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A DIMENSION THEOREM FOR SAMPLE FUNCTIONS OF PROCESSES WITH STABLE COMPONENTS[1]

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For processes $X(t)$ with stable components we calculate $\dim X(E)$ in terms of $\dim E$, where E is a fixed Borel subset of $[0, 1]$ of known Hausdorff-Besicovitch dimension, $\dim E$. Our results extend the earlier ones of Blumenthal and Getoor in the stable case.

1. Introduction. In [1], Blumenthal and Getoor proved that for stable processes $X(t)$ of stable index $\alpha \leqq 2$ in R^N and fixed Borel sets $E \subset R^1$,

$$\dim X(E) = \min\{N, \alpha \dim E\} \tag{1}$$

holds (with probability one), where $\dim A$ denotes the Hausdorff-Besicovitch dimension of A. The object of the present investigation is to prove results analogous to (1) for another type of stochastic process. Our process will be one with stable components as studied by Pruitt and Taylor in [10], and more recently in [5] and [6]. In [6] we obtained a partial solution to the problem and observed that processes of this type provide an interesting counterexample to some earlier conjectures.

2. Preliminaries. To define a process $X(t)$ with stable components in R^N (see [5] and [10]) we let $X_i(t)$ be a stable process of index α_i in Euclidean space of dimension d_i, for $i = 1, 2, \cdots, n$, $n \geqq 2$, assume the X_i are independent and let

$$X(t) \equiv (X_1(t), X_2(t), \cdots, X_n(t)) \qquad \text{and} \qquad N = d_1 + \cdots + d_n, \tag{2}$$

where the d_i-dimensional subspaces in which the $X_i(t)$ take their values are orthogonal. We assume that the stable indices satisfy: $0 < \alpha_n < \alpha_{n-1} < \cdots < \alpha_1 \leqq 2$. $X(t)$ will have the strong Markov property and stationary and independent increments. The sample functions $X(t, \omega)$ are right continuous and have left limits everywhere.

The d-dimensional characteristic function of a stable process in R^d of index α has the form $\exp[t\Psi(z)]$ where:

$$\Psi(z) = i(b, z) - c|z|^\alpha \int_{S^d} w_\alpha(z, \theta)\mu(d\theta)$$

with $b \in R^d$, $c > 0$,

$$w_\alpha(z, \theta) = [1 - i \operatorname{sgn}(z, \theta) \tan \prod \alpha/2]|(z/|z|, \theta)|^\alpha, \quad \alpha \neq 1,$$
$$w_1(z, \theta) = |(z/|z|, \theta)| + (2i/\prod)(z/|z|, \theta) \log|(z, \theta)|$$

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and μ is a probability measure on the surface of the unit sphere S^d in R^d [7]. We assume μ is not supported by a proper subspace of R^d, and that $b = 0$, $c = 1$. If μ is uniform the process is called symmetric.

Blumenthal and Getoor [2] used the characteristic function of a process with stationary and independent increments to define indices β, β', β'' which satisfy $0 \leqq \beta'' \leqq \beta' \leqq \beta \leqq 2$, are uniquely determined by a given process, and which can be used to characterize many aspects of the sample function behavior. We will not repeat the definitions here but point out that for stable processes of index α we have $\beta = \beta'' = \alpha$, and for processes with stable components as defined in (2) we have $\beta = \alpha_1$. However, some of the processes we shall consider will have:

$$\alpha_2 = \beta'' < 1 + \alpha_2 - \alpha_2/\alpha_1 = \beta' < \alpha_1 = \beta .$$

It has been proven, [2] and [9], that for an arbitrary process $X(t)$ with stationary and independent increments of index $\beta' \leqq d$ ($\beta' > d$ cannot occur in processes with stable components) and fixed Borel subset $E \subset [0, 1]$

$$\beta' \dim E \leqq \dim X(E) \leqq \beta \dim E \tag{3}$$

holds with probability 1. In [6] we observed a process with stable components which showed that the bounds in (3) are the best possible. For this same process we will show that, given a number ε such that $\beta' \leqq \varepsilon \leqq \beta$ we can find a Borel subset E (depending upon ε) of $[0, 1]$ for which $\dim X(E) = \varepsilon \dim E$.

The density function $p_\alpha(t, x)$ of a stable process $X(t)$ of index α in R^d is continuous and bounded in x for each fixed t. It also satisfies the scaling property (except for some nonsymmetric processes of index $\alpha = 1$, which we henceforth exclude)

$$p_\alpha(t, x) = p_\alpha(rt, r^{1/\alpha}x)r^{d/\alpha} \tag{4}$$

for all $r > 0$. The density of $X(t)$, a process with stable components as defined by (2), will be denoted by $p(t, x)$ and is given by:

$$p(t, x) \equiv \prod_{i=1}^{n} p_{\alpha_i}(t, x_i) ,$$

where $x = (x_1, x_2, \cdots, x_n) \in R^N$, $x_i \in R^{d_i}$, and $d_1 + \cdots + d_n = N$.

3. Dimension theorem. The result of this study is the following

THEOREM. *Let $X(t)$ be a process with stable components as defined in* (2), *and suppose that E is a fixed Borel subset of* $[0, 1]$. *Then* (*with probability one*):

(i) *If $\alpha_1 > d_1$*

$$\begin{aligned} \dim X(E) &= \alpha_1 \dim E && \text{if } 0 \leqq \dim E \leqq 1/\alpha_1 \\ &= 1 - \alpha_2/\alpha_1 + \alpha_2 \dim E && \text{if } 1/\alpha_1 \leqq \dim E \leqq 1 . \end{aligned}$$

(ii) *If $\alpha_1 \leqq d_1$ (see remark* (iii) *of Section* 4)

$$\dim X(E) = \alpha_1 \dim E .$$

PROOF. The proof of (ii) is shorter, so we will do it first. One way to prove (ii) is to observe that $\beta = \beta' = \alpha_1$ whenever $\alpha_1 \leqq d_1$ and use the inequalities in (3). A second method, which is also used in the proof of (i), is the following projection argument. If $\alpha_1 \leqq d_1$ we have:

$$\text{(5)}\qquad \begin{aligned} &\dim X(E) \leqq \beta \dim E = \alpha_1 \dim E \quad \text{by (3) and} \\ &\dim X(E) \geqq \dim X_1(E) = \min\{d_1, \alpha_1 \dim E\} = \alpha_1 \dim E \quad \text{by (1)}. \end{aligned}$$

This completes the proof of (ii). We remark in passing that the projection argument used in proving the last inequality could also have been used in [6] to obtain the lower bound on $\dim X(F)$ (page 692).

The more interesting result and proof is that of (i). If $0 \leqq \dim E \leqq 1|\alpha_1$ we have $\dim X(E) = \alpha_1 \dim E$ by application of Theorem 1 of [6], or by the same argument used above in the bounds in (5). Now suppose that $\alpha_1 > 1 = d_1$ and that E is a fixed Borel subset of $[0, 1]$ of Hausdorff dimension γ, where $1 \geqq \gamma > 1/\alpha_1$. To establish the upper bound we will use the following result of Pruitt and Taylor [10] (pages 282–283):

LEMMA 1. *Let $X(t)$ be a process with stable components for which $\alpha_1 > d_1$, and suppose that $\Lambda(a)$ is a fixed collection of abutting cubes of side $a \leqq 1$ in R^N. Let $M(a, s)$ denote the number of these cubes hit by the path $X(t)$ at some time $t \in [0, s]$. Then there is a positive constant c_1 such that, for all $a \leqq s^{1/\alpha_2}$,*

$$\text{(6)}\qquad E[M(a, s)] \leqq c_1 s a^{-\rho}, \qquad \textit{where}\ \ \rho = 1 + \alpha_2 - \alpha_2/\alpha_1.$$

We now let $\delta > \gamma = \dim E$ and select a sequence of coverings of E by intervals $E_{i,m}$ of length $d_{i,m}$ for which $E \subset \bigcup_{i=1}^\infty E_{i,m}$, $m = 1, 2, 3, \cdots$, and

$$\text{(7)}\qquad \sum_{i=1}^\infty d_{i,m}^\delta < 1/m \qquad \text{for}\ \ m = 1, 2, 3, \cdots.$$

We now consider $X(E_{i,m})$, the range of $X(t)$ on the interval $E_{i,m}$. Since the process has stationary and independent increments, $X(E_{i,m})$ has the same distribution as $X([0, d_{i,m}])$. Cover R^N by abutting cubes of edge $a = d_{i,m}^{1/\alpha_2}$. Lemma 1 then gives:

$$\text{(8)}\qquad E[M(a, d_{i,m})] \leqq c_1 d_{i,m} d_{i,m}^{-\rho/\alpha_2} = c_1 d_{i,m}^{1/\alpha_1 - 1/\alpha_2}.$$

Consider only thoes cubse of the above collection which are hit by the process in the time interval $[0, d_{i,m}]$. These cubes form a covering of $X([0, d_{i,m}])$ by cubes of side $d_{i,m;j}(\omega) = d_{i,m}^{1/\alpha_2}$ for $j = 1, 2, 3, \cdots$. From (8) we now obtain:

$$\text{(9)}\qquad E[\textstyle\sum_j d_{i,m;j}^{1-\alpha_2/\alpha_1+\alpha_2\delta}] = (d_{i,m}^{1/\alpha_2})^{1-\alpha_2/\alpha_1+\alpha_2\delta} E[M(a, d_{i,m})] \leqq c_1 d_{i,m}^\delta.$$

Consequently, we can cover $X(E)$ by a sequence of covers, $X(E) \subset \bigcup_{i=1}^\infty X(E_{i,m})$, $m = 1, 2, 3, \cdots$, in such a way that the diameters, $d_{i,m;j}(\omega)$, of the covering sets satisfy:

$$\text{(10)}\qquad E[\textstyle\sum_{i=1}^\infty \sum_{j=1}^\infty d_{i,m;j}^{1-\alpha_2/\alpha_1+\alpha_2\delta}] \leqq c_1 \sum_{i=1}^\infty d_{i,m}^\delta < \dfrac{c_1}{m} \to 0 \qquad \text{as}\ \ m \to \infty.$$

Hence, by Fatou's lemma, the lim inf of the double sum is (with probability one) zero. Thus, $\dim X(E) \leqq 1 - \alpha_2/\alpha_1 + \alpha_2\delta$. Since $\delta > \gamma$ was arbitrary, the proof of the upper bound is complete.

We now proceed to the lower bound proof when $\alpha_1 > d_1 = 1$ and $1 \geqq \dim E = \gamma > 1/\alpha_1$. First we state and prove the counterpart of Lemma 1 of [6].

LEMMA 2. *Let $X(t)$ be a process with stable components as defined in (2) for which $\alpha_1 > d_1 = 1$, and suppose $1 \geqq \gamma > 1/\alpha_1$. If $\delta \in (1, 1 + \gamma\alpha_2 - \alpha_2/\alpha_1)$ then:*

$$E|X(t) - X(s)|^{-\delta} \leqq c_2\tau^{1/\alpha_2 - 1/\alpha_1 - \delta/\alpha_2}, \tag{11}$$

where $\tau = t - s > 0$, and c_2 is a positive constant whose value does not depend upon t, s, or τ.

PROOF. The hypotheses on α_1, α_2 and γ imply that $\delta < 2$. Under the assumption that the scaling property (4) holds for each of the components we can express the density, $p(t, x)$ of $X(t)$ by

$$p(t, x) = \prod_{i=1}^{n} p_{\alpha_i}(1, t^{-1/\alpha_i}x_i)t^{-d_i/\alpha_i}, \qquad x \in R^N,\ x_i \in R^{d_i}. \tag{12}$$

Now let $\tau = t - s$, use the change of variable $u_i = \tau^{-1/\alpha_i}x_i$, and the boundedness of $p_{\alpha_i(1,u_i)}$ to obtain:

$$\begin{aligned} &E|X(t) - X(s)|^{-\delta} \\ &= \int_{R^{d_n}} \cdots \int_{R^{d_1}} |x|^{-\delta}p(\tau, x)\, dx_1 \cdots dx_n \\ &= \int_{R^{d_n}} \cdots \int_{R^{d_1}} \frac{p_{\alpha_1}(1, u_1)p_{\alpha_2}(1, u_2) \cdots p_{\alpha_n}(1, u_n)}{|(\tau^{1/\alpha_1}u_1, \tau^{1/\alpha_2}u_2, \cdots, \tau^{1/\alpha_n}u_n)|^{\delta}}\, du_1 \cdots du_n \\ &\leqq c_3 \int_{R^{d_2}} \int_0^{|u_2|\tau^{1/\alpha_2 - 1/\alpha_1}} p_{\alpha_2}(1, u_2)\tau^{-\delta/\alpha_2}|u_2|^{-\delta}\, du_1\, du_2 \\ &\quad + c_3 \int_{R^{d_2}} \int_{|u_2|\tau^{1/\alpha_2 - 1/\alpha_1}}^{\infty} p_{\alpha_2}(1, u_2)\tau^{-\delta/\alpha_1}|u_1|^{-\delta}\, du_1\, du_2 \\ &\leqq c_4\tau^{1/\alpha_2 - 1/\alpha_1 - \delta/\alpha_2} \int_{R^{d_2}} |u_2|^{1-\delta}p_{\alpha_2}(1, u_2)\, du_2, \end{aligned} \tag{13}$$

from which the lemma follows since $1 < \delta < 2$ and c_3 and c_4 are independent of τ.

The proof now parallels that of Blumenthal and Getoor [1] (pages 371–372) and we give only the outline. Denote the exponent of τ in (11) by $-\lambda$. Then the hypotheses on δ assure us that $0 < \lambda < \gamma$, and $\Lambda^{\lambda+\eta}(E) = +\infty$ if $0 < \eta < \gamma - \lambda$. According to Davies' [3] theorem there is a closed set F contained in E such that $\Lambda^{\lambda+\eta}(F) > 0$, so that $C_\lambda(F) > 0$ by Frostman's [4] theorem. Thus there is a probability measure m, concentrated on F such that

$$\int_F \int_F |x - y|^{-\lambda}m(dx)m(dy) < \infty. \tag{14}$$

Integrate (11) over $F \times F$ with respect to $m \times m$ and use Fubini's theorem to show that

$$\int_F \int_F |X(t, \omega) - X(s, \omega)|^{-\delta}m(dt)m(ds) < \infty$$

for almost all ω. Finally, the theorem of McKean [8] guarantees that $\Lambda^\delta X(F) > 0$ with probability one. Hence $\dim X(E) \geqq \delta$, and the proof is complete since $\delta < 1 + \alpha_2\gamma - \alpha_2/\alpha_1$ was arbitrary.

4. Remarks.

(i) By using a process with $\alpha_1 > d_1 = 1$ we can find sets E, depending upon ε, such that $\dim X(E) = \varepsilon \dim E$, for any ε in $[\beta', \beta]$.

(ii) If, in (i) of our dimension theorem, a graph is made of $\dim E$ vs $\dim X(E)$ one can think of $\dim X(E)$ using up the X_1 component first on sets E with $\dim E \leqq 1/\alpha_1$ and then the X_2 component. Thus, we see more intuitively how the number $\rho = 1 + \alpha_2 - \alpha_2/\alpha_1 = \dim X([0, 1])$ arises.

(iii) Our proofs of (i) of our theorem rely heavily upon the scaling property, so that processes with components for which (4) fails are excluded from our considerations.

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