$ denote the event in (3.12). Now the length

$$u_{n,i+1} + v_{n,i-1} = (1 + 2/\log n) a_n \sim (1 + 3/\log n) a_{n+1}$$

by (3.9). Hence (3.7) yields

(3.13) $$P(E_{n,i}) < \frac{c}{\varphi(a_n)} \exp\left[-\left(1 - \frac{3}{\log n}\right)\{\varphi(a_{n+1})\}^2\right] < \frac{c'}{\varphi(a_n)} \exp\left[-\{\varphi(a_n)\}^2\right]\,.$$

The last inequality follows easily from the fact that $\{\varphi(a_n)\}^2 \approx \log n$ since $\varphi \in \Phi_\varepsilon'$ and satisfies (3.5). Using this fact once again we get

(3.14) $$\sum_{n=1}^{\infty} \sum_{i=0}^{\log n} P(E_{n,i}) < c \sum_n \varphi(a_n) \exp\left[-\{a_n)\}^2\right]\,.$$

By Lemma 2.12 the assumed convergence of the integral in (3.4) implies the convergence of the right-hand side in (3.14). Then Lemma 2.15 (i) implies (3.11) for all i, n sufficiently large. Let us now assume that the integral in (3.4) diverges. To argue in this direction, define

$$F_{n,i} = \{\omega : X(t + v_{n,i}) - X(t - u_{n,i}) > 2^{\frac{1}{2}} a_n^{\frac{1}{2}} \varphi(a_n)\} . \tag{3.15}$$

By (3.6) we get $P(F_{n,i}) \approx \{\varphi(a_n)\}^{-1} \exp[-\{\varphi(a_n)\}^2]$. As before, since $\{\varphi(a_n)\}^2 \approx \log n$, we conclude from Lemma 2.12 that the divergence of the integral in (3.4) implies

$$\sum_{n=1}^{\infty} \sum_{i=0}^{\log n} P(F_{n,i}) = \infty . \tag{3.16}$$

From (3.16) we would like to conclude that infinitely many of the $F_{n,i}$ occur, by using Lemma 2.15 (ii). To this end it is evidently enough to check the following:

(a) If $m \geqq n + \{\log n\}^2$, then the correlation between the standardized random variables $Y_{n,i} = a_n^{-\frac{1}{2}}\{X(t + v_{n,i}) - X(t - u_{n,i})\}$ and $Y_{m,j} = a_m^{-\frac{1}{2}}\{X(t + v_{m,j}) - X(t - u_{m,j})\}$ is less than $a_m^{\frac{1}{2}} a_n^{-\frac{1}{2}}$ which in turn is less than $\varphi^{-1}(a_n)\varphi^{-1}(a_m)$ for all i, j, and n sufficiently large. Hence Lemma 2.9 gives

$$P(F_{n,i} \cap F_{m,j}) \leqq cP(F_{n,i})P(F_{n,j}) . \tag{3.17}$$

(b) For $n < m < n + \log^2 n$, we will show that

$$\sum_{m=n+1}^{n+\log^2 n} \sum_{\text{all } j} P(F_{m,j} \cap F_{n,i}) \leqq cP(F_{n,i}) . \tag{3.18}$$

We have for $Y_{n,i}$ and $Y_{m,j}$ as in (a),

$$\begin{aligned} P(F_{m,j} \cap F_{n,i}) &\leqq P\{Y_{n,i} > 2^{\frac{1}{2}}\varphi(a_n),\ Y_{m,j} > 2^{\frac{1}{2}}\varphi(a_n)\} \\ &\leqq c \exp[-\tfrac{1}{4}(1 - \rho^2)\{\varphi(a_n)\}^2] P(F_{n,i}) , \end{aligned} \tag{3.19}$$

where the first inequality comes from $\varphi(a_n) < \varphi(a_m)$ and the second from Lemma 2.10. Since φ satisfies (3.5) we have $\varphi^2(a_n) > \frac{1}{2} \log n$, hence we get from (3.19),

$$P(F_{m,j} \cap F_{n,i}) \leqq ce^{-(1-\rho^2)\log n/8} P(F_{n,i}) , \tag{3.20}$$

where it should be noted that ρ is the correlation between $Y_{m,j}$ and $Y_{n,i}$, hence it depends on n, m, i and j. We count the (m, j) according to the estimates of ρ. Let $I_{n,i} = [-u_{n,i}, v_{n,i}]$, $I_{m,j} = [-u_{m,j}, v_{m,j}]$. Then

$$\begin{aligned} \rho^2 &= a_n^{-1} a_m^{-1} (\text{length of } I_{n,i} \cap I_{m,j})^2 \\ &\leqq a_n^{-1} (\text{length of } I_{n,i} \cap I_{m,j}) . \end{aligned} \tag{3.21}$$

The number of intervals $I_{m,j}$ such that

$$a_n\left(1 - \frac{k}{\log n}\right) \leqq \text{length of } I_{n,i} \cap I_{m,j} \leqq a_n\left(1 - \frac{k-1}{\log n}\right) \tag{3.22}$$

is no more than $4k^2$, since m must be $\leqq n + 2k$ for the first inequality in (3.22) to hold, and for each m with $n \leqq m \leqq n + 2k \leqq n + \log^2 n$, there could not be more than $2k$ pairs (m, j) for which (3.22) holds. For such pairs (3.21) and

(3.22) give $1 - \rho^2 \geqq (k-1)/\log n$. Hence (3.20) gives

$$\sum_{m=n}^{n+\log^2 n} \sum_{\text{all } j} P(F_{n,i} \cap F_{m,j}) \leqq cP(F_{n,i}) \sum_{k=0}^{\infty} 4k^2 e^{-(k-1)/8} .$$

Since the event that infinitely many $F_{n,i}$ occur has probability 0 ,or 1 we can now apply Lemma 2.15 (ii) to deduce that $\varphi \in \mathscr{U}_2^{(1)}$. This completes the proof of Theorem 3.1.

4. Two-sided rate of escape. It is well known that for $d \geqq 2$, a Brownian motion process $X_d(t)$ in R^d does not return to its starting point. If we consider $|X_d(t)|$ we can ask how fast $f(s) = \min_{t \geqq s} |X_d(t)|$ grows. The asymptotic size of the small values of $f(s)$ as $s \to 0$ was obtained for $d \geqq 3$ by Dvoretzky and Erdös [4] and for $d = 2$ by Spitzer [10]. Let us recall their results.

If $\psi(h) \downarrow 0$ as $h \downarrow 0$ then, for each $t \geqq 0$ with probability 1 there exists a $\delta > 0$ such that $|X(t+h) - X(t)| > h^{\frac{1}{2}}\psi(h)$ for all $h < \delta$ if and only if

$$\int_{0+} \{\psi(h)\}^{d-2} \frac{dh}{h} < \infty , \qquad d \geqq 3 . \tag{4.1}$$

$$\int_{0+} |\log \psi(h)|^{-1} \frac{dh}{h} < \infty , \qquad d = 2 . \tag{4.2}$$

In order to formulate the two-sided results we first note that for $d \leqq 3$, the set of double points on the path is everywhere dense [5]. Our main objective in the present section is to obtain the asymptotic growth rate of

$$g(s) = \min_{u \geqq 0, v \geqq 0, u+v \geqq s} |X_d(t+u) - X_d(t-v)| . \tag{4.3}$$

THEOREM 4.1. *If $X_d(t)$ is a standard Brownian motion in R^d, $d \geqq 4$, and $\psi(h)$ is a monotonically increasing function on $(0, \varepsilon]$, $\varepsilon > 0$, then for each $t > 0$ with probability 1 there is a $\delta > 0$ such that*

$$|X_d(t+u) - X_d(t-v)| > (u+v)^{\frac{1}{2}}\psi(u+v) \tag{4.4}$$

for $u \geqq 0$, $v \geqq 0$, $0 \leqq u + v \leqq \delta$, if and only if

$$\int_{0+} \{\psi(h)\}^{d-4} \frac{dh}{h} < \infty , \qquad d \geqq 5 , \tag{4.5}$$

$$\int_{0+} |\log \psi(h)|^{-1} \frac{dh}{h} < \infty , \qquad d = 4 . \tag{4.6}$$

REMARK 4.1. The connection between our results and the corresponding conditions (4.1) and (4.2) for one-sided escape is obvious, and curious. The two-sided escape rate in dimension d has identical lower asymptotic growth rate as the one-sided escape rate in dimension $d - 2$. In Section 3 we noticed precisely the same phenomenon for the upper asymptotic growth at a fixed point t. We are not able to give any heuristic explanation for this further connection between Brownian motions in R^d and in R^{d+2}. An even more surprising result of this kind was noted in [3] and partially explained by Williams [13].

The following two lemmas give the necessary probability estimates to prove Theorem 4.1.

LEMMA 4.1. *If $X_d(t)$ and $Y_d(t)$ are independent Brownian motions in R^d, $d \geqq 5$, and $\Delta = o(h^{\frac{1}{2}})$ as $h \downarrow 0$, let*

$$E_{h,\Delta} = \{\min_{u \geqq h, v \geqq 0} |X_d(u) - Y_d(v)| \leqq \Delta\},$$
$$F_{h,\Delta} = \{\min_{h \leqq u \leqq 4h, h \leqq v \leqq 4h} |X_d(u) - Y_d(v)| \leqq \Delta\}.$$

Then

$$P(F_{h,\Delta}) \approx P(E_{h,\Delta}) \approx (\Delta h^{-\frac{1}{2}})^{d-4}. \tag{4.7}$$

PROOF. Since $F_{h,\Delta} \subset E_{h,\Delta}$, it is enough to get an upper bound for $P(E_{h,\Delta})$ and a lower bound for $P(F_{h,\Delta})$, each bound $\approx (\Delta h^{-\frac{1}{2}})^{d-4}$. First we get the upper bound for $P(E_{h,\Delta})$. Cover R^d by a fixed network of closed cubes of side $\Delta/2$ (whose interiors are disjoint and sides are parallel to coordinate axes). It is enough to get an estimate of the probability that one such cube is hit by $X_d(u)$ for some $u \geqq h$ and by $Y_d(v)$ for some $v \geqq 0$, and show that the sum over all cubes of these estimates is of the right order. Let ρ denote the distance from the center of the cube to the origin in R^d. For each integer k the number of such cubes in the annulus $2^k h^{\frac{1}{2}} < \rho \leqq 2^{k+1} h^{\frac{1}{2}}$ is of order $\Delta^{-d}(2^{k+1}h^{\frac{1}{2}})^d$. If C is a cube in this annulus with $k \geqq -2$, then by Lemma 2.1, using the independence of X_d and Y_d we get

$$P\{X_d(u) \in C,\ Y_d(v) \in C, \text{ for some } u \geqq 0, v \geqq 0\} \leqq c(\Delta 2^{-k-1} h^{-\frac{1}{2}})^{2(d-2)}. \tag{4.8}$$

Multiplying this bound by the number of cubes in the annulus and summing over $k \geqq -2$ we get for $d > 4$ the upper estimate $c(\Delta h^{-\frac{1}{2}})^{d-4}$. For $k < -2$, we do use the fact that $X_d(u)$ hits the cube C *after* time h. By Lemmas 2.1, 2.2 we get this time

$$P\{X_d(u) \in C,\ Y_d(v) \in C, \text{ for some } u \geqq h, v \geqq 0\} \leqq c(\Delta h^{-\frac{1}{2}})^{d-2}(\Delta 2^{-k} h^{-\frac{1}{2}})^{d-2}. \tag{4.9}$$

We multiply this bound by $\Delta^{-d}(2^{k+1}h^{\frac{1}{2}})^d$, the number of cubes in the annulus, to get the bound $c(\Delta h^{-\frac{1}{2}})^{d-4} 2^{2k}$, which summed over $k \leqq -3$ again gives the right upper estimate. To get the lower estimate for $P(F_{h,\Delta})$, consider the balls T_i of radius Δ centered at $X_d(t_i)$, where $t_i = h + ik\Delta^2$, k is a fixed positive integer to be chosen later and $0 \leqq i \leqq [3h\Delta^{-2}k^{-1}]$. If R_i is the event that $Y_d(t)$ hits the ball T_i for some t between h and 4h, then by Lemma 2.6, for all i

$$P\{R_i\} \approx (\Delta h^{-\frac{1}{2}})^{d-2}. \tag{4.10}$$

For $i \neq j$ let $r = |i - j|$, then

$$P\{R_i \cap R_j\} \leqq P\{Y_d(t) \text{ first hits } T_i, \text{ then } T_j \text{ for some } t \geqq 0\} + P\{Y_d(t) \text{ first hits } T_j, \text{ then } T_i \text{ for some } t \geqq 0\}. \tag{4.11}$$

The first probability on the right side in (4.11) is dominated by

$$P\{R_i\} P\{|Y_d(t) + X_d(t_i) - X_d(t_j)| \leqq 2\Delta \text{ for some } t \geqq 0\}, \tag{4.12}$$

where we apply the strong Markov property when $Y_d(t)$ first hits T_i and observe that $Y_d(t)$ is at distance Δ from $X_d(t_i)$ at the time it visits T_i. By Lemma 2.4, (4.12) is dominated by $cP(R_i)[\Delta(kr\Delta^2)^{-\frac{1}{2}}]^{d-2}$. The second term on the right side in (4.11) is of the same order in view of (4.10). Hence we get

$$P(R_i \cap R_j) \leqq cP(R_i)(rk)^{-(d-2)/2}. \tag{4.13}$$

If we now pick k sufficiently large, say $k = k_0$, then, since $d > 4$, we get from (4.13) for each fixed i,

$$\sum_{j \neq i} P(R_i \cap R_j) < \tfrac{1}{2} P(R_i). \tag{4.14}$$

Hence for $k = k_0$

$$P(F_{h,\Delta}) \geqq P(\textstyle\bigcup_i R_i) > \frac{1}{2} \sum_i P(R_i) > c(\Delta h^{-\frac{1}{2}})^{d-4}.$$

This completes the proof.

LEMMA 4.2. *If $X(t)$ and $Y(t)$ are independent Brownian motions in R^4 and $\Delta = o(h^{\frac{1}{2}})$ as $h \downarrow 0$. If*

$$E'_{h,\Delta} = \{\min_{h \leqq u \leqq 4h, v \geqq 0} |X(u) - Y(v)| \leqq \Delta\}$$

and $F_{h,\Delta}$ is defined as in Lemma 4.1, then

$$P(F_{h,\Delta}) \approx P(E'_{h,\Delta}) \approx |\log \Delta h^{-\frac{1}{2}}|^{-1}. \tag{4.15}$$

PROOF. For the lower estimate on $P(F_{h,\Delta})$ we consider the balls T_i of radius Δ centered at the points $X(t_i)$, where $T_i = h + ik\Delta^2 |\log \Delta h^{-\frac{1}{2}}|$, $i = 0, 1, \cdots$. Let $[3h\Delta^{-2}k^{-1} |\log \Delta h^{-\frac{1}{2}}|^{-1}] = M$. Now proceeding as in the second part of the proof of Lemma 4.1 and picking k sufficiently large at the last step we get the desired estimate. The upper estimate for $P(E'_{h,\Delta})$ is a little more difficult this time. Consider the "Wiener sausage"

$$S_{\Delta,i} = \{x \in R^4 : |X(t) - x| \leqq \Delta, \text{ for some } t, h \leqq t \leqq t_i\}.$$

We want an upper bound for the probability that $Y(t)$ hits $S_{\Delta,M+1}$ for some $t \geqq 0$. Let R_i be the event that $Y(t)$ hits the ball T_i (defined above in the proof). Then we will show that $P(\bigcup_{i=1}^{M+1} R_i) \approx P\{Y(t) \text{ hits } S_{\Delta,M+1} \text{ for some } t \geqq 0\}$. Let $E_i = \{Y(t) \in S_{\Delta,i} \text{ for some } t \geqq 0\}$. Note that $E_i \subset E_{i+1}$ and E_i depends on the X-process only up to time t_i. We have

$$\begin{aligned} P\{\textstyle\bigcup_{i=1}^{2M} R_i \cap E_{2M+1}\} &\geqq P\{\textstyle\bigcup_{j=1}^{2M} R_j \cap E_{M+1}\} \\ &= \textstyle\sum_{i=0}^{M} P(G_i) P\{\bigcup_{j=i+1}^{2M} R_j \mid G_i\}, \end{aligned} \tag{4.16}$$

where $G_i = E_{i+1} - E_i$. If we can show that for $0 \leqq i \leqq M$,

$$P\{\textstyle\bigcup_{j=i+1}^{2M} R_j \mid G_i\} \geqq c \tag{4.17}$$

for some $c > 0$, then we are done, for then from (4.16) we get

$$P(E_{M+1}) = \textstyle\sum_{i=0}^{M} P(G_i) \leqq c^{-1} \sum_{i=1}^{2M} P(R_i) \approx |\log \Delta h^{-\frac{1}{2}}|^{-1},$$

since $P(R_i) \approx (\Delta h^{-\frac{1}{2}})^2$ by Lemma 2.4 and $M = [3h\Delta^{-2}k^{-1} |\log \Delta h^{-\frac{1}{2}}|^{-1}]$. We will have to choose k suitably for the proof of (4.17). Observe that as we did in the second part of Lemma 4.1, we can pick $k = k_0$ sufficiently large so that

$$P\{\bigcup_{j=i+1}^{2M} R_j \mid G_i\} \approx \sum_{j=i+1}^{2M} P\{R_j \mid G_i\} . \tag{4.18}$$

To get a lower estimate for $P\{R_j \mid G_i\}$ we use a Fubini argument. Let $P = P_1 \times P_2$, where P_1 is the probability measure for $X_d(t)$, $t \geqq 0$, and P_2 for $Y_d(t)$, $t \geqq 0$. E_Q denotes expectation with respect to the probability measure Q on an appropriate space. Then using the strong Markov property,

$$P\{R_j \mid G_i\} = E_P\{P_2^{Y(\tau_i)}[Y(t + \tau_i) \in T_j \text{ for some } t \geqq 0]\} , \tag{4.19}$$

where τ_i denotes the first hitting time of $S_{\Delta,i}$ by Y_d given the event G_i; note that there is a random time θ_i (which depends on t_i) such that $t_{i-1} \leqq \theta_i \leqq t_i$ and $|X(\theta_i) - Y(\tau_i)| = \Delta$; moreover, τ_i is a stopping time for the Y-process which also depends on the X-process up to time t_i. Now by Lemma 2.11, $Y(t + \tau_i) - Y(\tau_i)$, $t \geqq 0$, is independent of $Y(\tau_i)$ and $X(t)$, $t \geqq 0$, hence by Lemma 2.4 we get

$$P_2^{Y(\tau_i)}\{Y(t + \tau_i) \in T_j \text{ for some } t \geqq 0\} = \frac{\Delta^2}{|X(t_j) - Y(\tau_i)|^2} , \tag{4.20}$$

where $t_j = h + jk\Delta^2 |\log \Delta h^{-\frac{1}{2}}|^{-1}$. Since

$$|X(t_j) - Y(\tau_i)| \leqq |X(t_j) - X(t_i)| + |X(t_i) - X(\theta_i)| + |X(\theta_i) - Y(\tau_i)| ,$$

where $t_{i-1} \leqq \theta_i \leqq t_i$, $|X(\theta_i) - Y(\tau_i)| = \Delta$, we get

$$\frac{\Delta^2}{|X(t_j) - Y(\tau_i)|^2} \geqq \Delta^2\{|X(t_j) - X(t_i)| + \sup_{t_{i-1} \leqq u \leqq t_i} |X(t_i) - X(u)| + \Delta\}^{-2} . \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) we get

$$P\{R_j \mid G_i\} \geqq \Delta^2 E_{P_1}\{|X(t_j) - X(t_i)| + \sup_{t_{i-1} \leqq u \leqq t_i} |X(t_i) - X(u)| + \Delta\}^{-2} . \tag{4.22}$$

We now integrate the right side in (4.22) over the set

$$\{|X(t_j) - X(t_i)| \geqq (t_j - t_i)^{\frac{1}{2}}\} \cap \{\sup_{t_{i-1} \leqq u \leqq t_i} |X(t_i) - X(u)| \geqq (t_i - t_{i-1})^{\frac{1}{2}}\} ;$$

this set has positive probability since the two sets involved are independent and each has a positive probability (independent of i, j). Hence

$$\begin{aligned} P\{R_j/G_i\} &\geqq c\Delta^2\{(t_j - t_i)^{\frac{1}{2}} + (t_i - t_{i-1})^{\frac{1}{2}} + \Delta\}^{-2} . \\ &\approx c\,|j - i|^{-1} k_0^{-2}\, |\log \Delta h^{-\frac{1}{2}}| . \end{aligned}$$

Hence $\sum_{j=i+1}^{2M} P\{R_j/G_i\} \approx c k_0^{-2} |\log \Delta h^{-\frac{1}{2}}|^{-1} \log (2M) = c' > 0$. This completes the proof.

PROOF OF THEOREM 4.1. Let $t > 0$. Then since ψ is non-decreasing, the event

$$E = \{|X(t + u) - X(t - v)| < (u + v)^{\frac{1}{2}} \psi(u + v) \text{ i.o.}, \ u + v \downarrow 0\}$$

is contained in the union of the events

$$E_1 = \{\min_{4^{-n} \leqq u \leqq 4^{-n+1}, 0 \leqq v \leqq u} |X(t+u) - X(t-v)| < (8.4^{-n})^{\frac{1}{2}}\psi(8.4^{-n}) \ \text{i.o.}\}$$
$$E_2 = \{\min_{4^{-n} \leqq v \leqq 4^{-n+1}, 0 \leqq u \leqq v} |X(t+u) - X(t-v)| < (8.4^{-n})^{\frac{1}{2}}\psi(8.4^{-n}) \ \text{i.o.}\} .$$

If $d \geqq 5$, then by Lemma 4.1, the probability of each of E_1 and E_2 is diminated by

$$\sum_{n=1}^{\infty} \{(8.4^{-n})^{\frac{1}{2}}\psi(8.4^{-n})4^{n/2}\}^{d-4} ,$$

which converges if and only $\sum_{n=1}^{\infty} \{\psi(4^{-n})\}^{d-4}$ converges. The convergence of this series in turn is equivalent to $\int_{0+} \{\psi(h)\}^{d-4}(dh/h) < \infty$. Hence it follows by Lemma 2.15 (i) that if $\int_{0+} \{\psi(h)\}^{d-4}(dh/h) < \infty$, then $P(E) = 0$. Conversely, the event

$$E_3 = \{\min_{4^{-n} \leqq u \leqq 4^{-n+1}, 4^{-n} \leqq v \leqq 4^{-n+1}} |X(t+u) - X(t-v)| < (4^{-n})^{\frac{1}{2}}\psi(4^{-n}) \ \text{i.o.}\}$$

implies the event E. Let

$$B_n = \{\min_{4^{-n} \leqq u, v \leqq 4^{-n+1}} |X(t+u) - X(t-v)| < (4^{-n})^{\frac{1}{2}}\psi(4^{-n})\} ,$$

then $E_3 = \bigcap_{m=1}^{\infty} \bigcup_{n \geqq m} B_n$. By Lemma 4.1 we get $P(B_n) \geqq c\{\psi(4^{-n})\}^{d-4}$. Since the divergence of $\int_{0+} \{\psi(h)\}^{d-4}(dh/h)$ is equivalent to $\sum_n \{\psi(4^{-n})\}^{d-4} = \infty$, we conclude that in this case $\sum_{n=1}^{\infty} P(B_n) = \infty$. To apply Lemma 2.15 (ii) to finish the proof, it is enough to show that there exists $c > 0$ such that $P(B_n \cap B_{n+2}) \leqq cP(B_n)P(B_{n+2})$ for all n. To see this, let n be fixed and let $h = 4^{-n-2}$. In the definition of B_n we will write $X(t+u) - X(t) = Y(u)$ and $X(t) - X(t-v) = Z(v)$ so that Y and Z are independent Brownian motions. Then

$$B_n \cap B_{n+2} = \{\min_{h \leqq u, v \leqq 4h} |Y(u) + Z(v)| \leqq h^{\frac{1}{2}}\psi(h) ; \quad \min_{16 \leqq u, v \leqq 64h} |Y(u) + Z(v)| \leqq (16h)^{\frac{1}{2}}\psi(16h)\} . \tag{4.23}$$

Let $h = t_0 < t_1 < \cdots < t_k = 4h$ be a partition of $[h, 4h]$ into k equal parts, where k will be chosen suitably later. Let

$$\tau = \inf\{v : h \leqq v \leqq 4h, \min |Y(u) + Z(v)| \leqq h^{\frac{1}{2}}\psi(h)\} \quad = \infty \quad \text{if the above set is empty.} \tag{4.24}$$

Define

$$U_i = \{\min_{h \leqq u \leqq t_i, h \leqq v \leqq 4h} |Y(u) + Z(v)| \leqq h^{\frac{1}{2}}\psi(h)\} . \tag{4.25}$$

Then

$$B_n \cap B_{n+2} = \bigcup_{i=1}^{k} \{U_i - U_{i-1} ; \quad \min_{16h \leqq u, v \leqq 64h} |Y(u) + Z(v)| \leqq (16h)^{\frac{1}{2}}\psi(16h)\} , \tag{4.26}$$

where the sets in the union are disjoint. We denote the ith set in this union by Λ_i. Since $h \leqq \tau \leqq 4h$ on each Λ_i, we have

$$\Lambda_i \subset \{U_i - U_{i-1} ; \quad \min_{16h \leqq u \leqq 64h, 12h \leqq v \leqq 68} |Y(u) + Z(\tau + v)| \leqq (16h)^{\frac{1}{2}}\psi(16h)\} . \tag{4.27}$$

If $U_i - U_{i-1}$ occurs, then there is a θ, $t_{i-1} \leqq \theta \leqq t_i$ such that $|Z(\tau) - Y(\theta)| = h^{\frac{1}{2}}\psi(h)$. Since $h^{\frac{1}{2}}\psi(h)$ is an increasing function of h we see that the bigger event in (4.27) is contained in the event

$$(4.28) \qquad \{U_i - U_{i-1};\ \min_{16h \leq u \leq 64h, 12h \leq v \leq 68h} |Y(u) - Y(t_i) + Z(\tau + v) - Z(\tau)| \leqq 2(16h)^{\frac{1}{2}}\psi(16h) + \sup_{t_{i-1} \leq \lambda \leq t_i} |Y(\lambda) - Y(t_i)|\} .$$

The event in (4.28) is contained in the union

$$(4.29) \qquad \{U_i - U_{i-1};\ \min_{12h \leq u, v \leq 68h} |Y(u) - Y(t_i) + Z(\tau + v) - Z(\tau)| \leqq 3(16h)^{\frac{1}{2}}\psi(16h)\} \cup \{\sup_{t_{i-1} \leq \lambda \leq t_i} |Y(\lambda) - Y(t_i) > (16h)^{\frac{1}{2}}\psi(16h)\} .$$

By Lemma 2.11, $Z(\tau + v) - Z(\tau)$ is independent of the $Y(u)$ process and of $Z(v)$ process up to time τ, and by the independent increments property $Y(u) - Y(t_i)$ is independent of the $Y(t)$ process up to time t_i and also of the $X(t)$ process. Hence $U_i - U_{i-1}$ is independent of the event

$$\min |Y(u) - Y(t_i) + Z(\tau + v) - Z(\tau)| \leqq 3(16h)^{\frac{1}{2}}\psi(16h) .$$

Hence the probability of the events in (4.29) is dominated by

$$(4.30) \qquad P\{U_i - U_{i-1}\}P\{\min_{12h \leq u, v \leq 68h} |Y(u) - Y(t_i) + Z(\tau + v) - Z(\tau) \leqq 3(16h)^{\frac{1}{2}}\psi(16h)\} + P\{\sup_{t_{i-1} \leq \lambda \leq t_i} |Y(\lambda) - Y(t_i)| > (16h)^{\frac{1}{2}}\psi(16h)\} .$$

Since $h \leqq t_i \leqq 4h$, it is easy to see by the arguments of Lemma 4.1 that the second probability in (4.30) is of order $\{\psi(16h)\}^{d-4}$ independently of t_i. The third probabilty in (4.30) clearly does not depend on i. Hence summing on i we see that

$$(4.31) \qquad P\{B_n \cap B_{n+2}\} \leqq cP(B_n)P(B_{n+2}) + kP\{\sup_{0 \leq \lambda \leq 3h/k} |Y(\lambda)| \geqq (16h)^{\frac{1}{2}}\psi(16h)\} .$$

By choosing k sufficiently large we can make the second expression on the right in (4.31) to be of the same order as the $P(B_n)P(B_{n+2})$. This completes the proof for $d \geqq 5$. The case $d = 4$ now follows by the same arguments by using Lemma 4.2 in place of Lemma 4.1.

REMARK 4.2. In the proof of Theorem 4.1 we only needed an upper bound for $P(E'_{h,\Delta})$ in Lemma 4.1 as well; however, we proved it in the stronger form since it involved no additional work. Note also that $P(E_{h,\Delta}) = 1$ when $d = 4$, since the infinite Wiener sausage is then a recurrent set; that is, two independent Brownian motions in 4-space approach arbitrarily closely for large times, even though they do not intersect.

REMARK 4.3. It is also clear that our arguments also deal with the close approach of two independent Brownian motion paths $X_d(t, \omega)$ and $Y_d(t, \omega)$ for large times. For $d \geqq 5$, if $\psi(h) \downarrow$ as $h \uparrow$, then there exists $\tau_0(\omega)$ such that $|X_d(t) - Y_d(s)| > \tau^{\frac{1}{2}}\psi(\tau)$ for $\tau \geqq \tau_0$ if and only if $\int^\infty \{\psi(h)\}^{d-4}(dh/h) < \infty$, where we can take $\tau = \min(t, s)$, or $\tau = t + s$, or $\tau = \max(t, s)$. For $d = 4$, the condition is $\int^\infty |\log \psi(h)|^{-1}h^{-1}dh < \infty$, but τ must be $\min(t, s)$ in this case.

5. First passage time process. Let $X_d(t)$ be standard Brownian motion in R^d and let $P_d(a)$ denote the first passage time of $X_d(t)$ out of the ball of radius a and center 0. The following tail estimate for $P_d(a)$ is available from [3],

$$P\{P_d(a) > c_d a^2 \lambda\} \approx e^{-\lambda} , \tag{5.1}$$

where $c_d = 2q_d^{-1}$, q_d being the first positive root of the Bessel function $J_{d/2-1}$. The other constants involved in (5.1) are independent of a.

As before, let Φ_ε denote the class of functions $\varphi: (0, \varepsilon) \to [0, \infty)$ such that $\varphi(t) \uparrow \infty$ as $t \downarrow 0$. Then we will prove

THEOREM 5.1. *Let $\varphi \in \Phi_\varepsilon$. Then there exists a positive constant $c_d = 2q_d^{-2}$, where q_d is the first positive root of the Bessel function $J_{d/2-1}$, such that*

$$P\{P_d(a) > c_d a^2 \varphi(a) \text{ i.o., } a \downarrow 0\} = 1 \quad or \quad 0$$

according as $\int_{0+} a^{-1}\varphi(a)e^{-\varphi(a)}da = \infty$ or $< \infty$.

REMARK 5.1. For $d = 1$ the constant c_d of Theorem 5.1 is simply $8\pi^{-2}$.

As in Section 3 we denote by $\{a_j\}$ the sequence

$$a_j = e^{-j/\log j} , \qquad j \geqq 2 . \tag{5.2}$$

We will also denote for $\varphi \in \Phi_\varepsilon$,

$$t_j = c_d a_j^2 \varphi(a_j) , \tag{5.3}$$

$$E_j = \{P_d(a_j) > t_j\} . \tag{5.4}$$

The following lemma will enable us to apply Lemma 2.15 in the proof of Theorem 5.1 later.

LEMMA 5.1. (i) *If $j + (\log j)^{\frac{3}{2}} < k$, then*

$$P(E_j \cap E_k) \leqq cP(E_j)P(E_k) .$$

(ii) *If $j < k$ and $t_j - t_k > c_d \lambda a_j^2$, $\lambda > 0$, then*

$$P(E_j \cap E_k) \leqq cP(E_j)e^{-\lambda/4} .$$

PROOF. We have

$$E_j \cap E_k \subset E_k \cap \{P_d'(a_j + a_k) > t_j - t_k\} ,$$

where $P_d'(a_j + a_k)$ is the first passage time for $X(u) - X(t_k)$, $u \geqq t_k$, which is independent of $P_d(a_k)$. Hence we have

$$P(E_j \cap E_k) \leqq P(E_k)P\{P_d'(a_j + a_k) > t_j - t_k\} . \tag{5.5}$$

Simple computation using the estimate in (5.1) shows that if $k > j(\log j)^{\frac{3}{2}}$, then the last probability in (5.5) is dominated by $cP(E_j)$, which proves (i). For (ii) we first dominate the last probability in (5.5) by $P\{P_d'(a_j + a_k) > c_d \lambda a_j^2\}$, since $t_j - t_k > c_d \lambda a_j^2$. Then get the upper estimate $ce^{-\lambda a_j^2/(a_j+a_k)^2}$ for this latter quantity from (5.1). This proves (ii) since $a_j \geqq a_k$, and $P(E_k)$ is dominated by $cP(E_j)$.

PROOF OF THEOREM 5.1. Let us define, writing $\log_m$ for the m-iterated logarithm,

$$\varphi_1(u) = \log_2 u^{-1} + \tfrac{1}{2}\log_3 u^{-1}, \quad \varphi_2(u) = \log_2 u^{-1} + 2\log_3 u^{-1} \tag{5.6}$$

Note that $\varphi_1, \varphi_2 \in \Phi_\varepsilon$. We now split the proof into several parts.

(i) Let $\varphi_1 \leqq \varphi \leqq \varphi_2$, $\varphi \in \Phi_\varepsilon$, such that $\sum_j e^{-\varphi(a_j)} < \infty$. Then we will show

$$P\{P_d(a) > c_d a^2 \varphi(a) \text{ i.o., } a \downarrow 0\} = 0 . \tag{5.7}$$

By using the estimate of (5.1), we get

$$\begin{aligned} \sum_j P\{P_d(a_j) > c_d a_{j+1}^2 \varphi(a_j)\} &\leqq c \sum_j \exp[-a_{j+1}^2 a_j^{-2} \varphi(a_j)] \\ &= c \sum_j \exp[-\varphi(a_{j+1}) - (a_j^2 - a_{j+1}^2) a_j^{-2} \varphi(a_j)] . \end{aligned} \tag{5.8}$$

Since $\varphi_1 \leqq \varphi \leqq \varphi_2$, $(a_j^2 - a_{j+1}^2)a_j^{-2}\varphi(a_j)$ is bounded in j, and we conclude that the series in (5.8) converges (by the assumed covergence of $\sum_j e^{-\varphi(a_j)}$). Hence by Lemma 2.15 (i) we have

$$P_d(a_j) \leqq c_d a_{j+1}^2 \varphi(a_j) \quad \text{eventually.} \tag{5.9}$$

If $a_{j+1} \leqq a < a_j$, then by the monotonicity of P_d, a_j and φ, we have

$$P_d(a) \leqq P_d(a_j) \leqq c_d a_{j+1}^2 \varphi(a_j) \leqq c_d a^2 \varphi(a) ,$$

which proves (5.7).

(ii) Let $\varphi_1 \leqq \varphi \leqq \varphi_2$, $\varphi \in \Phi_\varepsilon$, such that

$$u^2\varphi(u) \downarrow 0 \quad \text{as} \quad u \downarrow 0 , \qquad \sum_j \exp[-\varphi(a_j)] = \infty . \tag{5.10}$$

Then we will show

$$P\{P_d(a) > c_d a^2 \varphi(a) \text{ i.o., } a \downarrow 0\} = 1 . \tag{5.11}$$

Recalling the definition of E_n given by (5.4), it is clearly enough to prove

$$P\{E_n \text{ occur i.o.}\} = 1 . \tag{5.12}$$

We now proceed to apply Lemma 2.15 (ii) to the sequence of events E_n. First observe that by (5.1) we have $P(E_n) \geqq ce^{-\varphi(a_n)}$. Hence by (5.10) we have $\sum_n P(E_n) = \infty$. The problem now is that we cannot apply Lemma 2.15 (ii) to the sequence $\{E_n\}$ directly. For this reason we pick a subsequence $E_{n(j)}$ of $\{E_n\}$ such that $\sum_j P(E_{n(j)}) = \infty$ and Lemma 2.15 (ii) applies to this subsequence. Call a subscript "good" or "bad" according as

$$\varphi(a_{n+1}) - \varphi(a_n) \leqq \tfrac{1}{4} \qquad \text{or} \qquad \varphi(a_{n+1}) - \varphi(a_n) > \tfrac{1}{4} . \tag{5.13}$$

From the sequence $\{E_n\}$ drop the events with "bad" subscripts and relabel the remaining subscripts as $n(j), j \geqq 1$. Since $P(E_n) \approx \exp[-\varphi(a_n)]$ by (5.1), it is clear from (5.13) that $\sum_j P(E_{n(j)}) = \infty$. Let

$$j + \log^{\frac{1}{8}} j = j_1 , \qquad j + 4\log^{\frac{3}{2}} j = j_2 , \qquad P(E_{n(j)}E_{n(k)}) = b_{jk} . \tag{5.14}$$

Suppose there is a fixed j_0 such that for $j \geqq j_0$ the following estimates hold:

$$k > j_2 \Rightarrow b_{jk} \leqq cP(E_{n(j)})P(E_{n(k)}) , \tag{5.15}$$

$$j_1 \leqq k \leqq j_2 \Rightarrow b_{jk} \leqq cP(E_{n(j)}) \exp[-c'\{\log n(j)\}^{\frac{1}{8}}] \tag{5.16}$$

$$j < k < j_1 \Rightarrow b_{jk} \leqq cP(E_{n(j)})e^{-c'(k-j)} , \tag{5.17}$$

then it is easily verified that $\sum_{k=j+1}^{N} b_{jk} \leqq c''P(E_{n(j)}) \sum_{k=1}^{N} P(E_{n(k)})$, and Lemma 2.15 (ii) applies to the events $\{E_{n(j)}\}$ and we conclude that infinitely many of them occur a.s. Hence it remains to check (5.15), (5.16) and (5.17). Since $\varphi_1 \leqq \varphi \leqq \varphi_2$, first observe that starting at any $j \geqq j_0$ (large enough) there cannot be more than $\{\log j\}^{\frac{1}{8}}$ "bad" subscripts. Hence $j \leqq n(j) \leqq j\{\log n(j)\}^{\frac{1}{8}}$, which gives

$$\log n(j) \sim \log j . \tag{5.18}$$

If $k > j_2$, then $n(k) - n(j) \geqq k - j \geqq 4\{\log j\}^{\frac{3}{4}} \geqq \{\log n(j)\}^{\frac{3}{4}}$ for $j \geqq j_0$ by (5.18). Hence (5.15) follows from Lemma 5.1 (i). Since $\{t_n\}$ is decreasing, we have by (5.13)

$$\begin{aligned} c_d^{-1}(t_{n(j)} - t_{n(j+1)}) &\geqq c_d^{-1}(t_{n(j)} - t_{n(j)+1}) \\ &\geqq a_{n(j)}^2 \varphi(a_{n(j)}) - a_{n(j)+1}^2 [\varphi(a_{n(j)}) + \tfrac{1}{4}] \\ &\geqq a_{n(j)}^2 . \end{aligned}$$

The last inequality follows by noting that $\varphi(a_n)(a_n^2 - a_{n+1}^2)a_n^{-2} \geqq \frac{3}{2}$ and $a_n \downarrow 0$. Hence for $j < k$ we get

$$t_{n(j)} - t_{n(k)} \geqq c_d(a_{n(j)}^2 + \cdots + a_{n(k-1)}^2) . \tag{5.19}$$

Using (5.18) and the fact that $n(k) - n(j) \leqq (k - j)\{\log n(k)\}^{\frac{1}{8}}$, $j < k$, it is easily verified that if $j \geqq$ some j_0, then

$$a_{n(k-1)}^2 > \tfrac{1}{2}a_{n(j)}^2 \qquad \text{for } j \leqq k \leqq j_1 . \tag{5.20}$$

Hence if $j_1 \leqq k \leqq j_2$, then $t_{n(j)} - t_{n(k)} \geqq t_{n(j)} - t_{n(j_1)} \geqq \frac{1}{2}c_d a_j^2\{\log j\}^{\frac{1}{8}}$. The estimate (5.16) now follows from Lemma 5.1 (ii). Finally if $j < k < j_1$, then from (5.19) and (5.20) we get $t_{n(j)} - t_{n(k)} \geqq \frac{1}{2}c_d(k - j)a_{n(j)}^2$, and Lemma 5.1 (ii) gives (5.17).

(iii) Now let $\varphi \in \Phi_\varepsilon$ such that $\varphi_1 \leqq \varphi \leqq \varphi_2$. Combining (i) with Lemma 2.12 we conclude

$$\int_{0+} \frac{\varphi(u)}{u} e^{-\varphi(u)} du < \infty \Rightarrow P\{P_d(a) > c_d a^2 \varphi(a) \text{ i.o.}, \ a \downarrow 0\} = 0 . \tag{5.21}$$

Now define $\bar{\varphi}(u)$ by

$$u^2\bar{\varphi}(u) = \inf_{s \geqq u} s^2\varphi(s) . \tag{5.22}$$

Then $\bar{\varphi}(u) \leqq \varphi(u)$ and $\bar{\varphi}$ satisfies the conditions imposed on φ in (ii). Since $\bar{\varphi}(u) \leqq \varphi$, we have

$$\int_{0+} \frac{\varphi(u)}{u} e^{-\varphi(u)} du = \infty \Rightarrow \int_{0+} \frac{\bar{\varphi}(u)}{u} e^{-\bar{\varphi}(u)} du = \infty . \tag{5.23}$$

If $\int_{0+} u^{-1}\varphi(u)e^{-\varphi(u)}\,du = \infty$, then by (5.23) and (ii) we have

$$(5.24) \qquad P\{P_d(a) > c_d a^2\bar{\varphi}(a) \text{ i.o., } a \downarrow 0\} = 1 .$$

Since $a^2\bar{\varphi}(a)$ is the lower monotone envelope of $a^2\varphi(a)$ and $P_d(a)$ is monotone, we have

$$(5.25) \qquad \{P_d(a) > c_d a^2\bar{\varphi}(a) \text{ i.o., } a \downarrow 0\} \subset \{P_d(a) > c_d a^2\varphi(a) \text{ i.o., } a \downarrow 0\} .$$

Hence we conclude that

$$(5.26) \qquad \int_{0+} \frac{\varphi(u)}{u} e^{-\varphi(u)}\,du = \infty \Rightarrow P\{P_d(a) > c_d a^2\varphi(a) \text{ i.o., } a \downarrow 0\} = 1 .$$

Hence we have proved Theorem 5.1 for $\varphi \in \Phi_\varepsilon$, $\varphi_1 \leqq \varphi \leqq \varphi_2$. We can now apply Lemma 2.14 to derive a result similar to Lemma 3.1 to remove the restriction $\varphi_1 \leqq \varphi \leqq \varphi_2$. This proves Theorem 5.1.

REMARK 5.2. One can obviously formulate a result such as Theorem 5.1 when $a \to \infty$. The arguments outlined above apply to that case with obvious modifications.

Since the large values of $|X_d(t)|$ and $M_d(t)$ as $t \downarrow 0$ have the same behavior, the standard result about the large values of $|X_d(t)|$ as $t \downarrow 0$ gives us the following integral test for small values of $P_d(a)$ via Lemma 2.13.

THEOREM 5.2. *Let* $\varphi \in \Phi_\varepsilon$. *Then*

$$P\{P_d(a) < \tfrac{1}{2}a^2\varphi(a)^{-1} \text{ i.o., } a \downarrow 0\} = 1 \quad or \quad 0$$

according as $\int_{0+} (\varphi(x)^{d/2}/x)e^{-\varphi(x)}\,dx$ *diverges or converges.*

REMARK 5.3. It is not necessary to require φ to be in Φ_ε' in Theorem 5.2.

Two-sided results for $P_d(a)$: for $t > 0$ we can look at the length of the longest interval $(t - u, t + v)$ such that

$$(5.27) \qquad |X_d(s) - X_d(t)| \leqq a \qquad \text{for} \quad t - u < s < t + v .$$

Let $P_d{}^*(t, a)$ denote this random variable. For any fixed $t > 0$, it is clear that the asymptotic behavior of $P_d{}^*(t, a)$ as $a \downarrow 0$ is the same as that of $P_d{}^1(a) + P_d{}^2(a)$, where $P_d{}^1$, $P_d{}^2$ are independent first passage time processes of the type discussed in Theorem 5.1. By applying the method of Section 3 to modify the arguments of Theorem 5.1 we can prove

THEOREM 5.3. *For fixed* $t > 0$, c_d *as in Theorem* 5.1, *and* $\varphi \in \Phi_\varepsilon$

$$P\{P_d{}^*(t, a) > c_d a^2\varphi(a) \text{ i.o., } a \downarrow 0\} = 1(0)$$
$$\Leftrightarrow \int_{0+} \frac{\varphi(a)^2}{a} e^{-\varphi(a)}\,da = \infty(< \infty) .$$

6. Related processes. In [1] Chung obtained the asymptotic behavior of $\liminf_n \sup_{1\le j\le n} |S_j|$ for partial sums of independent random variables. If $X_d(t)$ is standard Brownian motion in R^d, it is clear that his results can be formulated for the process

$$M_d(t) = \sup_{0 \le u \le t} |X_d(u)| . \tag{6.1}$$

If Ψ_A denotes the class of functions $\psi\colon [A, \infty) \to [0, \infty)$ such that $\psi(t) \uparrow \infty$ as $t \uparrow \infty$, then we get

THEOREM 6.1. *Let $\psi \in \Psi_A$ and $M_d(t)$ be defined by* (6.1). *Then there exists a positive constant $c_d' = c_d^{-\frac{1}{2}}$, where c_d is the constant of Theorem* 5.1, *such that*

$$P\{M_d(t) < c_d' t^{\frac{1}{2}} \{\psi(t)\}^{-1} \text{ i.o., } t \uparrow \infty\} = 1(0)$$
$$\Leftrightarrow \int^{\infty} \frac{\{\psi(u)\}^2}{u} \exp[-\{\psi(u)\}^2]\, du = \infty .$$

Chung's method can also be used to give a local asymptotic result for $M_d(t)$. However it is clear that small values for $M_d(t)$ correspond to large values of $P_d(a)$, the first passage time considered in Section 5, since

$$\{P_d(a) > \lambda\} = \{M(\lambda) < a\} . \tag{6.2}$$

We exploit this connection to prove

THEOREM 6.2. *Let $\psi \in \Phi_\varepsilon$. Then there exists a constant c_d', same as in Theorem* 6.1, *such that*

$$P\{M_d(t) < c_d' t^{\frac{1}{2}} \{\psi(t)\}^{-1} \text{ i.o., } t \downarrow 0\} = 1(0)$$
$$\Leftrightarrow \int_{0+} \frac{\{\psi(a)\}^2}{a} \exp[-\{\psi(a)\}^2]\, da = \infty (< \infty) .$$

PROOF. Let $u(t) = c_d^{-\frac{1}{2}} t^{\frac{1}{2}} \{\psi(t)\}^{-1}$. Since u is strictly monotone in t, the inverse function $t(u)$ exists. Let $\varphi(u) = \{\psi(t(u))\}^2$. Then $\varphi \in \Phi_{\varepsilon'}$ for some $\varepsilon' > 0$. By the relationship (6.2) we have

$$\begin{aligned} P\{M_d(t) < c_d^{-\frac{1}{2}} t^{\frac{1}{2}} \{\psi(t)\}^{-1} \text{ i.o., } t \downarrow 0\} \\ = P\{P_d(u) > c_d u^2 \varphi(u) \text{ i.o., } u \downarrow 0\} . \end{aligned} \tag{6.3}$$

By Theorem 5.1 the right-hand side in (6.3) is equal to 1 or 0 according as the integral $\int_{0+} u^{-1} \varphi(u) e^{-\varphi(u)}\, du$ diverges or converges. By Lemma 2.13 the convergence of this integral is equivalent to the convergence of the integral $\int_{0+} a^{-1} \{\psi(a)\}^2 \exp[-\{\varphi(a)\}^2]\, da$, which proves Theorem 6.2.

REMARK 6.1. Because of (6.2) we could have adapted Chung's method [1] to prove the local result for $M_d(t)$ first and then used Lemma 2.13 to get our Theorem 5.1 for $P_d(a)$. However our approach seems to be considerably simpler.

It is not difficult to formulate and prove a two-sided growth law for $M_d(t)$ which can be obtained from Theorem 5.3 by applying Lemma 2.13. However the result is not a natural one so we omit it.

Consider

$$R(s, t) = \sup_{s \le u, v \le t} |X_d(u) - X_d(v)| .$$

Since the upper tail of the distribution of $R(t, t + h)$ differs asymptotically from the upper tail of $|X_d(h)|$ by only a constant multiple [12], it is clear that the standard proof of the integral test for the large values of $|X_d(h)|$ as $h \to 0$ gives precisely the same test for the large values of $R(t, t + h)|$ as $h \to 0$. An examination of the proof of Theorem 3.1 shows that we have also shown that the integral test of that theorem is valid for the large values of $R(t - u, t + v)$ as $(u + v) \to 0$.

There does not seem to be any easy way of obtaining results about the small values of $R(t, t + h)$ as $h \to 0$ in terms of the first passage process. It seems likely that a direct argument along the lines of Theorem 5.1 would lead to an integral test if the lower tail of the distribution of $R(t, t + h)$ were known: this tail is known for $d = 1$ [6], but not for $d \geqq 2$. Integral for the small values of $R(t - u, t + v)$ as $(u + v) \to 0$ could also be considered.

There is one other associated process which has more intrinsic interest. For $d \geqq 3$, Brownian motion is transient and we can consider

$$T_d(a) = \int_0^\infty I_a\{X(t)\}\, dt\,; \qquad I_a(x) = \begin{cases} 1 & \text{for} \quad |x| \leqq a\,, \\ 0 & \text{for} \quad |x| > a\,; \end{cases}$$

which is the total time spent by the process in a ball of radius a. For a fixed $a > 0$, it was shown in [3] that the random variable $T_{d+2}(a)$ has the same distribution as $P_d(a)$, so there is complete information about the distribution of $T_d(a)$. Standard methods [3] lead to iterated logarithm type results for both the large and small values of $T_d(a)$ as $a \to 0$, and again these are the same for $T_{d+2}(a)$ as for $P_d(a)$. However we cannot deduce any asymptotic results about $T_d(a)$ from those for $P_{d-2}(a)$ because the joint distributions $T_{d+2}(a_1)$, $T_{d+2}(a_2)$ are not the same as $P_d(a_1)$, $P_d(a_2)$. If one tries to obtain a precise integral test for either the large or the small asymptotic values of $T_d(a)$ as $a \to 0$, the independence difficulties are formidable, and we have not succeeded in finding a valid method.

In closing it is worth pointing out that very little is known about the problems considered in the present paper for other types of process. For independent increment processes it would be interesting to have information about the small values of $M(t)$ as $t \to 0$, but in general we do not even have an "iterated logarithm" type of result except for the monotone processes. For the large values of $|X(t + h) - X(t)|$ as $h \downarrow 0$ it is known that some processes other than Brownian motion have a correct function corresponding to the interated logarithm in (1.1), but no necessary and sufficient criterion is known for the existence of such a function. For most independent increment processes it is known that no exact upper growth function exists (for example, if $X(t)$ is stable of index α [11]), and in this case it is clear that the division into upper and lower classes will be the same for two-sided growth as it is for one-sided growth.

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