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CONVERGENCE TO ZERO OF QUADRATIC FORMS IN INDEPENDENT RANDOM VARIABLES[1]

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Let $\{X_n\}$ be a sequence of independent random variables none of which are degenerate and define for $y \geqq 0$, $F(y) = \sup_k P[|X_k| \geqq y]$ and $G(y) = \sup_{j \neq k} P[|X_j X_k| \geqq y]$. Relationships between the rate of convergence of F and G to zero are investigated.

Set $Q_N = \sum_{j,k} a_{jk,N} X_j X_k$ for $N = 1, 2, \cdots$. If the X's are symmetric then it is shown that Q_N converges to zero in probability for a large class of weights $\{a_{jk,N}\}$ if and only if $\lim_{y\to\infty} yG(y) = 0$. Convergence results are also given for the case when the random variables are not symmetric.

1. Introduction and summary. Let $X_1, X_2, \cdots$ be a sequence of independent random variables, none of which are degenerate, and define for $y \geqq 0$

$$F(y) = \sup_k P[|X_k| \geqq y] \qquad \text{and} \qquad G(y) = \sup_{j \neq k} P[|X_j X_k| \geqq y] .$$

If the random variables are identically distributed then the functions F and G measure the probabilities in the tails of X_1 and $X_1 X_2$, respectively. We will investigate the relationship between F and G.

Let $s > 0$. In Section 2 it is shown that

$$(1) \qquad y^s F(y) \to 0 \qquad \text{as} \quad y \to \infty$$

is implied by

$$(2) \qquad y^s G(y) \to 0 \qquad \text{as} \quad y \to \infty ,$$

but not conversely, even in the identically distributed case. However, it is shown that $y^s(\log y)^{\frac{1}{2}} F(y) \to 0$ as $y \to \infty$ does imply (2) and the tightness of this result is discussed. Comparisons are also made when F converges "exponentially" to zero. In this case with $p > 0$, we show that if $F(y) \leqq M \int_y^\infty \exp\{-Cx^p\}\, dx$ for all $y > 0$, then $G(y) \leqq L \int_y^\infty \exp\{-Dx^{p/2}\}\, dx$ for all $y > 0$. The converse holds in the identically distributed case.

Jamison, Orey and Pruitt (1965) investigated, in the identically distributed case, the relationship between (1) with $s = 1$ and convergence in probability of

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weighted averages of the X's. We now restate their Theorem 1. For $k = 1, 2, \cdots$ let w_k be positive constants and for $N = 1, 2, \cdots$ set $W_N = \sum_{k=1}^N w_k$ and $S_N = \sum_{k=1}^N w_k X_k$. They assume

$$\max_{1 \leq k \leq N} w_k / W_N \to 0 \qquad \text{as } N \to \infty . \tag{3}$$

Their Theorem 1 can, with the technique of symmetrization, be interpreted as follows.

THEOREM 1.1. *For each sequence of weights satisfying* (3) *there exsist constants* μ_N *such that* $S_N/W_N - \mu_N \to_P 0$ *if and only if* $yF(y) \to 0$ *as* $y \to \infty$, *and in this case* μ_N *can be chosen to be* $\sum_{k=1}^N (w_k/W_N) E(X_k I_{[|X_k| < W_N/w_k]})$. *Furthermore, if* $\lim_{T \to \infty} E(X_1 I_{[|X_1| < T]}) = \mu$, *then* $\mu_N \to \mu$.

In Section 3 we consider the analogous problem of relating (2) with $s = 1$ to convergence in probability of quadratic forms in the X's. Let $a_{jk,N}$ be real numbers for $N, j, k = 1, 2, \cdots$ and let $Q_N = \sum_{j,k} a_{jk,N} X_j X_k$. If $a_{jk,N} = 0$ for all $j \neq k$, $a_{jj,N} = w_j/W_N$ for $j = 1, 2, \cdots, N$ and $a_{jj,N} = 0$ for $j > N$, then Q_N is a weighted average of $X_1^2, \cdots, X_N^2$. If the X's are identically distributed, the results of Jamison, Orey and Pruitt show that if there exist constants μ_N such that $Q_N - \mu_N \to_P 0$ then $yP[X_1^2 \geqq y] \to 0$ as $y \to \infty$ which is more restrictive than (2) with $s = 1$. So we will first consider quadratic forms for which $a_{jj,N} = 0$ for all $N, j = 1, 2, \cdots$ and then we will show how the diagonal terms can be handled in some cases. If the random variables are identically distributed and symmetric and the diagonal elements are zero, then $yG(y) \to 0$ as $y \to \infty$ is equivalent to $Q_N \to_P 0$. Convergence results for Q_N are also given in the cases when the random variables are not necessarily identically distributed or symmetric and when the diagonal elements are not necessarily zero.

2. Rates of convergence of F and G to zero. With the same notation as in Section 1, we will investigate the rates at which F and G converge to zero. In this section, we consider two types of convergence rates. The first might be called algebraic rates. Let $s \geqq 0$. We begin with the following remark.

REMARK 2.1. If $\lim_{y \to \infty} y^s G(y) = 0$, then $\lim_{y \to \infty} y^s F(y) = 0$.

PROOF. Since the random variables under consideration are not degenerate, we choose $\varepsilon > 0$ such that $\delta = \min(P[|X_1| \geqq \varepsilon], P[|X_2| \geqq \varepsilon]) > 0$. For $y > 0$, $G(y) \geqq \sup_{k \neq 1} P[|X_1| \geqq \varepsilon] P[|X_k| \geqq y/\varepsilon]$, $G(y) \geqq P[|X_1| \geqq y/\varepsilon] P[|X_2| \geqq \varepsilon]$ and hence $G(y) \geqq \delta F(y/\varepsilon)$. The desired result follows.

It is easy to construct examples for which neither F nor G converge to zero as $y \to \infty$. However, the following remark shows that $\lim_{y \to \infty} F(y) = 0$ is equivalent to $\lim_{y \to \infty} G(y) = 0$.

REMARK 2.2. $\lim_{y \to \infty} F(y) = 0$ if and only if $\lim_{y \to \infty} G(y) = 0$.

PROOF. Clearly for $y > 0$ and positive integers j and k, $P[|X_j X_k| \geqq y^2] \leqq P[|X_j| \geqq y] + P[|X_k| \geqq y]$ and so $G(y^2) \leqq 2F(y)$. This inequality and Remark 2.1 with $s = 0$ complete the proof.

Unless specified otherwise assume $s > 0$.

THEOREM 2.3. *If* $\lim_{y\to\infty} y^s(\log y)^{\frac{1}{2}}F(y) = 0$, *then* $\lim_{y\to\infty} y^s G(y) = 0$.

PROOF. We first observe that for $2^n < y \leqq 2^{n+1}$, $y^s G(y) \leqq 2^{(n+1)s}G(2^n)$ and so it suffices to show $\lim_{n\to\infty} 2^{ns}G(2^n) = 0$. For positive integers j and k with $j \neq k$,

$$\begin{aligned} P[|X_j X_k| \geqq 2^{n+1}] &\leqq P[|X_j| < 1, |X_k| \geqq 2^{n+1}] \\ &\quad + \textstyle\sum_{\nu=0}^{n-1} P[2^\nu \leqq |X_j| < 2^{\nu+1}, |X_k| \geqq 2^{n-\nu}] \\ &\quad + P[|X_j| \geqq 2^n] \\ &\leqq 3F(2^n) + \textstyle\sum_{\nu=1}^{n-1} F(2^\nu)F(2^{n-\nu}) . \end{aligned}$$

If $b_n = 2^{ns}(\log 2^n)^{\frac{1}{2}}F(2^n)$ and $a_n = \sup_{k\geqq n} b_k$, then by hypothesis $\lim_{n\to\infty} a_n = 0$. So

$$2^{(n+1)s}G(2^{n+1}) \leqq o(1) + 2^2(\log 2)^{-1} \sum_{\nu=1}^{n-1} \frac{a_\nu a_{n-\nu}}{(\nu(n-\nu))^{\frac{1}{2}}} ,$$

and since a_n is a non-increasing sequence the above is bounded by

$$o(1) + 2^s(\log 2)^{-1}a_1 a_{[n/2]} \textstyle\sum_{\nu=1}^{n-1} (\nu(n-\nu))^{-\frac{1}{2}} .$$

We conclude the proof by showing $\sum_{\nu=1}^{n-1} (\nu(n-\nu)^{-\frac{1}{2}}$ is bounded. Since for each $n = 1, 2, \cdots, f_n(x) = (x(n-x))^{-\frac{1}{2}}$ is non-increasing for $0 < x \leqq [n/2]$,

$$\begin{aligned} \textstyle\sum_{\nu=1}^{n-1} (\nu(n-\nu))^{-\frac{1}{2}} &\leqq 2 \textstyle\sum_{\nu=1}^{[n/2]} (\nu(n-\nu))^{-\frac{1}{2}} \\ &\leqq 2 \textstyle\int_0^{[n/2]} (x(n-x))^{-\frac{1}{2}}\, dx \\ &\leqq 2 \textstyle\int_0^{n/2} (nx)^{-\frac{1}{2}}(1-x/n)^{-\frac{1}{2}}\, dx \\ &= 2 \textstyle\int_0^{n/2} (nx)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k}(-1)^k(x/n)^k\, dx . \end{aligned}$$

Since the terms of this series are nonnegative we may rewrite the above as

$$\begin{aligned} 2 \textstyle\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k}(-1)^k n^{-(k+\frac{1}{2})} \int_0^{n/2} x^{k-\frac{1}{2}}\, dx &= 2 \textstyle\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k}(-1)^k 2^{-(k+\frac{1}{2})}(k+\frac{1}{2})^{-1} \\ &\leqq 2^{\frac{3}{2}}(1-\tfrac{1}{2})^{\frac{1}{2}} = 4 . \end{aligned}$$

We now give an example to show that the converse of Theorem 2.3 is not valid.

EXAMPLE. Let $X_1, X_2, \cdots$ be independent and identically distributed nonnegative random variables with $F(y) = 1$ for $0 \leqq y \leqq 1$ and $F(y) = C2^{-sk^3}k^{-\frac{3}{2}}$ for $2^{(k-1)^3} < y \leqq 2^{k^3}$ and $k = 1, 2, \cdots$ where $0 < C \leqq 2^s$. We first show that $\lim_{y\to\infty} y^s F(y) = 0$. If $2^{(k-1)^3} < y \leqq 2^{k^3}$, then $y^s F(y) \leqq 2^{sk^3}F(2^{k^3}) \to 0$ as $k \to \infty$. However, $2^{sk^3}(\log 2^{k^3})^{\frac{1}{2}}F(2^{k^3}) = C(\log 2)^{\frac{1}{2}}$ and so $y^s(\log y)^{\frac{1}{2}}F(y) \nrightarrow 0$ as $y \to \infty$. The example is completed by showing $\lim_{y\to\infty} y^s G(y) = 0$ or equivalently $\lim_{n\to\infty} 2^{ns}G(2^n) = 0$. We observe that

$$\begin{aligned} 2^{ns}P[|X_1X_2| \geqq 2^n] &\leqq 2^{ns}P[X_1 = 1, X_2 \geqq 2^n] + 2^{ns} \textstyle\sum_{\nu=1}^{n-1} P[X_1 = 2^\nu, X_2 \geqq 2^{n-\nu}] \\ &\quad + 2^{ns}P[X_1 \geqq 2^n] \\ &= o(1) + 2^{ns} \textstyle\sum_{\nu=1}^{n-1} P[X_1 = 2^\nu]F(2^{n-\nu}) \\ &\leqq o(1) + 2^{ns} \textstyle\sum_{\nu=1}^{[(n-1)^{\frac{1}{3}}]} F(2^{\nu^3})F(2^{n-\nu^3}) . \end{aligned}$$

For n fixed, let j_ν be such that $(j_\nu - 1)^3 < n - \nu^3 \leqq j_\nu^{\,3}$ and then we can rewrite the last term of the above as

$$\begin{aligned} 2^{ns} \textstyle\sum_{\nu=1}^{[(n-1)^{\frac{1}{3}}]} F(2^{\nu^3})F(2^{j_\nu^3}) &= 2^{ns}C^2 \textstyle\sum_{\nu=1}^{[(n-1)^{\frac{1}{3}}]} 2^{-s\nu^3}\nu^{-\frac{3}{2}}2^{-sj_\nu^3}j_\nu^{-\frac{3}{2}} \\ &\leqq C^2 \textstyle\sum_{\nu=1}^{[(n-1)^{\frac{1}{3}}]} (\nu^3(n-\nu^3))^{-\frac{1}{2}} . \end{aligned}$$

The summand in the last sum is maximized when $\nu = 1$ and so $2^{ns}G(2^n) \leqq C^2(n-1)^{-\frac{1}{6}} \to 0$ as $n \to \infty$.

While the converse of Theorem 2.3 is not true, the next result shows that in a certain sense it is tight.

THEOREM 2.4. *Let* $g\colon (0, \infty) \to (-\infty, \infty)$ *such that* $\lim_{y\to\infty} g(y) = \infty$. *There exist independent and identically distributed random variables* $X_1, X_2, \cdots$ *for which* $\lim_{y\to\infty} y^s(\log y)^{\frac{1}{2}}(g(y))^{-1}F(y) = 0$ *and* $y^sG(y) \nrightarrow 0$ *as* $y \to \infty$.

PROOF. Let $\{a_n\}$ be a non-increasing sequence of real numbers with $a_1 = 1$ and $\lim_{n\to\infty} a_n = 0$, and set $p_n = (2^sa_n - a_{n+1})2^{-ns}$ for $n = 1, 2, \cdots$. Clearly, $p_n \geqq 0$ and $\sum_{n=1}^\infty p_n = 1$. So there exist independent and identically distributed discrete random variables $X_1, X_2, \cdots$ such that $P[X_1 = 2^n] = p_n$ for $n = 1, 2, \cdots$. We want to choose the sequence $\{a_n\}$ so that $\lim_{y\to\infty} y^s(\log y)^{\frac{1}{2}}(g(y))^{-1}F(y) = 0$ and so it would suffice if $y^s(\log y)^{\frac{1}{2}}F(y)$ were bounded or equivalently $2^{ns}(\log 2^n)^{\frac{1}{2}}F(2^n)$ were bounded. However, $F(2^n) = \sum_{k=n}^\infty p_k = 2^{-(n-1)s}a_n$ and so we will choose $\{a_n\}$ so that $n^{\frac{1}{2}}a_n$ is bounded. We also wish to choose $\{a_n\}$ so that $y^sG(y) \nrightarrow 0$ as $y \to \infty$. For $n \geqq 1$

$$\begin{aligned} 2^{ns}G(2^n) &= 2^{ns}P[|X_1X_2| \geqq 2^n] \geqq 2^{ns} \textstyle\sum_{\nu=1}^{n-1} P[X_1 = 2^\nu]P[X_2 \geqq 2^{n-\nu}] \\ &= 2^s \textstyle\sum_{\nu=1}^{n-1} (2^sa_\nu - a_{\nu+1})a_{n-\nu} \geqq 2^s(2^s - 1) \textstyle\sum_{\nu=1}^{n-1} a_\nu a_{n-\nu} \\ &\leqq 2^s(2^s-1)(n-1)a_n^{\,2} . \end{aligned}$$

The proof is completed by observing that the choice $a_n = n^{-\frac{1}{2}}$ satisfies all of the above requirements.

The second type of convergence rate we have investigated might be called exponential. Let M, C and p be positive real numbers. We will be concerned with statements like

$$F(y) \leqq M \textstyle\int_y^\infty \exp\{-Cx^p\}\,dx \qquad \text{for all } y > 0 . \tag{4}$$

With $1 \leqq p \leqq 2$, Hanson (1967) studied the relationship between condition (4) and bounds on the moment generating functions of the sequence $\{X_n\}$.

THEOREM 2.5. *If there exist M and C such that* (4) *holds, then there exist L and D such that*

$$G(y) \leqq L \textstyle\int_y^\infty \exp\{-Dx^{p/2}\}\,dx \qquad \textit{for all } y > 0 . \tag{5}$$

Furthermore if there exist positive integers $j_1 \neq j_2$ and a positive constant γ such that $P[|X_{j_i}| \geqq y] \geqq \gamma F(y)$ for $i = 1, 2$ and all $y > 0$, then the converse is valid.

Before we give the proof of the theorem, we state the following lemma.

LEMMA 2.6. *If $p > 0$, then*

$$\int_y^\infty \exp\{-Cx^p\}\,dx \sim (Cp)^{-1}y^{1-p}\exp\{-Cy^p\}\,. \tag{6}$$

The lemma may be proved by applying L'Hospital's rule to the ratio of the two expressions in (6).

PROOF OF THEOREM 2.5. For $y \geqq 1$ and positive integers j and k

$$\begin{aligned} P[|X_jX_k| \geqq y] &\leqq P[|X_j| \geqq y^{\frac{1}{2}}] + P[|X_k| \geqq y^{\frac{1}{2}}] \\ &\leqq 2M\int_{y^{\frac{1}{2}}}^\infty \exp\{-Cx^p\}\,dx = M\int_y^\infty x^{-\frac{1}{2}}\exp\{-Cx^{p/2}\}\,dx \\ &\leqq M\int_y^\infty \exp\{-Cx^{p/2}\}\,dx\,. \end{aligned}$$

Condition (5) holds with $D = C$ and $L = \max(M, (\int_1^\infty \exp\{-Cx^{p/2}\}\,dx)^{-1})$, since $L\int_y^\infty \exp\{-Cx^{p/2}\}\,dx \geqq 1$ for $0 < y < 1$.

We begin the proof of the second conclusion in the theorem by noting that for $y > 0$

$$\begin{aligned} \gamma^2F^2(y) &\leqq \prod_{i=1}^2 P[|X_{j_i}| \geqq y] \leqq P[|X_{j_1}X_{j_2}| \geqq y^2] \\ &\leqq L\int_{y^2}^\infty \exp\{-Dx^{p/2}\}\,dx\,. \end{aligned}$$

Using the lemma, we see that $\int_{y^2}^\infty \exp\{-Dx^{p/2}\}\,dx \sim 2(Dp)^{-1}y^{2-p}e^{-Dy^p}$ and since both of these expressions are continuous, there exists a constant $k_1 > 0$ such that

$$\int_{y^2}^\infty \exp\{-Dx^{p/2}\}\,dx \leqq k_1y^{2-p}\exp\{-Dy^p\} \qquad \text{for } y \geqq 1\,. \tag{7}$$

Since $y^p\exp\{-(D/2)y^p\}$ is bounded there exists $k_2 > 0$ such that (7) is bounded by

$$16k_2(Dp)^{-2}y^{2(1-p)}\exp\{-(D/2)y^p\} \sim k_2(\int_y^\infty \exp\{-(D/4)x^p\}\,dx)^2\,.$$

Hence we can choose $k_3 > 0$ such that

$$F(y) \leqq k_3\int_y^\infty \exp\{-(D/4)x^p\}\,dx \qquad \text{for } y \geqq 1$$

and the proof is completed by choosing $C = D/4$ and

$$M = \max(k_3, (\int_1^\infty \exp\{-Cy^p\}\,dx)^{-1})\,.$$

It is easy to construct examples to show that the converse of Theorem 2.5 is not true in general. One obtains such an example by choosing M and $X_1, X_2, \cdots$ such that $P[|X_1| \geqq y] = M\int_y^\infty \exp\{-x^{p/2}\}\,dx$ for all $y > 0$ and $|X_k| \equiv 1$ for $k = 2, 3, \cdots$. However, if $X_1, X_2, \cdots$ are identially distributed, the condition imposed in the second part of Theorem 2.5 holds and so the converse is valid.

3. Convergence of quadratic forms. We now investigate relationship between $yG(y) \to 0$ as $y \to \infty$ and convergence in probability, to a degenerate limit, of quadratic forms in $X_1, X_2, \cdots$. Let $a_{jk,N}$ be real numbers for $N, j, k = 1, 2, \cdots$ with

$$a_{jj,N} = 0 \qquad \text{for all } N, j = 1, 2, \cdots \tag{8}$$

and

$$\sum_{j,k=1}^\infty |a_{jk,N}| \leqq M \qquad \text{for all } N = 1, 2, \cdots\,. \tag{9}$$

We will relax assumption (8) in Theorems 3.4 and 3.5. Without loss of generality we may assume $a_{jk,N} = a_{kj,N}$ for all N, j and k. Set $Q_{N,m} = \sum_{j,k=1}^{m} a_{jk,N} X_j X_k$.

THEOREM 3.1. *Let $X_1, X_2, \cdots$ be identically distributed. If there exist constants $\mu_{N,m}$ such that for each N, $Q_{N,m} - \mu_{N,m}$ converges in distribution to, say, Q_N and $Q_N \to_P 0$, then* $\sup_{j,k} |a_{jk,N}| \to 0$ *as* $N \to \infty$.

PROOF. Let $X_1, X_1', X_2, X_2', \cdots$ be independent identically distributed with characteristic function $\phi(t)$. For each N we may choose integers $j_N \neq k_N$ and m_N such that $j_N, k_N \leqq m_N$, $|a_{j_N k_N,N}| \geqq \sup_{j,k} |a_{jk,N}|/2$, and $Q_{N,m_N} - \mu_{N,m_N} \to_P 0$. So $|E(\exp\{itQ_{N,m_N}\})| \to 1$ as $N \to \infty$. We let $\mathscr{B} = \mathscr{B}\{X_k : k \neq k_N\}$ and $E^{\beta}(Y)$ denote the conditional expectation of Y with respect to the σ-field β. Then

$$\begin{aligned} |E(\exp\{itQ_{N,m_N}\})| &= |E(\exp\{it \textstyle\sum_{j,k=1; j\neq k_N \neq k}^{m_N} a_{jk,N} X_j X_k\} \\ &\qquad \times E^{\mathscr{B}}(\exp\{2it \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j X_{k_N}\}))| \\ &\leqq E|\phi(2t \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j)| \\ &\leqq (E|\phi(2t \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j)|^2)^{\frac{1}{2}} \leqq 1 . \end{aligned}$$

Hence $E|\phi(2t \sum_{j=1}^{m_N} a_{k_N j,N} X_j)|^2 \to 1$ for all t as $N \to \infty$. Now

$$|\phi(2t \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} x_j)|^2 = E(\exp\{2it(X_{k_N} - X'_{k_N}) \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} x_j\})$$

or

$$|\phi(2t \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j)|^2 = E^{\mathscr{B}}(\exp\{2it(X_{k_N} - X'_{k_N}) \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j\})$$

almost surely and hence

$$E|\phi(2t \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j)|^2 = E(\exp\{2it(X_{k_N} - X'_{k_N}) \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j\}) .$$

So $(X_{k_N} - X'_{k_N}) \sum_{j=1}^{m_N} a_{k_N j,N} X_j \to_P 0$ and using (8) and the fact that the X's all have the same nondegenerate distribution,

$$\textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j \to_P 0 \qquad \text{or} \qquad |E(\exp\{it \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j\})| \to 1 .$$

However,

$$|E(\exp\{it \textstyle\sum_{j=1}^{m_N} a_{k_N j,N} X_j\})| \leqq |\phi(t a_{k_N j_N,N})| \leqq 1$$

and so $|a_{k_N j_N,N}| \to 0$ as $N \to \infty$.

NOTE. We cannot omit the assumption $a_{jj,N} = 0$ in Theorem 3.1. For all N let $a_{jj,N} = (-1)^j$ if $j = 1, 2$ and $a_{jj,N} = 0$ if $j > 2$, let $|X_k| \equiv 1$ for all k, and for $j \neq k$ let $a_{jk,N}$ be such that $\sum_{j\neq k} a_{jk,N} X_j X_k \to_P 0$ as $N \to \infty$. (See Theorems 3.2 and 3.3 for nontrivial examples.) Clearly $\sum_{j,k=1}^{\infty} a_{jk,N} X_j X_k$ converges to zero in probability but $\sup_{j,k} |a_{jk,N}| \geqq 1$ for all N.

So we assume

$$a_N = \sup_{j,k} |a_{jk,N}| \to 0 \qquad \text{as } N \to \infty \tag{10}$$

THEOREM 3.2. *Let $X_1, X_2, \cdots$ be identically distributed. If for every sequence of weights satisfying conditions* (8), (9) *and* (10) *there exist constants $\mu_{N,m}$ such that $Q_{N,m} - \mu_{N,m}$ converges in distribution to, say, Q_N and $Q_N \to_P 0$, then $yG(y) \to 0$ as $y \to \infty$. Furthermore, if $X_1, X_2, \cdots$ are symmetric the converse holds with $\mu_{N,m} \equiv 0$*

and in fact $Q_{N,m} \rightarrow_{\text{a.s.}} Q_N$ *for each* N *and the random variables need not be identically distributed.*

PROOF. We first show that $yG(y) \to 0$ as $y \to \infty$ is a necessary condition. Let $w_k > 0$ for $k = 1, 2, \cdots$ $W_N = \sum_{k=1}^N w_k$ and suppose (3) holds. For each N set $a_{2k-1,2k,N} = a_{2k,2k-1,N} = w_k/(2W_N)$ for $k = 1, 2, \cdots, N$ and set the other weights equal to zero. Now this sequence of weights satisfy (8), (9) and (10). So by hypothesis there exist constants $\mu_{N,m}$ such that $Q_{N,m} - \mu_{N,m}$ converge in distribution. However, $Q_{N,m}$ converges to $W_N^{-1} \sum_{k=1}^N w_k X_{2k-1} X_{2k}$ and so $\mu_{N,m}$ converges to, say, μ_N. Also by hypothesis $W_N^{-1} \sum_{k=1}^N w_k Z_k - \mu_N \to_P 0$ where $Z_k = X_{2k-1}X_{2k}$. By Theorem 1.1 $yG(y) \to 0$ as $y \to \infty$.

For the second part of the proof, we will first show that for each $\varepsilon > 0$ there exists a sequence $c_N \to 0$ as $N \to \infty$ for which $P[\sup_{1 \le m \le L} |Q_{N,m}| > \varepsilon] \leqq c_N$. Furthermore the sequence c_N will not depend on L. For each pair (j, k) for which $a_{jk,N} \neq 0$ let $I_{j,k}$ be the indicator function of the set $[|X_j X_k| < |a_{jk,N}|^{-1}]$. Clearly,

$$(11) \qquad P[\sup_{1 \le m \le L} |Q_{N,m}| > \varepsilon] \leqq \textstyle\sum'_{1 \le j,k < \infty} G(|a_{jk,N}|^{-1})$$

$$(12) \qquad + P[\sup_{1 \le m \le L} |\textstyle\sum'_{1 \le j,k \le m} a_{jk,N} X_j X_k I_{jk}| > \varepsilon]$$

where the prime on these summations indicates they are taken over those pairs (j, k) for which $a_{jk,N} \neq 0$. We first consider expression (12). We wish to show that $\hat{Q}_{N,m} = \sum'_{1 \le j,k \le m} a_{jk,N} X_j X_k I_{jk}$ is a martingale sequence. Let $\mathscr{A}_{m-1} = \mathscr{B}(\hat{Q}_{N,1}, \cdots, \hat{Q}_{N,m-1})$ and $\mathscr{B}_{m-1} = \mathscr{B}(X_1, \cdots, X_{m-1})$. Now $E^{\mathscr{B}_{m-1}}(\hat{Q}_{N,m}) = \hat{Q}_{N,m-1} + 2 \sum'_{1 \le j \le m-1} a_{jm,N} E^{\mathscr{B}_{m-1}}(X_j X_m I_{jm})$ almost surely and if $a_{jm,N} \neq 0$, $E^{\mathscr{B}_{m-1}}(X_j X_m I_{jm}) = E(x_j X_m I_{[|x_j X_m| < a_{jm,N}|^{-1}]})|_{x_j = X_j}$ almost surely. But this last term is zero since X_m is symmetric. Since $\mathscr{A}_{m-1} \subset \mathscr{B}_{m-1}$ and $\hat{Q}_{N,m-1}$ is measurable with respect to $\mathscr{A}_{m-1}$ we can conclude $\hat{Q}_{N,m}$ is a martingale sequence. (Vareberg (1966) showed that these quadratic forms are martingales in a slightly different setting.) By the martingale inequality, expression (12) is bounded by $\varepsilon^{-2} E(\sum'_{1 \le j,k \le L} a_{jk,N} X_j X_k I_{jk})^2$. However, if in the term $E(\prod_{i=1}^2 a_{j_i k_i,N} X_{j_i} X_{k_i} I_{j_i k_i})$ any index, say, j_1 appears only once this term is zero. For let $\mathscr{B} = \mathscr{B}(X_{k_1}, X_{j_2}, X_{k_2})$ and then we can write

$$E(\textstyle\prod_{i=1}^2 a_{j_i k_i,N} X_{j_i} X_{k_i} I_{j_i k_i}) = E(E^{\mathscr{B}}(\prod_{i=1}^2 a_{j_i k_i,N} X_{j_i} X_{k_i} I_{j_i k_i}))$$

and then apply techniques similar to those used to show $\hat{Q}_{N,m}$ is a martingale sequence. Hence, expression (12) is bounded by

$$\begin{aligned} &2\varepsilon^{-2} \textstyle\sum'_{1 \le j,k \le L} a_{jk,N}^2 E(X_j^2 X_k^2 I_{jk}) \\ &\qquad = 2\varepsilon^{-2} \textstyle\sum'_{1 \le j,k \le L} a_{jk,N}^2 \int_{[0,|a_{jk,N}|^{-1})} x^2 \, dP[|X_j X_k| < x] \\ &\qquad = -2\varepsilon^{-2} \textstyle\sum'_{1 \le j,k \le L} a_{jk,N}^2 \int_0^{|a_{jk,N}|^{-1}} x^2 \, dP[|X_j X_k| \geqq x] \\ &\qquad \leqq 4\varepsilon^{-2} \textstyle\sum'_{1 \le j,k \le L} a_{jk,N}^2 \int_0^{|a_{jk,N}|^{-1}} xG(x) \, dx \, . \end{aligned}$$

If we let $H(x) = \sup_{y \ge x} yG(y)$, the above is bounded by

$$4\varepsilon^{-2} \textstyle\sum'_{1 \le j,k \le L} |a_{jk,N}| \int_0^1 H(x/|a_{jk,N}|) \, dx \leqq 4M\varepsilon^{-2} \int_0^1 H(x/a_N) \, dx \, .$$

Since H is non-increasing, expression (11) is bounded by $M \cdot H(1/a_N) \leqq M \int_0^1 H(x/a_N)\,dx$. We now choose $c_N = M(4\varepsilon^{-2} + 1) \int_0^1 H(x/a_N)\,dx$. Clearly H is bounded and for $x \in (0, 1]$, $H(x/a_N) \to 0$ as $N \to \infty$. So the Dominated Convergence Theorem shows that $c_N \to 0$ as $N \to \infty$.

With N fixed, we will show that $Q_{N,m}$ converges almost surely as $m \to \infty$ by showing $\lim_{m\to\infty} \lim_{L\to\infty} P[\sup_{1\leqq\alpha\leqq L} |Q_{N,m+\alpha} - Q_{N,m}| > \varepsilon] = 0$. Since $A_{N,m} = \sup\{|a_{jk,N}| : j > m \text{ or } k > m\} \to 0$ as $m \to \infty$, we see from the first part of the proof that

$$\lim_{L\to\infty} P[\sup_{1\leqq\alpha\leqq L} |Q_{N,m+\alpha} - Q_{N,m}| > \varepsilon] \leqq M(4\varepsilon^{-2} + 1) \int_0^1 H(x/A_{N,m})\,dx\,.$$

Using the Dominated Convergence Theorem, $\int_0^1 H(x/A_{N,m})\,dx \to 0$ as $m \to \infty$ and so $Q_{N,m}$ converges almost surely.

The proof is completed by observing that $Q_{N,m} \to_{\text{a.s.}} Q_N$ and $P[|Q_{N,m}| > \varepsilon] \leqq c_N$ for all m imply $P[|Q_N| > \varepsilon] \leqq c_N$. (See Theorem 2.1 of Billingsley (1968).)

It would be of interest to know if the converse of the first conclusion of Theorem 3.2 holds when the random variables are not assumed to be symmetric. In the nonsymmetric case, the following theorem gives conditions which ensure $Q_N \to_P 0$; however, they are more restrictive than those in Theorem 3.2. Let $b_{j,N} = \sum_{k=1}^{\infty} |a_{jk,N}|$. In the next theorem we will assume

$$b_N = \sup_j b_{j,N} \to 0 \qquad \text{as } N \to \infty\,. \tag{13}$$

THEOREM 3.3. *If* $G(y) \to 0$ *as* $y \to \infty$ *and* $-\int_0^\infty y\,dG(y) < \infty$ *and if the weights satisfy* (8), (9) *and* (13), *then there exist constants* $\mu_{N,m}$ *such that* $Q_{N,m} - \mu_{N,m}$ *converges almost surely for each* N *to, say,* Q_N *and* $Q_N \to_P 0$. *In fact we may choose* $\mu_{N,m} = \sum_{j,k=1}^{m} a_{jk,N} E(X_j)E(X_k)$.

NOTE. We state without proof the following result. If F and G are non-negative, non-increasing functions with $F(y) \leqq G(y)$ and $G(y) \to 0$ as $y \to \infty$ then $-\int_a^\infty y\,dF(y) \leqq -\int_a^\infty y\,dG(y)$ for $a \geqq 0$. Using this result and the result established in Section 2 that there exist positive constants ε and δ such that $F(y) \leqq \delta^{-1}G(\varepsilon \cdot y)$ we see that the hypotheses of Theorem 3.3 imply that $-\int_0^\infty y\,dF(y) < \infty$. Furthermore, $-\int_0^\infty y\,dG(y) < \infty$ and $G(y) \to 0$ as $y \to \infty$ imply that $yG(y) \to 0$ and then $yF(y) \to 0$ as $y \to \infty$. Using the result stated above $-\int_0^\infty y\,dF(y) < \infty$ implies that there exists a D such that $E|X_k| \leqq D$ for all k.

In the identically distributed case the moment assumption of Theorem 3.3 is equivalent to $|E(X_1)| < \infty$.

PROOF. We consider first the case $E(X_k) \equiv 0$. Now if $Y_j = 2\sum_{k=1}^{j-1} a_{jk,N} X_j X_k$ for $j = 2, 3, \cdots$, then $Q_{N,m} - \mu_{N,m} = \sum_{j=2}^{m} Y_j$. So $E\sum_{j=2}^{\infty} |Y_j| = \sum_{j=2}^{\infty} E|Y_j| \leqq 2MD^2$ which implies that $\sum_{j=2}^{\infty} |Y_j|$ is finite almost surely and so $Q_{N,m} - \mu_{N,m}$ converges almost surely.

Let I_{jk} be the indicator function of $[|a_{jk,N} X_j X_k| < 1]$ and I_j be the indicator function of $[|b_{j,N} X_j| < 1]$. Set $Z_{jk} = X_j X_k I_{jk} I_j I_k$ and $\hat{Q}_{N,m} = \sum_{j,k=1}^{m} a_{jk,N} Z_{jk}$.

Clearly for $\varepsilon > 0$, $P[|Q_{N,m}| > \varepsilon]$ is bounded by

$$\sum'_{1\leq j,k\leq m} G(|a_{jk,N}|^{-1}) + \sum'_{1\leq j\leq m} F(b_{j,N}^{-1}) \tag{14}$$

$$+ P[|\hat{Q}_{N,m}| > \varepsilon]\,. \tag{15}$$

Letting $J_{N,m} = \{(j_1, \cdots, j_4)\colon 1 \leqq j_i \leqq m, j_i \neq j_k \text{ for } i \neq k, a_{j_1j_2,N} \neq 0, \text{ and } a_{j_3j_4,N} \neq 0\}$, we see that expression (15) is bounded by

$$\varepsilon^{-2} \sum_{J_{N,m}} a_{j_1j_2,N}a_{j_3j_4,N} E(Z_{j_1j_2})E(Z_{j_3j_4}) \tag{16}$$

$$+ 4\varepsilon^{-2} \sum'_{1\leq j,k,l\leq m; k\neq l} a_{jk,N}a_{jl,N} E(Z_{jk}Z_{jl}) \tag{17}$$

$$+ 2\varepsilon^{-2} \sum'_{1\leq j,k\leq m} a^2_{jk,N} E(Z^2_{jk})\,. \tag{18}$$

As in the proof of Theorem 3.2, the sum of the first expression in (14) and (18) is bounded by c_N.

Expression (16) is bounded by $\varepsilon^{-2}(\sum_{j,k}|a_{jk,N}||EZ_{jk}|)^2$. However, since $E(X_jX_k) = 0$ for $j \neq k$,

$$\begin{aligned} |E(Z_{jk})| &\leqq E|X_jX_k|(1 - I_{jk}) + D(E|X_j|(1 - I_j) + E|X_k|(1 - I_k)) \\ &\leqq -\textstyle\int_{a_N^{-1}}^{\infty} y\,dG(y) - 2D \int_{b_N^{-1}}^{\infty} y\,dF(y)\,, \end{aligned} \tag{19}$$

and if we set d_N equal to the square of the product of M, ε^{-1} and expression (19), then expression (16) is bounded by d_N and $d_N \to 0$ as $N \to \infty$.

Expression (17) is bounded by $4D^2\varepsilon^{-2} \sum_{j,k} |a_{jk,N}|b_{j,N}EX_j^{\,2}I_j$ and $EX_j^{\,2}I_j$ can be written

$$\begin{aligned} -\textstyle\int_0^{b_{j,N}^{-1}} y^2\,dP[|X_j| \geqq y] &\leqq 2\textstyle\int_0^{b_{j,N}^{-1}} yF(y)\,dy \\ &\leqq 2b_{j,N}^{-1} \textstyle\int_0^1 J(y/b_{j,N})\,dy \end{aligned}$$

where $J(x) = \sup_{y\geq x} yF(y)$. Also the second expression in (14) is bounded by $\sum'_{1\leq j\leq m} b_{j,N} \int_0^1 J(y/b_{j,N})\,dy \leqq M \int_0^1 J(y/b_N)\,dy$ and so if we set $e_N = M(1 + 8D^2\varepsilon^{-2}) \int_0^1 J(y/b_N)\,dy$, then the sum of the second term in (14) and (17) is bounded by e_N and $e_N \to 0$ as $N \to \infty$.

The proof of this case is completed by observing that c_N, d_N and e_N do not depend on m.

We consider the case when the means are not necessarily zero. First rewrite

$$\begin{aligned} Q_{N,m} - \mu_{N,m} &= \textstyle\sum_{j,k=1}^m a_{jk,N}(X_j - E(X_j))(X_k - E(X_k)) \\ &\quad + 2\textstyle\sum_{j,k=1}^m a_{jk,N}(X_j - E(X_j))E(X_k)\,. \end{aligned} \tag{20}$$

Using the results of Section 2 and those in the above note, it can be shown that $\sup_{j\neq k} P[|(X_j - E(X_j))(X_k - E(X_k))| \geqq y]$ also satisfies the hypotheses of Theorem 3.3. In the preceding case the first term in (20) was shown to have an almost sure limit for each N and the limits were shown to converge to zero in probability. An argument similar to that at the beginning of the proof of this theorem shows that the second term in (20) converges absolutely almost surely for each N. We will denote this limit by S_N. We now appeal to a modified version of Theorem 2b of Hanson and Wright (1971a). To avoid confusion with notation we will write their $S_N = \sum_j a_{N,j}Z_j$ and so to apply their theorem to our S_N we set $Z_j = X_j - E(X_j)$ and $a_{N,j} = \sum_{k=1}^{\infty} a_{jk,N} E(X_k)$. Assumption (2.5) and all the hypotheses of that theorem hold in this case except ρ_N might not converge to zero. We will show that in our case their result obtains if we require that ρ_N is bounded and $\sup_k |a_{N,k}| \to 0$ as $N \to \infty$. The proof given of this theorem consists of showing that expressions (4.5), (4.6) and (4.7) converge to zero as $N \to \infty$. The proofs given for (4.5) and (4.7) remain valid even if ρ_N is bounded and $\sup_k |a_{N,k}| \to 0$ as $N \to \infty$. For expression (4.6) we note that since Z_k is centered at its mean

$$\begin{aligned}\textstyle\sum_k |a_{N,k} E(Z_k I_{[|a_{N,k} Z_k|<1]})| &= \textstyle\sum_k |a_{N,k} E(Z_k I_{[|a_{N,k} Z_k| \geqq 1]})| \\ &\leqq -\textstyle\sum_k' |a_{N,k}| \int_{(\sup_k |a_{N,k}|)^{-1}}^{\infty} x d(\sup_k P[|Z_k| \geqq x])\end{aligned}$$

which converges to zero if ρ_N is bounded and $\sup_k |a_{N,k}| \to 0$ because $\sup_k P[|Z_k| \geqq x] \leqq F(x - D)$ for $x \geqq D$ and so $-\int_0^\infty x d(\sup_k P[|Z_k| \geqq x]) < \infty$. The proof of Theorem 3.3 is completed by observing $\sum_j |\sum_k a_{jk,N} E(X_k)| \leqq DM$ and $\sup_j |\sum_k a_{jk,N} E(X_k)| \leqq Db_N \to 0$ as $N \to \infty$.

We now consider quadratic forms whose diagonal elements are not necessarily zero. We will assume

$$\sum_{j=1}^{\infty} |a_{jj,N}|^{\frac{1}{2}} \leqq M' \qquad \text{for all } N \tag{21}$$

instead of (8). The following two theorems represent one way of combining Theorems 3.2 and 3.3 with known convergence results for weighted sums of random variables. There are, of course, other ways this could be done. The proofs of these theorems will be given simultaneously.

THEOREM 3.4. *Let $X_1, X_2, \cdots$ be symmetric random variables with $yG(y) \to 0$ as $y \to \infty$ and let the weights satisfy conditions* (9), (10) *and* (21). *For each N, $Q_{N,m}$ converges almost surely to, say, Q_N and $Q_N \to_P 0$.*

THEOREM 3.5. *If we replace* (8) *by* (21) *in the statement of Theorem* 3.3 *it remains valid.*

PROOFS OF THEOREMS 3.4 AND 3.5. In both theorems we have assumed $yG(y) \to 0$ as $y \to \infty$ and by Remark 2.1 this implies $yF(y) \to 0$ as $y \to \infty$ or $y^{\frac{1}{2}} \sup_k P[X_k^2 \geqq y] \to 0$ as $y \to \infty$. Also in both theorems we have assumed $\sup |a_{jj,N}| \to 0$ as $N \to \infty$. With $t = \frac{1}{2}$ the lemma in Wright (1972) shows that $\sum_{j=1}^{m} a_{jj,N} X_j^2$ converges almost surely. We denote the limit by S_N. Again with $t = \frac{1}{2}$, Theorem 2 of this same paper shows that $S_N \to_P 0$. This result combined with Theorems 3.2 and 3.3 gives Theorems 3.4 and 3.5, respectively.

In conclusion we would like to raise the following questions. Is it necessary to assume the random variables have first moments in Theorem 3.3? Is it necessary to assume $b_N \to 0$ in Theorem 3.3? These two questions could be restated as follows: does the assumption of symmetric random variables make a real difference in convergence of quadratic forms? Along this line we wish to point out to the reader that the Theorem of Hanson and Wright (1971b) giving exponential rates of convergence for quadratic forms has only been proven for symmetric random variables.

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