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A PROBABILISTIC PROOF OF THE NORMAL CONVERGENCE CRITERION[1]

BY D. ROOT AND H. RUBIN

Purdue University

By embedding partial sum processes into Brownian motion, it is well known that the deMoivre–Laplace central limit theorem is a consequence of the strong law of large numbers. It is the purpose here to show that the embedding technique can be used to establish both the degenerate convergence criterion and the normal convergence criterion for triangular arrays of uniformly asymptotically negligible random variables.

Let X_{nk}, $k = 1, \cdots, k(n)$, $n = 1, 2, \cdots$ be a triangular array of independent random variables.

THEOREM. *If X_{nk} are independent summands then for every $\varepsilon > 0$*

$$\mathscr{L}(\sum_k X_{nk}) \to N(\alpha, \sigma^2) \qquad \text{and} \qquad \max_k P[|X_{nk}| > \varepsilon] \to 0$$

as $n \to \infty$, if and only if for every $\delta > 0$ and every $\tau > 0$,

(i) $\sum_k P[|X_{nk}| \geqq \delta] \to 0$ *as* $n \to \infty$
(ii) $\sum_k \sigma^2(X_{nk}^\tau) \to \sigma^2$ *as* $n \to \infty$
(iii) $\sum_k E(X_{nk}^\tau) \to \alpha$ *as* $n \to \infty$.

PROOF. First we prove sufficiency by embedding the sums into Brownian motion. Let $\Omega = \{\omega : [0, \infty) \to (-\infty, +\infty) \mid \omega \text{ is continuous and } \omega(0) = 0\}$, and define $l_t(\omega) = \omega(t)$ and $\mathscr{B}_t =$ the σ-field defined on Ω by $\{l_s; s \leqq t\}$ and let P be a probability measure such that $(\Omega, l, \mathscr{B}_t, P)$ is standard Brownian motion. By (i) we assume without loss of generality that $P[|X_{nk}| < \varepsilon_n] = 1$ where $\varepsilon_n \downarrow 0$ for all n and k. Define stopping times $T(n, k) : \Omega \to [0, \infty)$ so that $\mathscr{L}(l_{T(n,k)}) = \mathscr{L}(X_{nk} - E(X_{nk}))$ [see [2] for definition and properties of the $T(n, k)$]. Then $E(T(n, k)) = \sigma^2(X_{nk})$ and there is a constant C such that $E(T^2(n, k)) \leqq CE((X_{nk} - E(X_{nk}))^4)$. If $\omega \in \Omega$, define ω_t by $\omega_t(s) = \omega(t + s) - \omega(t)$ and if S, T are two stopping times define $(T + S)(\omega) = T(\omega) + S(\omega_{T(\omega)})$. Setting $S(n) = \sum_k T(n, k)$ we have $\mathscr{L}(l_{S(n)}) = \mathscr{L}(\sum_k (X_{nk} - E(X_{nk}))$. Clearly $(E((X_{nk} - E(X_{nk}))^4))/(\sigma^2(X_{nk})) \to 0$ so that

$$\sum_k \sigma^2(T(n, k)) \leqq \sum_k E(T^2(n, k)) \leqq C \sum_k E((X_{nk} - E(X_{nk}))^4) \to 0 .$$

So by Chebyshev's inequality $\mathscr{L}(\sum_k T(n, k)) \to \mathscr{L}(\sigma^2)$. By continuity of

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Brownian paths, $\mathscr{L}(\sum(X_{nk} - E(X_{nk})) = \mathscr{L}(l_{S(n)}) \to N(0, \sigma^2)$ where $S(n) = \sum T(n, k)$. Hence by (iii) $\mathscr{L}(\sum_k X_{nk}) \to N(\alpha, \sigma^2)$. Sufficiency is established. To prove necessity let X_{nk}, $k = 1, \cdots, k(n)$, $n = 1, 2, \cdots$ be a triangular array such that $\mathscr{L}(\sum_k X_{nk}) \to N(0, 1)$ and $\max_k P[|X_{nk}| > \varepsilon] \to 0$. First we show (i). Let $Y_{nk} = 2^{-\frac{1}{2}}(X_{nk} - X'_{nk})$ where X_{nk} and X'_{nk} are independent and $\mathscr{L}(X_{nk}) = \mathscr{L}(X'_{nk})$. Clearly $\mathscr{L}(\sum_k Y_{nk}) \to N(0, 1)$. To establish (i) it is sufficient to show that for any $a > 0$, $\sum_k P[|Y_{nk}| > a] \to 0$. Define M_n to be the number of Y_{nk}, $k = 1, \cdots, k(n)$ such that $|Y_{nk}| > a$. Let

$$C_n = \{|\textstyle\sum_k Y_{nk}| < \frac{1}{2}a\}\,, \qquad P[C_n] \to \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-a/2}^{a/2} e^{-x^2/2}\, dx = 2(\Phi(a/2) - \tfrac{1}{2}) > 0\,.$$

If $M_n = r$ then of the 2^r possible ways of assigning positive signs to the $|Y_{nk}| > a$ at most $\binom{r}{l}$, $l = [r/2]$ (this is the maximum number of incomparable subsets of a set of size r) allow $|\sum_k Y_{nk}| < a/2$. Hence $P[C_n] \leqq P[M_n = 0] + \sum_{r=1}^{\infty} b_r P[M_n = r]$ where $b_r = \binom{r}{l}2^{-r}$. If $\limsup_n \sum_k P[|Y_{nk}| > a] = \eta > 0$, then we can, by passing to a subsequence, assume $\lim_n \sum P[|Y_{nk}| > a] = \eta$. First $\eta \neq \infty$, because in that case $P[M_n = r] \to 0$ for $r = 0, 1, 2, \cdots$ which implies $P[C_n] \to 0$, which is a contradiction. Next we show that $\eta = 0$. Suppose not. Let $I(n) = \{k : |Y_{nk}| < a\}$, and $J(n)$ be the complement of $I(n)$. Then

$$\begin{aligned} P[\textstyle\sum_k Y_{nk} > Na] &\geqq P[M_n = N]P[\textstyle\sum_{k\in I(n)} Y_{nk} \geqq 0]P[\textstyle\sum_{k\in J(n)} Y_{nk} > Na] \\ &\geqq P[M_n = N]\tfrac{1}{2}P[Y_{nk} > 0 \quad \text{for all} \quad k \in J(n)] \\ &\to \frac{e^{-\eta}(\eta)^N}{(N)!}\,\frac{1}{2}\left(\frac{1}{2}\right)^N = p(\eta, N); \end{aligned}$$

also

$$\lim_{n\to\infty} P[\textstyle\sum_k Y_{nk} > Na] \leqq \frac{1}{2^{\frac{1}{2}}\pi} \int_{Na}^{\infty} e^{-x^2/2}\, dx \leqq \frac{1}{2^{\frac{1}{2}}\pi}\,\frac{e^{-N^2a^2/2}}{Na} = q(N)\,.$$

Hence $\lim_{N\to\infty} p(\eta, N)/q(N) = \infty$ which implies $\eta = 0$.

Thus (i) is established. To show (ii) we may assume without loss of generality that $P[|X_{nk}| \leqq \tau_n] = 1$ for all n and k where $\tau_n \downarrow 0$. Let $(\Omega, l_t, \mathscr{B}_t, P)$ be standard Brownian motion and let $T(n, k)$ be a stopping time such that $\mathscr{L}(l_{T(n,k)}) = \mathscr{L}(X_{nk} - E(X_{nk}))$ and $E(T(n, k)) = \sigma^2(X_{nk})$, and $E(T^2(n, k)) \leqq CE((X_{nk} - E(X_{nk}))^4)$. By passing to a subsequence, if necessary, we can assume that $\lim \sum_k \sigma^2(X_{nk}) \to \lambda$, $0 \leqq \lambda \leqq +\infty$. Suppose $\lambda > 1$. We can now chose integers $r(n)$ such that $\sum_{k=1}^{r(n)} \sigma^2(X_{nk}) \to 1 + \varepsilon$ for some $\varepsilon > 0$. Then we have

$$\textstyle\sum_{k=1}^{r(n)} \sigma^2(T(n, k)) \leqq \sum_{k=1}^{r(n)} E(T^2(n, k)) \leqq C \sum_{k=1}^{r(n)} E((X_{nk} - E(X_{nk}))^4) \to 0\,,$$

because (i) implies that $(\sigma^2(X_{nk}))^{-1}E((X_{nk} - E(X_{nk}))^4) \to 0$ as $n \to \infty$. So by Chebyshev's inequality $\mathscr{L}(\sum_{k=1}^{r(n)} T(n, k)) \to \mathscr{L}(1 + \varepsilon)$ hence by the continuity of Brownian paths $\mathscr{L}(\sum_{k=1}^{r(n)} (X_{nk} - E(X_{nk}))) \to N(0, 1 + \varepsilon)$. Thus

$$\begin{aligned} &\lim_{n\to\infty} \max_x P[x - a < \textstyle\sum_{k=1}^{k(n)} X_{nk} < a + x] \\ &\qquad\qquad \leqq (2\pi(1 + \varepsilon))^{-\frac{1}{2}} \int_{-a}^{a} \exp -\left[\frac{x^2}{2(1 + \varepsilon)}\right] dx \end{aligned}$$

and hence $\mathscr{L}(\sum_k X_{nk}) \nrightarrow N(0, 1)$. By a similar argument we show $\lambda < 1$ implies $\mathscr{L}(\sum_k (X_{nk} - E(X_{nk}))) \to N(0, \lambda)$, which is impossible. Thus (ii) is established. From (ii) we have that $\mathscr{L}(\sum_k T(n, k)) \to \mathscr{L}(1)$ and thus $\mathscr{L}(\sum_k (X_{nk} - E(X_{nk}))) \to N(0, 1)$ and hence $\sum_k E(X_{nk}) \to 0$. The theorem is proved.

REMARK. The degenerate convergence criterion is a special case of the normal convergence criterion (see [1] pages 316–317) and can be proved by the same embedding technique.

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DEPARTMENT OF STATISTICS
MATHEMATICAL SCIENCES BUILDING
PURDUE UNIVERSITY
LAFAYETTE, INDIANA 47907