The Annals of Probability
1973, Vol. 1, No. 5, 788-809

LIMIT DISTRIBUTIONS OF SELF-NORMALIZED SUMS

BY B. F. LOGAN, C. L. MALLOWS,
S. O. RICE AND L. A. SHEPP

Bell Laboratories, Murray Hill

If X_i are i.i.d. and have zero mean and arbitrary finite variance the limiting probability distribution of $S_n(2) = (\sum_{i=1}^n X_i)/(\sum_{j=1}^n X_j^2)^{\frac{1}{2}}$ as $n \to \infty$ has density $f(t) = (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)$ by the central limit theorem and the law of large numbers. If the tails of X_i are sufficiently smooth and satisfy $P(X_i > t) \sim rt^{-\alpha}$ and $P(X_i < -t) \sim lt^{-\alpha}$ as $t \to \infty$, where $0 < \alpha < 2$, $r > 0$, $l > 0$, $S_n(2)$ still has a limiting distribution F even though X_i has infinite variance. The density f of F depends on α as well as on r/l. We also study the limiting distribution of the more general $S_n(p) = (\sum_{i=1}^n X_i)/(\sum_{j=1}^n |X_j|^p)^{1/p}$ where X_i are i.i.d. and in the domain of a stable law G with tails as above. In the cases $p = 2$ (see (4.21)) and $p = 1$ (see (3.7)) we obtain exact, computable formulas for $f(t) = f(t, \alpha, r/l)$, and give graphs of f for a number of values of α and r/l. For $p = 2$, we find that f is always symmetric about zero on $(-1, 1)$, even though f is symmetric on $(-\infty, \infty)$ only when $r = l$.

1. Introduction. Consider the statistic

$$S_n(p) = (\textstyle\sum_{i=1}^n X_i)/(\sum_{j=1}^n |X_j|^p)^{1/p} \tag{1.1}$$

where X_i are independent and identically distributed (i.i.d.) and $0 < p < \infty$. For $p = \infty$, we define

$$S_n(\infty) = \textstyle\sum_{i=1}^n X_i/\max_{1 \le j \le n} |X_j| \tag{1.2}$$

which is the limit of $S_n(p)$ as $p \to \infty$. Darling [2], in his study of the influence of the maximum term on a sum of i.i.d. random variables, obtained the limiting *characteristic function* of $S_n(\infty)$ when X_i is in the domain of attraction of a positive stable law. The methods used by Darling for $p = \infty$ fail when $p < \infty$, and those we use for $p < \infty$ fail when $p = \infty$. A further difference between Darling's work and ours is that we obtain the limiting *distribution function* (for $p = 1, 2$) rather than the limiting characteristic function (which is not simply obtainable and usually of less interest).

Student's T-statistic T_n and $S_n(2)$ are closely related, T_n being defined ($=_\Delta$) by

$$\begin{aligned} T_n &=_\Delta \frac{1}{n^{\frac{1}{2}}} \textstyle\sum_{i=1}^n X_i \Big/ \Big(\frac{1}{n-1} \sum_{j=1}^n (X_j - \bar{X})^2\Big)^{\frac{1}{2}} \\ &= S_n(2) \Big(\frac{n-1}{n - S_n^2(2)}\Big)^{\frac{1}{2}} . \end{aligned} \tag{1.3}$$

It follows that whenever T_n has a limiting distribution, $S_n(2)$ has a limiting

Received November 3, 1972; revised February 2, 1973.
AMS 1970 *subject classifications.* Primary 60F0F; Secondary 60G50.
Key words and phrases. Limit theorems, stable laws, maxima of i.i.d. characteristic function, domains of attraction.

distribution, and the two coincide. This fact was pointed out by Efron [3], who studied the limiting behavior of $S_n(2)$ for X_i in the domain of a stable law, among other cases. Hotelling [6] also studied the asymptotics of T_n for long-tailed X_i and has many additional references.

In the sequel, whenever X_i are assumed to belong to the domain of attraction of a stable law G, the parameter of the attracting stable law G is denoted by α, following customary usage [5]. The fact that $0 < \alpha \leqq 2$ and other well-known facts about the stable laws may be found in [3].

Under certain conditions, the denominator of $S_n(p)$ may be equivalently replaced by a sequence of constants; for example, when $p < \alpha$, X_i has a finite pth moment, and the law of large numbers applies. In particular, if $p = 2$ and X_i has zero mean and finite variance, $S_n(2)$ is asymptotically normal. (One is tempted to conjecture that $S_n(2)$ is asymptotically normal if [and perhaps only if] X_i are in the domain of the normal law. However, these questions are not of primary interest for us here.) Thus the case $p < \alpha$ reduces to normalization of partial sums by constants which has been thoroughly studied [5]. The case $p = \alpha$ is also handled in this way, and so we shall assume throughout that

$$\alpha < p \,. \tag{1.4}$$

If X_i is symmetric, $S_n(2)$ automatically has zero mean and unit variance, which shows that $S_n(2)$ is a natural statistic in the sense that it is normalized.

Suppose that X_i have as common distribution the stable distribution G itself, with density g satisfying

$$x^{\alpha+1}g(x) \to r\,, \qquad x^{\alpha+1}g(-x) \to l \tag{1.5}$$

where $0 < \alpha < 2$, $r + l > 0$ and G is centered so that

$$U_n =_\Delta \frac{X_1 + \cdots + X_n}{n^{1/\alpha}} \quad \text{has df} \quad G \tag{1.6}$$

for each n. By [5] page 544, $|X_i|^p$ belong to the domain of the positive stable law H with parameter α/p *provided* $\alpha < p$, so that if (1.4) holds,

$$V_n^{\,p} =_\Delta (V_n(p))^p =_\Delta \frac{|X_1|^p + \cdots + |X_n|^p}{n^{p/\alpha}} \tag{1.7}$$

has limiting distribution H. It will be seen that (U_n, V_n) has a *joint limiting distribution* and we are interested in the limiting distribution of their ratio

$$S_n(p) = \frac{U_n}{V_n} \,. \tag{1.8}$$

We remark that $S_n(p)$ will have the same limiting distribution even if X_i is merely in the domain of the corresponding stable law G. Indeed, working with the joint characteristic function of U_n and V_n and ([5], page 544) it may be shown with some effort that for appropriately chosen constants, a_n, $U_n a_n$ and $V_n^{\,p} a_n^{\,p}$ have the same joint limiting distribution as in the case when X_i have exactly the distribution G. The ratio $S_n(p)$ then has the same limiting distribution, and so we may and do restrict our attention to the case when X_i are exactly stable which simplifies the choice of the normalizing constants in (1.6) and (1.7). That is, we assume henceforth,

$$(1.9) \qquad X_i \text{ has the stable distribution } G\,.$$

For $1 < \alpha < 2$, if $EX_i \neq 0$, (1.6) fails and $S_n(p)$ has no limiting distribution. Thus we are forced to assume henceforth,

$$(1.10) \qquad EX_i = 0 \qquad \text{for } 1 < \alpha < 2\,.$$

Finally when $\alpha = 1$, G must either be symmetric (the Cauchy law), or a translate of the Cauchy law, for $S_n(p)$ to have a limiting distribution which is not simply degenerate at zero, because the suitable normalization in (1.6) for an unsymmetric stable law with $\alpha = 1$ is $n \log n$, ([5] page 167).

We should mention what happens when the tails of the distribution of X_i are even fatter than stable tails, for example if $P(X_i > x)$ and $P(X_i < -x)$ are slowly varying as $x \to \infty$, say as $x \to \infty$

$$(1.11) \qquad P(X_i > x) \sim \frac{r}{\log x}\,, \qquad P(X_i < -x) \sim \frac{l}{\log x}\,.$$

As Theorem (3.2) of [2] easily implies, $S_n(p)$ then degenerates asymptotically to the two point law with mass $r/(r+l)$ and $l/(r+l)$ at 1 and -1 respectively. In such cases, all terms except the one of maximum modulus in the numerator and denominator of $S_n(p)$ are asymptotically negligible. It seems worthy of conjecture from the above results that the only possible nontrivial limiting distributions of $S_n(p)$ are those obtained when X_i follows a stable law. Of course, it is entirely possible that $S_n(p)$ has no limiting distribution at all as would be the case if the density of X_i in (1.5) oscillated sufficiently slowly as $x \to \infty$ between those of two different stable laws.

For $p = 1$ and $0 < \alpha < 1$, the limiting distribution F has a very simple explicit formula (3.6). It is seen that F is concentrated on $(-1, 1)$ (because $|S_n(1)| \leqq 1$), with infinite singularities at ± 1.

Graphs of the density f of F are given in Section 6.

For $p = 2$ and $0 < \alpha < 2$, f is given by (4.21) which involves an integral of a ratio of parabolic cylinder functions and so is much more complicated. Nevertheless based on (4.21) we can compute f and F and it is seen in Section 5 that

(i) the tails of F are Gaussian-like at $\pm\infty$,

(ii) the density f has infinite singularities $|1 \mp x|^{-\alpha}$ at ± 1 for $0 < \alpha < 1$ (except if $r = 0$ or $l = 0$),

(iii) f is smooth for $1 < \alpha < 2$ and converges to the standard Gaussian density as $\alpha \to 2$.

Graphs of f are given in Section 6 for various parameter values.

In addition to the characteristics noted above of the limiting density f in the case $p = 2$, one phenomenon, which was first discovered on the basis of numerical (computer) analysis of (4.21) and only later proved by analytical techniques, still appears unexplainable from a purely probabilistic viewpoint. Namely, f is always symmetric in $(-1, 1)$ although only for $\alpha = 2$ or $r = l$ is f actually symmetric on $(-\infty, \infty)$. Another curiosity is that f has noticeable bumps around ± 1, $\pm 2^{\frac{1}{2}}$, $\pm 3^{\frac{1}{2}}$, at least for small α. It seems reasonable to explain the bumps as follows: It is known [2] that the sums in the numerator and denominator of $S_n(p)$ are essentially determined by a few summands of largest modulus for $\alpha < 2$. If there are say k summands dominating the rest, and if these are approximately equal, then $S_n(2)$ is approximately $\pm k^{\frac{1}{2}}$, and this occurs with enough probability to be apparent in the density.

2. The limiting distribution of $S_n(p)$, $0 < \alpha < 1$. Suppose $0 < \alpha < 1$ and X_i, $i = 1, \cdots, n$ are independent identically distributed stable variables with density g satisfying $\alpha < p$. We show first that (U_n, V_n) has a limiting joint distribution. Denoting X_1 by X, we have

$$\begin{aligned} & E\exp(iU_n s + iV_n{}^p t) \\ (2.1) \qquad & = \{E\exp(iXs/n^{1/\alpha} + i|X|^p t/n^{p/\alpha})\}^n \\ & = \{1 + \textstyle\int_{-\infty}^{\infty}[\exp(ixs/n^{1/\alpha} + i|x|^p t/n^{p/\alpha}) - 1]g(x)\,dx\}^n . \end{aligned}$$

The integral in the last term in (2.1) is, setting $x = n^{1/\alpha}y$

$$\begin{aligned} (2.2) \qquad & \frac{1}{n}\int_{-\infty}^{\infty}[\exp(iys + i|y|^p t) - 1]\frac{1}{|y|^{1+\alpha}}(n^{1/\alpha}|y|)^{1+\alpha}g(n^{1/\alpha}y)\,dy \\ & \sim \frac{1}{n}\int_{-\infty}^{\infty}[\exp(iys + i|y|^p t) - 1]\frac{1}{|y|^{1+\alpha}}K(y)\,dy , \qquad \text{as } n \to \infty \end{aligned}$$

because of dominated convergence, (1.4), and the fact that $|x|^{1+\alpha}g(x)$ is bounded in $-\infty < x < \infty$, noting that the integral on the right side converges if $\alpha < 1$ and $\alpha < p$, and from (1.4)

$$\begin{aligned} (2.3) \qquad K(y) &= r , \qquad y > 0 \\ K(y) &= l , \qquad y < 0 . \end{aligned}$$

Putting (2.2) into (2.1) and letting $n \to \infty$, we obtain

$$\begin{aligned} (2.4) \qquad & \lim_{n\to\infty} E\exp(iU_n s + iV_n{}^p t) \\ & = \exp\left\{\int_{-\infty}^{\infty}[\exp(iys + i|y|^p t) - 1]\frac{1}{|y|^{1+\alpha}}K(y)\,dy\right\} . \end{aligned}$$

Since (2.4) holds for s real and $\operatorname{Im} t > 0$, the continuity theorem ([5] page 481) shows that $(U_n, V_n{}^p)$ has a limiting distribution. It then follows that $S_n(p) = U_n/V_n$ has a limiting distribution since V_n has a limiting distribution concentrated on the positive half-line. Let (U, V) denote random variables whose joint distribution is that of the limiting joint distribution of (U_n, V_n). Thus,

$$(2.5) \qquad E\exp(iUs + iV^p t) = \lim_{n\to\infty} E\exp(iU_n s + iV_n{}^p t) .$$

Setting first $t = is^p a$ then $t = is^p b$ with a and b positive, in (2.4) and (2.5), subtracting, dividing by s and integrating over $0 < s < \infty$, we obtain

$$(2.6) \qquad \int_0^\infty E \frac{1}{s} e^{iUs}\{e^{-V^p s^p a} - e^{-V^p s^p b}\}\, ds = \int_0^\infty \frac{1}{s}\left(e^{s^\alpha \psi(a)} - e^{s^\alpha \psi(b)}\right) ds$$

where we have used the change of variable $ys = x$ on the right in (2.4), and

$$(2.7) \qquad \psi(a) = \int_{-\infty}^{\infty} [\exp(ix - |x|^p a) - 1] \frac{1}{|x|^{\alpha+1}} K(x)\, dx\,.$$

Noting that V is with probability one strictly positive as remarked above, we may set $Vs = t$ to obtain

$$(2.8) \qquad E \int_0^\infty \frac{1}{s}\{e^{-V^p s^p a} - e^{-V^p s^p b}\}\, ds = E \int_0^\infty \frac{1}{s}\{e^{-s^p a} - e^{-s^p b}\}\, ds < \infty\,.$$

From (2.8) it follows that the necessary absolute convergence holds and we may interchange the expectation and integration in the left side of (2.6), which becomes upon setting $Vs = t$ in the integral,

$$(2.9) \qquad E \int_0^\infty \frac{1}{t} e^{i(U/V)t}\{e^{-t^p a} - e^{-t^p b}\}\, dt = \int_0^\infty \frac{1}{s}\left(e^{s^\alpha \psi(a)} - e^{\alpha \psi(b)}\right) ds\,.$$

Noting that U/V is a random variable with the limiting distribution of $S_n(p)$, we obtain from (2.9) after again interchanging integration and expectation

$$(2.10) \qquad \int_0^\infty \frac{1}{t} \varphi(t)\{\exp(-t^p a) - \exp(-t^p b)\}\, dt = \int_0^\infty \frac{1}{s}\left(e^{s^\alpha \psi(a)} - e^{s^\alpha \psi(b)}\right) ds$$

where φ is the characteristic function (ch.f.) of the distribution function (df) F,

$$(2.11) \qquad \varphi(t) = E e^{i(U/V)t} = \lim_{n\to\infty} E e^{iS_n(p)t}\,.$$

Finally, differentiating both sides of (2.10) with respect to a, justified because the formally differentiated integrands are absolutely integrable, we obtain for $a > 0$ since $\operatorname{Re}\psi(a) < 0$,

$$(2.12) \qquad \int_0^\infty t^{p-1}\varphi(t) \exp(-t^p a)\, dt = -\psi'(a) \int_0^\infty s^{\alpha-1} e^{s^\alpha \psi(a)}\, ds = \frac{1}{\alpha}\frac{\psi'(a)}{\psi(a)}\,.$$

Our next task is to invert (2.12) to obtain the distribution F corresponding to φ. That this is possible in principle can be shown as follows:

(i) substitute $t^p = u$ in the left side of (2.12) which then becomes the Laplace transform of $\varphi(u^{1/p})$,

(ii) by uniqueness of the Laplace transform $\varphi(u^{1/p})$ is obtainable by inverting the Laplace transform and so φ is known,

(iii) an inversion of the Fourier transform finally yields F.

Although the method sketched just above does produce a mathematical formula for F, it is not suitable for numerical analysis. However, in the cases

$p = 1$ and $p = 2$ there are other inversions of (2.12), which are given below and which yield computationally usable formulas for F.

These inversions are rigorous forms of the following heuristic development. Assuming the probability density $f(x)$ corresponding to the ch.f. $\varphi(t)$ exists, let

$$\Phi(s) = \int_0^\infty \varphi(t)e^{-st}\,dt\,, \qquad \operatorname{Re}(s) > 0\,. \tag{2.13}$$

Then $\varphi(-t) = \varphi^*(t)$ when t is real and formally,

$$\begin{aligned} f(x) &= \pi^{-1} \operatorname{Re} \int_0^\infty \varphi(t)e^{-ixt}\,dt \\ &= \pi^{-1} \lim_{\varepsilon\to 0} \operatorname{Re} \Phi(\varepsilon + ix)\,. \end{aligned} \tag{2.14}$$

For $p = 1$, (2.12) is already of the form (2.13) and (2.14) can be applied directly with $\Phi(s) = \alpha^{-1}\psi'(s)/\psi(s)$. When $p = 2$, (2.12) may be put in the form (2.13) by multiplying both sides of (2.12) by a function $\theta(a, s)$, integrating from $a = 0$ to $a = \infty$, and requiring that

$$\int_0^\infty \theta(a, s)t \exp(-t^2a)\,da = e^{-st}\,.$$

Regarding this as a Laplace transform and inverting gives $\theta(a, s) = 2(\pi a)^{-\frac{1}{2}} \exp(-s^2/4a)$. Comparison with (2.13) gives $\Phi(s)$ as a definite integral with limits $a = 0$ and $a = \infty$ (shown on the right side of (4.4) below). The integral converges in the sector $|\arg s| \leqq \pi/4$, but not at $s = ix$, the point at which $\Phi(s)$ is needed to get $f(x)$ from (2.14). The value of $\Phi(ix)$ can be obtained by analytic continuation. The procedure is, in effect, to rotate the path of integration from the positive real a-axis to the positive (assuming $x > 0$) imaginary axis. Actually, it is more convenient to set $a = 1/(2\tau^2)$ so that the positive imaginary a-axis corresponds to the ray $\arg \tau = -\pi/4$ as indicated by the upper limit of integration $\tau = \infty/i^{\frac{1}{2}}$ in (4.21) (with τ for t). The resulting integral converges when $s = ix$ and $f(x)$ is given by (2.14).

We recall that since p must be $> \alpha$, as remarked above, when $p = 1$ we must have $0 < \alpha < 1$.

3. The case $p = 1$, $0 < \alpha < 1$. We need the following slight variation of the usual inversion formula ([5] page 484), where the second equality follows from the fact that $\varphi(-t)$ is the complex conjugate of $\varphi(t)$.

LEMMA. *If φ is the* ch.f. *of the* df F, *then*

$$\begin{aligned} F(x) - F(y) &= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t) \frac{(e^{-iyt} - e^{ixt})}{it} e^{-\varepsilon|t|}\,dt \\ &= \lim_{\varepsilon\to 0} \frac{1}{\pi} \operatorname{Re} \int_0^\infty \varphi(t) \frac{(e^{-iyt} - e^{-ixt})}{it} e^{-\varepsilon t}\,dt \end{aligned} \tag{3.1}$$

whenever x and y are points of continuity of F.

Set $p = 1$ in (2.12) and obtain for all a > 0,

$$\int_0^\infty \varphi(t)e^{-at}\,dt = \frac{1}{\alpha}\frac{\psi'(a)}{\psi(a)} \tag{3.2}$$

where ψ, given by (2.7) with $p = 1$, is after a short calculation,

$$\psi(a) = \Gamma(-\alpha)(r(a-i)^\alpha + l(a+i)^\alpha)\,, \tag{3.3}$$

where the phases are chosen so that $\arg(a-i)$ and $\arg(a+i)$ tend to 0 as $a \to +\infty$ and branch cuts are taken in the complex a-plane from $-\infty \pm i$ to $\pm i$. Both sides of (3.2) are analytic in the half-plane $\operatorname{Re}(a) > 0$ and so (3.2) must continue to hold throughout $\operatorname{Re}(a) > 0$. Set $a = \varepsilon + iu$ with $\varepsilon > 0$, $-\infty < u < \infty$ in (3.2) to obtain

$$\frac{1}{\pi}\int_0^\infty \varphi(t)e^{-iut}e^{-\varepsilon t}\,dt = \frac{1}{\pi}\frac{1}{\alpha}\frac{\psi'(\varepsilon+iu)}{\psi(\varepsilon+iu)}\,. \tag{3.4}$$

Integrating (3.4) on u from x to y and applying (3.1) we see that the limiting distribution function F of $S_n(1)$ satisfies for $-\infty < y < x < \infty$.

$$F(x) - F(y) = \lim_{\varepsilon\to 0}\frac{1}{\pi\alpha}\operatorname{Im}[\log\psi(\varepsilon+ix) - \log\psi(\varepsilon+iy)]\,. \tag{3.5}$$

Since $F(-\infty) = 0$ by convention, a short calculation from (3.5) gives for $0 < \alpha < 1$, $|x| < 1$, $r \geqq 0$, $l \geqq 0$, $r + l > 0$, the limiting distribution for $p = 1$,

$$\begin{aligned} F(x) &= \lim_{n\to\infty} P(S_n(1) < x) \\ &= \tfrac{1}{2} + \frac{1}{\pi\alpha}\tan^{-1}\left[\frac{l(1+x)^\alpha - r(1-x)^\alpha}{l(1+x)^\alpha + r(1-x)^\alpha}\tan\frac{\pi\alpha}{2}\right], \end{aligned} \tag{3.6}$$

while $F(x) = F(1) = 1$ for $x \geqq 1$, $F(x) = F(-1) = 0$ for $x < -1$ which is as expected since $|S_n(1)| \leqq 1$. For $r = 0$ or $l = 0$ F degenerates to the one point distribution at -1 or $+1$ respectively which is not surprising because in these cases $S_n(1) \equiv \pm 1$ since all X_i are of one sign. On the other hand, for $r > 0$, $l > 0$, $0 < \alpha < 1$, $p = 1$, F has a density f given for $|x| < 1$ from (3.6) by

$$\begin{aligned} f(x) = \frac{2rl\sin\pi\alpha}{\pi}\Big/[(1-x^2)^{1-\alpha}(r^2(1-x)^{2\alpha} + 2rl\cos\pi\alpha(1-x^2)^\alpha \\ + l^2(1+x)^{2\alpha})]\,. \end{aligned} \tag{3.7}$$

Of course, f vanishes for $|x| > 1$. We have from (3.7)

$$f(x) \sim \frac{r}{l}\,2^{-\alpha}\,\frac{\sin\pi\alpha}{\pi}(1-x)^{\alpha-1}\,, \qquad \text{as } x \uparrow 1 \tag{3.8}$$

$$f(x) \sim \frac{l}{r}\,2^{-\alpha}\,\frac{\sin\pi\alpha}{\pi}(1+x)^{\alpha-1}\,, \qquad \text{as } x \downarrow -1\,, \tag{3.9}$$

showing that f has infinite singularities at ± 1. From the graphs of f, given in Section 6, it seems that f has one minimum for $0 < \alpha \leqq \frac{1}{2}$ and two minima for $\frac{1}{2} < \alpha < 1$.

4. The case $p = 2$, $0 < \alpha < 2$. Setting $p = 2$ in (2.12) with $0 < \alpha < 1$ we obtain for $a > 0$,

$$\int_0^\infty t\varphi(t)\exp(-t^2 a)\,dt = \frac{1}{\alpha}\frac{\psi'(a)}{\psi(a)} \tag{4.1}$$

where φ is the limiting characteristic function of $S_n(2)$ and ψ, given by (2.7), is

$$\psi(a) = \int_{-\infty}^{\infty} [\exp(ix - x^2a) - 1] \frac{1}{|x|^{\alpha+1}} K(x)\, dx \,, \tag{4.2}$$

K being given by (2.3). Multiplying (4.1) by $(\pi a)^{-\frac{1}{2}} \exp(-s^2/4a)$, integrating over a from 0 to ∞, and using the identity

$$\frac{t}{\pi^{\frac{1}{2}}} \int_0^\infty \exp[-(s^2/4a) - at^2]a^{-\frac{1}{2}}\, da = e^{-st} \,, \qquad s > 0,\ t > 0 \,, \tag{4.3}$$

we obtain for $s > 0$,

$$\int_0^\infty \varphi(t)e^{-st}\, dt = \frac{1}{\alpha} \int_0^\infty (\pi a)^{-\frac{1}{2}} \exp(-s^2/4a) \frac{\psi'(a)}{\psi(a)}\, da \,. \tag{4.4}$$

Note that (4.4) gives the Laplace transform of the Fourier transform of the distribution function we are seeking. We can apply the same method to (4.4) that we used to invert (3.2) provided we can analytically continue the right side of (4.4), which only converges for $|\arg s| < \pi/4$, to $s = iy$, $-\infty < y < \infty$. To do the continuation, it is convenient to use the parabolic cylinder functions [7], $D_\nu(z)$, which are entire functions of z for each real ν. We shall make use of the following properties of $D_\nu(z)$, [7]:

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)} \int_0^\infty e^{-zt-t^2/2}t^{-\nu-1}\, dt \,, \qquad \nu < 0 \tag{4.5}$$

$$D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) = 0 \tag{4.6}$$

$$\frac{d}{dz} D_\nu(z) + \tfrac{1}{2}zD_\nu(z) - \nu D_{\nu-1}(z) = 0 \tag{4.7}$$

$$D_\nu(iz) = \frac{\Gamma(\nu+1)}{(2\pi)^{\frac{1}{2}}} [e^{i\nu\pi/2}D_{-\nu-1}(-z) + e^{-i\nu\pi/2}D_{-\nu-1}(z)] \tag{4.8}$$

$$D_\nu(z) \sim z^\nu \exp(-z^2/4) \,, \qquad \text{as} \quad z \to \infty \,, \quad |\arg z| < 3\pi/4 \tag{4.9}$$

$$\frac{d^2}{dz^2} D_\nu(z) + (\nu + \tfrac{1}{2} - \tfrac{1}{4}z^2)D_\nu(z) = 0 \,. \tag{4.10}$$

Differentiating (4.2) and using the substitutions

$$a = 1/(2b^2) \,, \qquad x = \xi b \tag{4.11}$$

we obtain from (4.5) for $b > 0$,

$$\psi'(a) = -b^{2-\alpha}e^{-b^2/4}\Gamma(2-\alpha)[rD_{\alpha-2}(-ib) + lD_{\alpha-2}(ib)] \,. \tag{4.12}$$

Integrating (4.2) by parts and using (4.5), (4.6), and (4.11) we obtain

$$\psi(a) = b^{-\alpha}e^{-b^2/4}\Gamma(-\alpha)[rD_\alpha(-ib) + lD_\alpha(ib)] \,. \tag{4.13}$$

Putting (4.11), (4.12), and (4.13) into (4.4) we obtain for $s > 0$,

$$\frac{1}{\pi} \int_0^\infty \varphi(t)e^{-st}\,dt = \int_0^\infty e^{-s^2t^2/2}\mathscr{D}(t)\,dt\,. \tag{4.14}$$

where

$$\mathscr{D}(t) = (1-\alpha)(2\pi^{-3})^{\frac{1}{2}}\,\frac{rD_{\alpha-2}(-it) + lD_{\alpha-2}(it)}{rD_\alpha(-it) + lD_\alpha(it)}\,. \tag{4.15}$$

We note that $\mathscr{D}(t)$ is analytic (free of poles) in the region $|\arg t| \leqq \pi/4$ because the denominator of $\mathscr{D}$ is given by (4.13) and $\psi(a)$ has negative real part if $\operatorname{Re}(a) > 0$ as is clear from (4.2). Verifying from (4.8) and (4.9) that for $|\theta| \leqq \pi/4$,

$$|\mathscr{D}(\rho e^{i\theta})| = O(\rho^{1-2\alpha}) \tag{4.16}$$

we may rotate the contour on the right of (4.14) to the ray $\arg t = \theta$, for $s > 0$ and fixed $|\theta| < \pi/4$. Denote, for $|\theta| \leqq \pi/4$, the corresponding integral by

$$H_\theta(s) = \int_0^{\infty\cdot e^{i\theta}} e^{-s^2t^2/2}\mathscr{D}(t)\,dt \tag{4.17}$$

which is convergent and analytic for

$$\operatorname{Re}(s^2e^{2i\theta}) > 0\,. \tag{4.18}$$

From Cauchy's theorem we see that $H_\theta(s)$ are merely different representations of a single analytic function

$$H(s) = H_\theta(s)\,, \qquad \operatorname{Re}(s^2e^{2i\theta}) > 0\,. \tag{4.19}$$

Thus for $y \geqq 0$, letting $s \to iy$ with $\operatorname{Re}(s) > 0$,

$$\begin{aligned} H(iy) &= H_{-\pi/4}(iy+0) \\ &= \int_0^{\infty/i^{\frac{1}{2}}} e^{y^2t^2/2}\mathscr{D}(t)\,dt\,, \quad i^{\frac{1}{2}} = e^{-\pi i/4}\,. \end{aligned} \tag{4.20}$$

We can now obtain the limiting df F by the method of Lemma (3.1). Passing directly (to avoid more cumbersome formulas) to the density f of F we obtain for $y \geqq 0$, from (4.20)

$$\begin{aligned} f(y) &= \lim\nolimits_{s\to iy} \operatorname{Re}\frac{1}{\pi}\int_0^\infty \varphi(t)e^{-st}\,dt \\ &= \operatorname{Re} H(iy) = \operatorname{Re}\int_0^{\infty/i^{\frac{1}{2}}} e^{y^2t^2/2}\mathscr{D}(t)\,dt\,, \quad y \geqq 0\,. \end{aligned} \tag{4.21}$$

Since

$$\mathscr{D}(t)^* = \mathscr{D}(-t^*) \tag{4.22}$$

with * denoting complex conjugate, it is easy to check that for $y \geqq 0$,

$$\begin{aligned} f(-y) &= \operatorname{Re}\int_0^{\infty\cdot i^{\frac{1}{2}}} e^{y^2t^2/2}\mathscr{D}(t)\,dt \\ &= \operatorname{Re}\int_0^{\infty/i^{\frac{1}{2}}} e^{y^2t^2/2}\mathscr{D}(-t)\,dt\,, \quad y \geqq 0\,. \end{aligned} \tag{4.23}$$

Since $\mathscr{D}(\rho/i^{\frac{1}{2}}) \sim \text{constant}\cdot\rho^{1-\alpha}\exp(\rho^2 i/2)$ as $\rho\to\infty$, the integrals in (4.21) and (4.23) converge only conditionally. The corresponding formula for F, obtained by a formal integration of (4.21) and (4.23), involves absolutely convergent integrals but we omit writing the formula for F since we have not used it for computation.

It is clear from (4.21)—(4.23) that interchanging r and l reflects f, that is,

$$f(y; r, l) = f(-y; l, r) . \tag{4.24}$$

It is *not* transparent from (4.21)—(4.23) that

$$f(y) = f(-y) \qquad \text{for } -1 < y < 1 \tag{4.25}$$

To see this we first note that from (4.21) and (4.23),

$$f(y) - f(-y) = \operatorname{Re} \int_0^{\infty/i^{\frac{1}{2}}} \exp(y^2 t^2/2)[\mathscr{D}(t) - \mathscr{D}(-t)]\, dt \tag{4.26}$$

and from (4.15), (4.6) and (4.7), for all values of t,

$$\begin{aligned} \mathscr{D}(t) - \mathscr{D}(-t) \\ = \frac{2(r^2 - l^2)it}{\pi\Gamma(1-\alpha)[rD_\alpha(-it) + lD_\alpha(it)][rD_\alpha(it) + lD_\alpha(-it)]} \end{aligned} \tag{4.27}$$

where we have used the Wronskian ([7] page 117)

$$D_\alpha(-z)D_\alpha'(z) + D_\alpha'(-z)D_\alpha(z) = -(2\pi)^{\frac{1}{2}}\Gamma(-\alpha) \tag{4.28}$$

in which z may be complex and D_α' denotes the derivative of D_α. From (4.27) and (4.9) we have as $|t| \to \infty$ in the sector $|\arg t| < \pi/4$

$$\mathscr{D}(t) - \mathscr{D}(-t) \sim \frac{2(r^2 - l^2)it^{1-2\alpha}\exp(-t^2/2)}{\pi\Gamma(1-\alpha)(r^2 + l^2 + 2rl\cos\pi\alpha)} . \tag{4.29}$$

This result shows that when $|y| < 1$ the contour in (4.26) can be shifted from the ray $\arg t = -\pi/4$ to the positive real axis, where from (4.27), $\mathscr{D}(t) - \mathscr{D}(-t)$ is purely imaginary. Hence $f(y) - f(-y)$ is zero and (4.25) follows.

We will see in Section 5 that f is symmetric in $-\infty < y < \infty$ only if $r = l$.

We turn next to the case $1 < \alpha < 2$. Suppose then that X_i are stable with parameter α, $1 < \alpha < 2$, and that $EX_i = 0$ and $\alpha < p$ which are necessary, (1.4) and (1.10), for $S_n(p)$ to have a proper, non-stable limiting distribution. The integrand in (2.1) can be written simply as

$$\int_{-\infty}^{\infty} [\exp(ixs/n^{1/\alpha} + i|x|^p t/n^{p/\alpha}) - 1 - ixs/n^{1/\alpha}]g(x)\, dx \tag{4.30}$$

since $\int xg(x)\, dx = 0$. The remainder of the proof carries through as before except that ψ in (2.7) must be replaced by

$$\tilde{\psi}(a) = \int_{-\infty}^{\infty} [\exp(ix - |x|^p a) - 1 - ix] \frac{1}{|x|^{\alpha+1}} K(x)\, dx , \tag{4.31}$$

which converges for $1 < \alpha < 2$ and $\alpha < p$.

In particular, for $p = 2$, (4.1) holds with ψ replaced by $\tilde{\psi}$. A calculation shows however that (4.12) and (4.13) are still valid if ψ is replaced by $\tilde{\psi}$. Thus the formulas (4.21) and (4.23) for the limiting density f are valid without change also for $1 < \alpha < 2$.

The case $\alpha = 1$ is somewhat special. As we noted in the introduction, $S_n(p)$ has a proper limiting distribution only if $r = l$, that is only if X_i follow a Cauchy distribution. However, in contrast to the case when $\alpha < 1$, we will see that for $\alpha = 1$, translating X_i to $X_i + \mu$ changes the limiting distribution. Suppose then that $\tilde{X}_i = X_i + \mu$ have the noncentral Cauchy df.

$$G_\mu(x) = \frac{1}{\pi} \int_{-\infty}^{x-\mu} (1 + u^2)^{-1}\, du \tag{4.32}$$

where $-\infty < \mu < \infty$. We will show that the limiting density of $\tilde{S}_n(2)$ (that is, $S_n(2)$ with X_i replaced by $\tilde{X}_i$), $\tilde{f}(y; \mu)$, is given by (analogues of (4.21)—(4.23))

$$\tilde{f}(y; \mu) = \operatorname{Re} \int_0^{\infty/i^{\frac{1}{2}}} e^{y^2t^2/2} \tilde{\mathscr{D}}(t)\, dt\,, \qquad y \geqq 0 \tag{4.33}$$

$$\begin{aligned} \tilde{f}(-y; \mu) &= \operatorname{Re} \int_0^{\infty/i^{\frac{1}{2}}} e^{y^2t^2/2} \tilde{\mathscr{D}}(-t)\, dt\,, \\ &= \operatorname{Re} \int_0^{\infty \cdot i^{\frac{1}{2}}} e^{y^2t^2/2} \tilde{\mathscr{D}}(t)\, dt\,, \qquad y \geqq 0\,, \end{aligned} \tag{4.34}$$

where

$$\tilde{\mathscr{D}}(t) = \left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \{1 + te^{t^2/2}(\textstyle\int_0^t e^{-u^2/2}\, du - i\mu/(2\pi)^{\frac{1}{2}})\}^{-1}\,. \tag{4.35}$$

We note that (in contrast to the case $\alpha \neq 1$) the contour in (4.33) may be rotated to the real axis for $-1 < y < 1$ because $\tilde{\mathscr{D}}(t) = O(\exp(-t^2/2))$ as real $t \to \infty$ from (4.35), thus

$$\tilde{f}(y; \mu) = \operatorname{Re} \int_0^\infty e^{y^2t^2/2} \tilde{\mathscr{D}}(t)\, dt\,, \qquad -1 < y < 1\,. \tag{4.36}$$

It follows immediately from (4.33) and (4.36) that (analogues of (4.24)—(4.25))

$$\tilde{f}(y; \mu) = \tilde{f}(-y; -\mu)\,, \qquad -\infty < y < \infty \tag{4.37}$$

$$\tilde{f}(y; \mu) = \tilde{f}(-y, \mu)\,, \qquad -1 < y < 1\,. \tag{4.38}$$

We remark that for $\mu = 0$, $\tilde{\mathscr{D}}$ is the limit of $\mathscr{D}$ as $\alpha \to 1$ keeping $r = l$.

The proof of (4.33) and (4.34) is again similar to that of (4.21)—(4.23), and so we only give the new aspects. Let X_i denote (symmetric) Cauchy random variables and (U, V) random variables with the limiting joint distribution of (U_n, V_n) as in (1.6) and (1.7). Then

$$\tilde{S}_n(2) \approx \frac{U + \mu}{V} \qquad \text{as } n \to \infty \tag{4.39}$$

where $\tilde{S}_n(2)$ is $S_n(2)$ with X_i replaced by $\tilde{X}_i = X_i + \mu$, and $\approx$ denotes that the right side has the limiting distribution of the left side as $n \to \infty$. To find this limiting distribution set $\alpha = 1$, $p = 2$ in (2.4) and (2.5), to obtain

$$E \exp[iUs + iV^2t] = \exp\left\{\int_{-\infty}^{\infty} [(e^{iy^2t} \cos ys) - 1] \frac{1}{y^2}\, dy\right\}. \tag{4.40}$$

Multiplying (4.40) by $\exp(i\mu s)$ and proceeding as before, we obtain (4.4) with

φ replaced by $\tilde{\varphi}$, the ch.f. of $\tilde{f}$, and ψ replaced by

$$\tilde{\psi}(a) = \int_{-\infty}^{\infty} [e^{-x^2 a} (\cos x) - 1] \frac{1}{x^2}\, dx + i\mu . \tag{4.41}$$

A calculation shows that, with the substitution $a = 1/2b^2$, (4.4) with ψ replaced by $\tilde{\psi}$ reduces to (4.14) with $\mathscr{D}$ replaced by $\tilde{\mathscr{D}}$. The remainder of the proof of (4.33) and (4.34) is as before.

5. Asymptotic and numerical analysis of the exact formulas. We first obtain an alternate formula for the limiting density f in the case $p = 2$, $0 < \alpha < 2$, by shifting the contour in (4.21) and (4.36) still further to the negative imaginary axis. The resulting formula for $y > [1/(2\alpha + 1)]^{\frac{1}{2}}$ is, as we shall see

$$f(y) = \operatorname{Re} \frac{(2\pi i)}{\alpha} \left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \sum\nolimits'_{t \in A} e^{y^2 t^2/2} t \tag{5.1}$$

where the sum is taken over the set A of poles of $\mathscr{D}$ ($\tilde{\mathscr{D}}$ in the case $\alpha = 1$) in the sector $\{-\pi/2 \leqq \arg t \leqq -\pi/4\}$, with the special provision (indicated by the prime on the summation sign in (5.1)) that the term corresponding to the (as it turns out, unique) pole t_0 of $\mathscr{D}$ on the negative imaginary axis is summed with weight $\frac{1}{2}$. The series in (5.1) converges absolutely for $y > (3/(2\alpha + 1))^{\frac{1}{2}}$, and converges conditionally, if the terms are ordered according to increasing modulus of the poles t, for $y > [1/(2\alpha + 1)]^{\frac{1}{2}}$, $y \neq 1$. For $\alpha = 1$, (5.1) gives $\tilde{f}(y, \mu)$ if $\tilde{\mathscr{D}}$ is used in place of $\mathscr{D}$ in determining the index set A, with the same convergence criteria.

We begin the proof of (5.1) by observing that the numerator and denominator of $\mathscr{D}$ cannot vanish simultaneously, so that $t \in A$ if and only if

$$rD_\alpha(-it) + lD_\alpha(it) = 0 . \tag{5.2}$$

LEMMA A. *Unless* $0 < \alpha < 1$ *and* $r = 0$, *there is one and only one value* t_0 *of* t *such that* $t = t_0$ *satisfies* (5.2) *and* t_0 *is on the negative imaginary axis, that is*

$$t_0 = -i\tau_0 , \qquad \tau_0 > 0 . \tag{5.3}$$

For $\alpha = 1$ *there is one and only one* t_0 *satisfying* (5.3) *and* $\tilde{\mathscr{D}}(t_0) = \infty$ (*see* (4.35)); t_0 *may be approximated using a table of Dawson's integral* [1].

Assuming the lemma true (it will be proved in Section 7) consider the closed contour C made up of three arcs: C_1—from 0 to $\rho \exp(-i\pi/2)$ along the negative imaginary axis with an indentation into the fourth quadrant around t_0; C_2—from $\rho \exp(-i\pi/2)$ to $\rho \exp(-i\pi/4)$ along the shorter arc of the circle of radius ρ centered at 0; C_3—from $\rho \exp(-i\pi/4)$ to 0 along the straight line joining these points. By Cauchy's formula,

$$\int_C e^{y^2t^2/2} \mathscr{D}(t)\, dt = (2\pi i) \sum\nolimits_{t \in A \cap I(C)} \operatorname{res} (e^{y^2t^2/2} \mathscr{D}(t)) , \tag{5.4}$$

where $I(C)$ is the region interior to C, and res denotes the residue at t.

The residue of $\exp(y^2t^2/2)\mathscr{D}(t)$ at t is

$$e^{y^2t^2/2}(1-\alpha)\left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \frac{rD_{\alpha-2}(-it)+lD_{\alpha-2}(it)}{r(-i)D_\alpha'(-it)+liD_\alpha'(it)} \tag{5.5}$$

where the prime denotes the derived function. Using (4.6) and (4.7), the expression in (5.5) becomes

$$\operatorname{res}\,(e^{y^2t^2/2}\mathscr{D}(t)) = -\frac{1}{\alpha}\left(\frac{2}{\pi^2}\right)^{\frac{1}{2}} e^{y^2t^2/2}t\,. \tag{5.6}$$

The real part of the integral along C_1 in (5.4) vanishes because $\mathscr{D}(t)$ is real on the imaginary axis; the indentation at t_0 contributes $\frac{1}{2}\times$ the residue at t_0 because the contour goes half-way around t_0; the contribution of the integral along C_2 is zero provided that $\rho\to\infty$ along a properly chosen sequence according to lemma B, below. Putting the above statements together we obtain (5.1) from (4.21) and (5.4) except for the assertion about absolute convergence of the series in (5.1), which is proved in Section 7, along with lemma B.

LEMMA B. *If $y > [1/(2\alpha+1)]^{\frac{1}{2}}$, the integral in* (5.4) *over $C_2(\rho)$ tends to zero for a sequence $\rho=\rho_n^*\to\infty$ for which the region between $C_2(\rho_n^*)$ and $C_2(\rho_{n+1}^*)$ contains only one pole of $\mathscr{D}$.*

The behavior of $f(y)$ as $y\to\infty$. The expansion (5.1) shows that as $y\to\infty$, for some a and τ,

$$f(y)\sim ae^{-\frac{1}{2}y^2\tau^2} \tag{5.7}$$

where a and τ correspond to the term of (5.1) for which $\operatorname{Re} t^2$ is a maximum (note that $\operatorname{Re} t^2<0$ for $t\in A$). We have not proved but conjecture, on the basis of both mathematical simplicity and numerical evidence, that $\operatorname{Re} t^2$ is a maximum over $t\in A$ for $t=t_0=-i\tau_0$ in (5.3). If this conjecture is true, then as $y\to\infty$, from (5.1),

$$f(y)\sim\frac{1}{\alpha}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\tau_0 e^{-\frac{1}{2}y^2\tau_0^2}\,. \tag{5.8}$$

However, (5.8) must be regarded as an unproved conjecture although (5.7) holds rigorously for some a and $\tau>0$. Because of (4.24), as $y\to-\infty$, (5.8) holds with τ_0 replaced by $\tau_0'>0$ where $t_0'=i\tau_0'$ satisfies (5.2) (equivalently, $-i\tau_0'$ satisfies (5.2) with r and l interchanged). It is easily seen that $\tau_0\neq\tau_0'$ if $r\neq l$ and so f is not symmetric on $(-\infty,\infty)$ unless $r=l$.

For $r=0$ and $0<\alpha<1$, A is empty and in this case, $f(y)\equiv 0$ for $y>0$. Of course this conclusion is obvious probabilistically anyway because if $r=0$ and $0<\alpha<1$, all X_i are negative. In every other case, the tails of f and F are asymptotically Gaussian (5.7).

We can also use (4.21) to determine the behavior of f at $y=1$. We have:

(i) for $0 < \alpha < 1$, $r > 0$, as $y \downarrow 1$,

$$f(y) \sim \frac{1}{(y-1)^{1-\alpha}} \frac{1}{\pi} \frac{r^2 \sin \pi\alpha}{|C|^2}, \qquad y \downarrow 1 \tag{5.9}$$

where

$$C = r\exp(-i\pi\alpha/2) + l\exp(i\pi\alpha/2); \tag{5.10}$$

(ii) for $0 < \alpha < 1$, $rl > 0$, as $y \uparrow 1$,

$$f(y) \sim \frac{1}{(1-y)^{1-\alpha}} \frac{1}{\pi} \frac{rl \sin \pi\alpha}{|C|^2}, \qquad y \uparrow 1; \tag{5.11}$$

(iii) for $\alpha = 1$, $-\infty < \mu < \infty$, as $y \to 1$,

$$\bar{f}(y, \mu) \sim \frac{1}{\pi^2} \frac{1}{1+\mu^2} \log \frac{1}{|1-y|}; \tag{5.12}$$

(iv) for $1 < \alpha < 2$, $f(y)$ is continuous at $y = 1$.

The case $l = 0$, $0 < \alpha < 1$ is interesting. The right side of (5.11) vanishes. Indeed, in this case $f(y) \equiv 0$ for $y < 1$ because as we noted before, X_i are positive in this case and so $S_n(2) \geqq 1$, as is easily verified. Similarly, if $r = 0$, $0 < \alpha < 1$, $f(y) \equiv 0$ for $y > -1$.

(v) for $0 < \alpha \leqq 1$, $y \neq \pm 1$ and for $1 < \alpha < 2$, $-\infty < y < \infty$, f is continuous at y;

(vi) for $-\infty < y < \infty$, as $\alpha \to 2$,

$$f(y) \to \varphi(y) = \frac{1}{(2\pi)^{-\frac{1}{2}}} e^{-\frac{1}{2}y^2} \tag{5.13}$$

Assertion (vi) is not surprising because as we noted in the introduction $S_n(2)$ is asymptotically normal when X_i are normal. We begin proving (i)—(vi) by noting that from (4.8) we have

$$\begin{aligned} &e^{\alpha\pi i/2}[rD_\alpha(-it) + lD_\alpha it] \\ &\qquad = (re^{-\alpha\pi i/2} + le^{\alpha\pi i/2})D_\alpha(it) - ir\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\alpha)} D_{-1-\alpha}(t), \end{aligned} \tag{5.14}$$

and from (4.9) we find that as $t \to \infty$, $0 < \alpha < \frac{3}{2}$

$$D(t/i^{\frac{1}{2}}) \sim \frac{2ir}{\pi C} \frac{1}{i^{\frac{1}{2}}} \frac{1}{\Gamma(1-\alpha)} t^{1-2\alpha} e^{t^2 i/2}. \tag{5.15}$$

It follows from (4.21) that as $y \to 1$, for any T,

$$f(y) \sim \text{Re} \int_T^\infty e^{(i/2)t^2(1-y^2)} t^{1-2\alpha}\, dt \cdot \frac{2r}{\pi C \Gamma(1-\alpha)} \tag{5.16}$$

at least for $0 < \alpha < 1$, $r \cdot l > 0$ because, as we shall see, in this case the right side becomes infinite as $y \to 1$ and finite quantities can be neglected. We may set $T = 0$ and shift the contour to $\arg t = \pi/4$ for $y < 1$, $\arg t = -\pi/4$ for $y > 1$. A short calculation gives (i) and (ii) and (iii) follows similarly.

To prove (iv), we note from (5.15) for $\alpha < \frac{3}{2}$ and from (5.14) and (4.9) for $\alpha \geqq \frac{3}{2}$ that as real $t \to \infty$,

$$\mathscr{D}(t/i^{\frac{1}{2}}) = O(t^{1-2\alpha}), \qquad 1 < \alpha < \tfrac{3}{2} \tag{5.17}$$

$$\mathscr{D}(t/i^{\frac{1}{2}}) = O(t^{-2}), \qquad \tfrac{3}{2} \leqq \alpha < 2 \tag{5.18}$$

so that the last integral in (4.21) converges uniformly in y (note that $\exp(y^2t^2/2)$ has modulus one on the contour). Thus f is uniformly bounded and continuous for $1 < \alpha < 2$. Using the asymptotic expansion for $\mathscr{D}$ (or $\tilde{\mathscr{D}}$) for $0 < \alpha \leqq 1$ in a similar way, (v) follows.

To prove (vi), note that as $\alpha \to 2$, from (4.15)

$$\mathscr{D}(t) \to -\left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \frac{rD_0(-it) + lD_0(it)}{rD_2(-it) + lD_2(it)} = \left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \frac{1}{1+t^2}\,. \tag{5.19}$$

Thus, from (4.21), we obtain easily

$$\lim_{\alpha\to 2} f(y) = \left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \operatorname{Re} \int_0^{\infty\cdot i^{-\frac{1}{2}}} e^{y^2t^2/2} \frac{dt}{1+t^2} = \varphi(y)\,. \tag{5.20}$$

Moments. The moments of f, which all exist by (5.7) can be computed recursively from (4.14) as follows. Expand $\varphi(t)$ and $\mathscr{D}(t)$ in (4.14) into power series about $t = 0$ and integrate term by term. Equating coefficients of like powers of $1/s$ gives for $\alpha \neq 1$, after a laborious calculation,

$$\lim_{n\to\infty} ES_n(2) = \frac{1}{\pi^{\frac{1}{2}}} \frac{r-l}{r+l} \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)} \tag{5.21}$$

$$\lim_{n\to\infty} \sigma^2(S_n(2)) = 1 + \left(\frac{\alpha}{2} - \frac{1}{\pi}\right)\left(\frac{r-l}{r+l}\right)^2 \left(\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\right)^2. \tag{5.22}$$

We see from (5.22) that the variance of $S_n(2)$ is $\leqq 1$ if and only if $\alpha \leqq 2/\pi$ and checks with known results as $\alpha \to 0$, $\alpha \to 2$, and when $r = l$.

When $r = l$, $S_n(2)$ is symmetric about zero and as $n \to \infty$, we find in the same way,

$$\begin{aligned} \lim ES_n^2(2) &= 1 \\ \lim ES_n^4(2) &= 1 + \alpha \\ \lim ES_n^6(2) &= 1 + 3\alpha + 2\alpha^2 \\ \lim ES_n^8(2) &= \tfrac{1}{3}(3 + 20\alpha + 34\alpha^2 + 17\alpha^3)\,. \end{aligned} \tag{5.23}$$

In the same way as (5.21) and (5.22) were obtained, we find for $\alpha = 1$, $-\infty < \mu < \infty$,

$$\lim_{n\to\infty} E\tilde{S}_n(2) = \int_{-\infty}^{\infty} xf(x, \mu)\, dx = \frac{\mu}{\pi} \tag{5.24}$$

$$\lim_{n\to\infty} \sigma^2(\tilde{S}_n(2)) = \int_{-\infty}^{\infty} \left(x - \frac{\mu}{\pi}\right)^2 f(x, \mu)\, dx = 1 + \frac{\mu^2}{\pi}\left(\frac{1}{2} - \frac{1}{\pi}\right). \tag{5.25}$$

Numerical analysis of (4.21). In computing $f(y)$ from (4.21) by numerical integration, the rather slow rate of convergence was increased by subtracting an asymptotic expression for the tail from the original integrand. The asymptotic expression, which could be integrated in closed form, was obtained from the properties of $D_\alpha(z)$.

For $|t| \leqq 4$, $\mathscr{D}(t)$ was found by using the power series for $D_\nu(t)$ in (4.15); for $|t| > 4$ the asymptotic series for $\mathscr{D}(t)$ was used. Both series were stopped when the absolute value of the last computed term became less than 0.0001. "Single precision" computation with a machine error of 1 part in 10^8 was used.

Numerical analysis of (5.1). To compute the poles of $\mathscr{D}$ one must compute the zeros of the expression in (5.2) which we shall call $D(t)$.

The zeros of D nearest the origin may be estimated by the roots of polynomials obtained by truncating the power series of D. More exact results can be found using tables of D_ν. In this way we found for the case $\alpha = 1$, $\mu = 0$, which was our original case of interest, that $\tau_0 \sim 1.31$.

A zero of D, $\hat{\beta}$, with large modulus may be estimated by the roots of the asymptotic expansion of D using (5.14) and (4.9). More exact values of β were computed by Newton's method. Thus for small values of $\beta - \hat{\beta}$,

$$D(\beta) \approx D(\hat{\beta}) + (\beta - \hat{\beta})\left(\frac{dD}{d\beta}\right)_{\hat{\beta}}. \tag{5.26}$$

Setting $D(\beta) = 0$ and approximating the derivative by $(D(\hat{\beta} + \Delta) - D(\hat{\beta}))/\Delta$ gives the iteration formula

$$\beta = \hat{\beta} + D(\hat{\beta})\Delta/(D(\hat{\beta}) - D(\hat{\beta} + \Delta)). \tag{5.27}$$

In the calculations the increment Δ was set equal to 0.01, and the initial value of $\hat{\beta}$ (in the case $\alpha = 0$, $\mu = 0$) for the nth zero in modulus, β_n, was taken as

$$\beta_n \approx i^{\frac{1}{2}}(4\pi n + \pi/2)^{\frac{1}{2}} \tag{5.28}$$

obtained as described above by setting the asymptotic expansion of D equal to zero. It was found that about 10 iterations were required to reduce $|D(\beta)|$ to less than .0001 (the accuracy to which $D(\beta)$ is known).

6. Graphing the limiting density f. In Fig 1, $p = 1$. See (3.7). The curve with the local maximum represents f in the case $p = 1$, $\alpha = .8$ $r/l = 1.5$. For $p = 1$, $.5 < \alpha < 1$, f seems to always have a local maximum. The other curve represents f in the case $p = 1$, $\alpha = .5$, $r/l = 3$. For $p = 1$, $0 < \alpha \leqq .5$, f seems to have no local maximum. Both of these assertions are unproved but are conjectured on the basis of numerical evidence.

In Fig. 2, $p = 2$, $\alpha = .15$, $l = 0$, $f(x)$ is plotted on semi-log scale for $1 < x < 2$ to show the bumps near $2^{\frac{1}{2}}$ and $3^{\frac{1}{2}}$. The value 2.03 for $f(2^{\frac{1}{2}})$ was actually computed for $x = 1.414000$. It may differ somewhat from the true $f(2^{\frac{1}{2}})$ because of the rapid change in $f(x)$. The bump at $3^{\frac{1}{2}}$ was not apparent for $\alpha = .30$. It seems consistent with the proposed explanation of this phenomenon given in the introduction that the bumps would be more pronounced for smaller α, which seems to be the case.

In Fig. 3, graphs of f are shown in the cases $p = 2$, $\alpha = .5$, with r/l having each of the values 1, $1 + 2^{\frac{1}{2}}$, ∞. The value $1 + 2^{\frac{1}{2}}$ was used because during the computational stage of the work reported here we were using the more usual [5], (α, γ) normalization of the stable distributions; the case $r/l = 1 + 2^{\frac{1}{2}}$ corresponds to $\gamma = .25$. Note that f is symmetric only for $r/l = 1$, but is always symmetric in $(-1, 1)$. Note also the bump of f near $\pm 2^{\frac{1}{2}}$, most pronounced in the $r/l = \infty$ case where there is even a local maximum near 1.41. Note finally that $f \equiv 0$ for $x < 1$ when $r/l = \infty$ because $l = 0$ and $\alpha < 1$. See remark below (5.12).

In Fig. 4, $f_\infty(x) \equiv \tilde{f}(x, 0)$, that is, f_∞ is the density in the case $p = 2$, $\alpha = 1$, $\mu = 0$. The other curves, f_2 and f_3, are the densities of $\tilde{S}_2(2)$, $\tilde{S}_3(2)$ in the case $\alpha = 1$, $\mu = 0$. Note that f_2 and f_3 show bumps at $x = \pm 1$, $\pm 2^{\frac{1}{2}}$ but f_∞ only shows a bump at $x = \pm 1$.

In Fig. 5, the three graphs of f correspond to the case $p = 2$, $\alpha = 1.5$, with

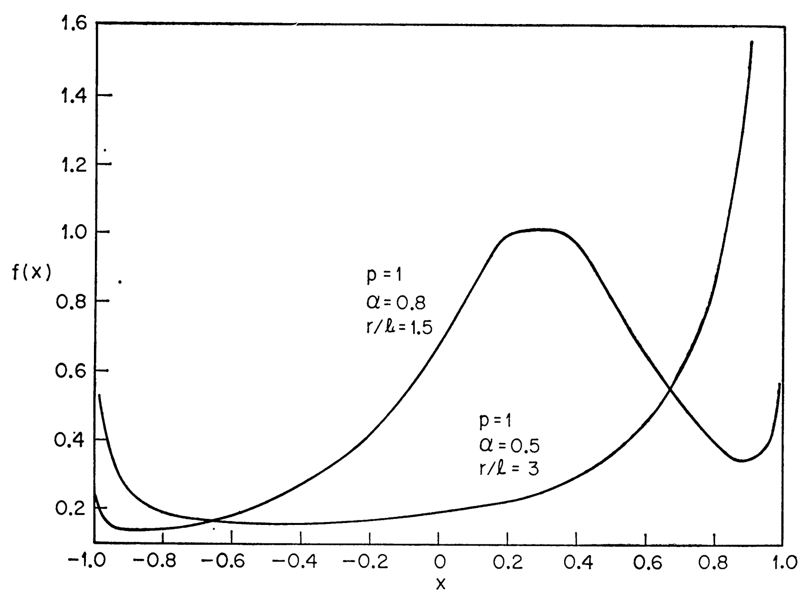

FIG. 1. Probability density $f(x)$ for $p = 1$, $\alpha = .8$, $r/l = 1.5$ and $p = 1$, $\alpha = .5$, $r/l = 3$. For $|x| > 1$, $f(x)$ is zero (see (3.7)).

r/l having each of the values 1, $1 + 2^{\frac{1}{2}}$ ∞. The symmetry in $(-1, 1)$ is quite clear. Note the bumps of f at ± 1 are now finite and disappear in the case $r/l = \infty$.

As α increases to 2, the bumps of f at ± 1 disappear and f tends to the normal density (5.13). For $p = 2$, $\alpha = 1.99$ and $r/l = 1$, $1 + 2^{\frac{1}{2}}$, ∞ the graph of f is indistinguishable from that of the normal density.

Summarizing the case $p = 2$ qualitatively we see: as α increases from 0 to 1 there are noticeable finite bumps in f near $x = \pm 3^{\frac{1}{2}}$ (disappearing before $\alpha = .3$); there are bumps near $x = \pm 2^{\frac{1}{2}}$ (disappearing about $\alpha = .5$); there is an infinite singularity at $x = \pm 1$ (thinning out as $\alpha \uparrow 1$ but remaining at $\alpha = 1$). In the above we have assumed $r > 0$ and $l > 0$; in the cases $r = 0$ or $l = 0$, f is one-sided (if $r = 0$, $f(x) \equiv 0$ for $x > -1$). As α increases beyond one, the bumps at $x = \pm 1$, now finite, disappear between $\alpha = 1.5$ and $\alpha = 1.99$. f evolves into the normal curve.

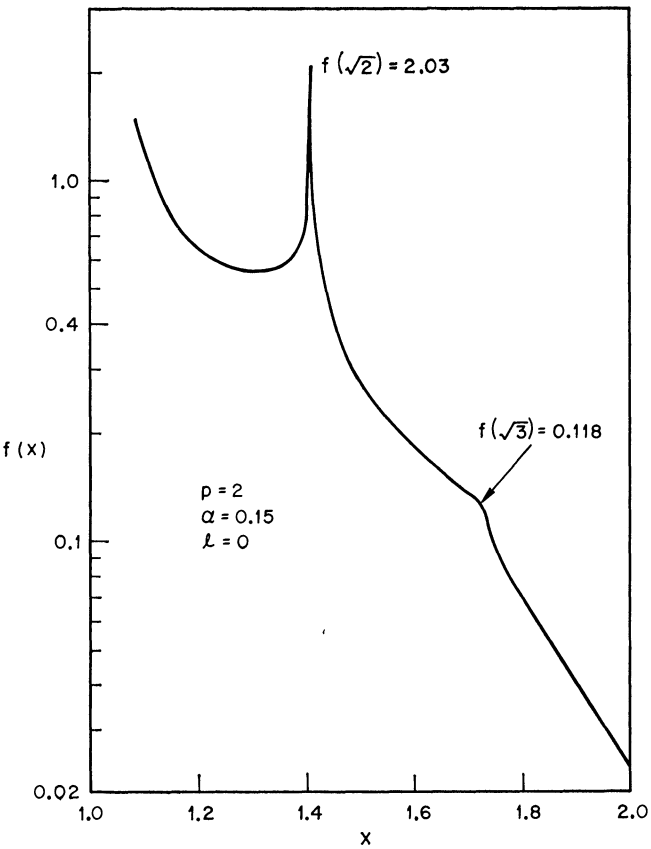

FIG. 2. Probability density $f(x)$ for $p = 2$, $\alpha = 0.15$, and $l = 0$. Note the bumps at $x = 2^{\frac{1}{2}}$ and $3^{\frac{1}{2}}$. When $\alpha < 1$ and $l = 0$, $f(x) \equiv 0$ for $x < 1$ and $f(x) \sim \pi^{-1}(x - 1)^{\alpha-1} \sin \pi\alpha$ as $x \to 1$ from above (see (5.9)).

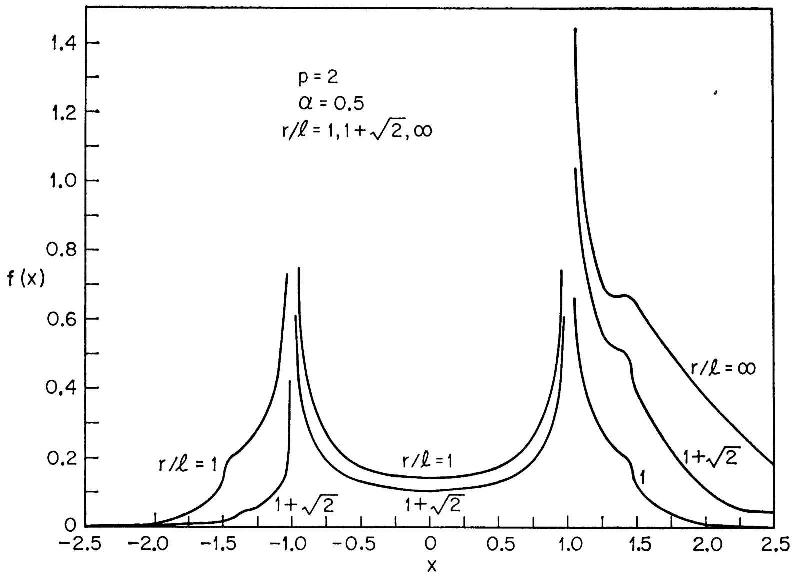

FIG. 3. Probability density $f(x)$ for $p = 2$, $\alpha = 0.5$, and $r/l = 1,\ 1 + 2^{\frac{1}{2}},\ \infty$. Note the symmetry for $|x| < 1$. The behavior of $f(x)$ near $x = 1$ is given by (5.9) and (5.11). As $x \to \infty$, $f(x)$ approaches the Gaussian tail (5.8) with $\tau_0 = 1.65, 1.18, 0.77$, for $r/l = 1,\ 1 + 2^{\frac{1}{2}}$, ∞ respectively. As in Fig. 2, $f(x) = 0$ if $x < 1$ and $r/l = \infty$.

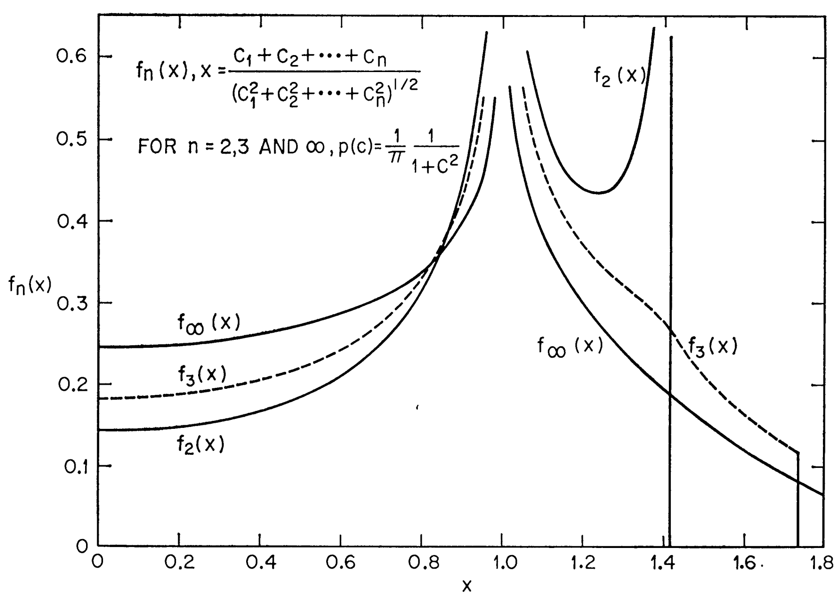

FIG. 4. Probability density $f_n(x)$ of $\tilde{S}_n(2)$, $n = 2, 3, \infty$ and $\alpha = 1$. When $\alpha = 1$, X_i becomes a Cauchy random variable ((4.32) to (4.41)) and $f_\infty(x) \equiv \tilde{f}(x, 0)$. As $x \to 1$, $f_\infty(x) = -\pi^{-2} \log |1 - x^2| + 0.226 \cdots + o(1)$ and as $x \to \infty$, $f_\infty(x)$ asymptotically approaches the Gaussian tail (5.8) with $\alpha = 1$ and $\tau_0 = 1.31 \cdots$.

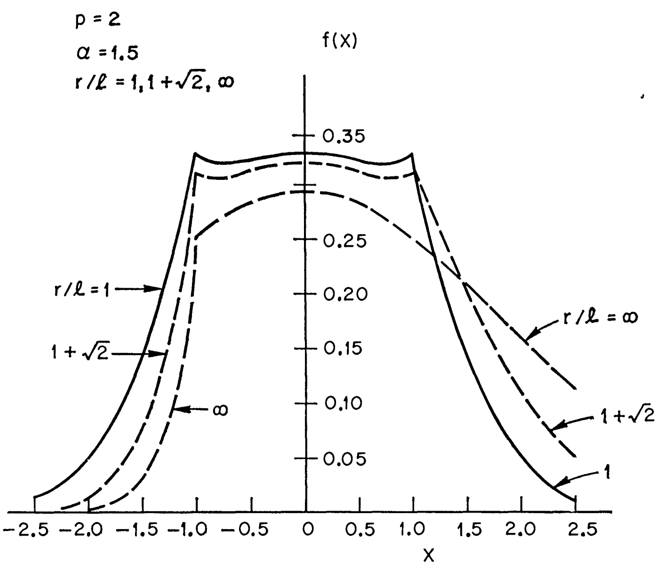

FIG. 5. Probability density $f(x)$ for $p = 2$, $\alpha = 1.5$ and r/l the same as in Fig. 3. Again note the symmetry for $|x| < 1$. If α exceeds 1, $f(x)$ is continuous at $x = \pm 1$ (see discussion below (5.18)). As $x \to \infty$, $f(x)$ approaches the Gaussian tail (5.8).

7. The poles of $\mathscr{D}$. We begin the proof Lemma A of Section 5 with the case $0 < \alpha < 1$. Suppose that $t = -i\tau$ is a zero of (5.2) with $\tau \geqq 0$, so that

$$rD_\alpha(\tau) + lD_\alpha(-\tau) = 0 . \tag{7.1}$$

We may use (4.5) and (4.6) to obtain an integral representation of D_α for $0 < \alpha < 1$. Doing this, (7.1) becomes

$$\int_0^\infty e^{-u^2/2}u^{-\alpha-1}[r(e^{\tau u} - 1) + l(e^{-\tau u} - 1)]\, du = 0 . \tag{7.2}$$

We note that the left side is a convex function of τ which is negative at $\tau = 0$. Thus (7.2) and also (7.1) has one and only one zero for $\tau \geqq 0$.

For $1 < \alpha < 2$, (7.1) becomes after a two-fold use of (4.6) and (4.5).

$$\int_0^\infty [r(e^{\tau u - u^2/2} - 1 - \tau u) + l(e^{-\tau u - u^2/2} - 1 + \tau u)]u^{-\alpha-1}\, du = 0 \tag{7.3}$$

and the existence and uniqueness of τ_0 follows from convexity in the same way as before.

For $\alpha = 1$, $\tilde{\mathscr{D}}(-i\tau) = \infty$ if and only if

$$(1/\tau)e^{\tau^2/2} - \int_0^\tau e^{v^2/2}\, dv = \mu/(2\pi)^{\frac{1}{2}} \tag{7.4}$$

from (4.35). The left side of (7.4) strictly decreases from ∞ to $-\infty$ in $0 \leqq \tau \leqq \infty$ and so Lemma A also holds for $\alpha = 1$.

To prove Lemma B, we first determine the approximate location of the poles t of large modulus in the region $\{-\pi/2 \leqq \arg t \leqq -\pi/4\}$, so that

$$t = \rho e^{-i\theta}/i^{\frac{1}{2}} , \qquad 0 \leqq \theta \leqq \pi/4, \quad \rho \text{ large.} \tag{7.5}$$

Using (5.14) and (4.9) we see that as $\rho \to \infty$,

$$rD_\alpha(-it) + lD_\alpha(it) \sim C((it)^{\frac{1}{2}})^\alpha e^{t^2/4}(1 - b((it)^{\frac{1}{2}})^{-2\alpha-1}e^{-t^2/2}) \tag{7.6}$$

where C is given by (5.10) and

$$b = i^{\frac{3}{2}} r(2\pi)^{\frac{1}{2}}/C\Gamma(1-\alpha)\,. \tag{7.7}$$

In terms of ρ and θ, the right side of (7.6) is zero if and only if for some integer n,

$$\tfrac{1}{2}\rho^2 \sin 2\theta - (2\alpha+1)\log\rho + \log|b| = 0 \tag{7.8}$$

and

$$\tfrac{1}{2}\rho^2 \cos 2\theta + (2\alpha+1)\theta + \arg b = 2\pi n\,. \tag{7.9}$$

Denote by ρ_n, θ_n a solution of (7.8) and (7.9) for a given large value of n. It follows easily that

$$\rho_n \sim (4\pi n - 2(\arg b))^{\frac{1}{2}} \tag{7.10}$$

$$\theta_n \sim \frac{1}{2}\,\frac{(2\alpha+1)\log 4\pi n}{4\pi n} \tag{7.11}$$

Choosing $\rho = \rho_n{}^*$ between the ρ_n's say

$$\rho_n{}^* = (4\pi n - 2(\arg b) + 2\pi)^{\frac{1}{2}} \tag{7.12}$$

should cause the contour C_2 of Lemma B to fall in the valley between the poles of $\mathscr{D}$ near t_n and t_{n+1}, where

$$t_n = \rho_n e^{-i\theta_n}/i^{\frac{1}{2}}\,, \qquad n = 1, 2, \cdots\,. \tag{7.13}$$

We verify from (5.14) and (4.9) that for $t = \rho_n{}^* e^{-i\theta}/i^{\frac{1}{2}}$, $0 \leqq \theta \leqq \pi/4$,

$$|\mathscr{D}(t)| \leqq (\text{const.}) \cdot \rho^{1-2\alpha} e^{\frac{1}{2}\rho^2 \sin 2\theta}\,, \qquad \theta \leqq \theta_n \tag{7.14}$$

$$|\mathscr{D}(t)| \leqq (\text{const.})\rho^{-2}\,, \qquad \theta > \theta_n\,. \tag{7.15}$$

Thus with

$$C_2 = \{\rho_n{}^* e^{-i\theta}/i^{\frac{1}{2}} : 0 \leqq \theta \leqq \pi/4\} \tag{7.16}$$

we have, writing $\rho_n{}^* = \rho$ for simplicity,

$$\begin{aligned} \left|\int_{C_2} e^{y^2t^2/2}\mathscr{D}(t)\,dt\right| &\leqq \rho \int_0^{\pi/4} e^{-\frac{1}{2}y^2\rho^2 \sin 2\theta}\mathscr{D}(\rho e^{-i\theta}/i^{\frac{1}{2}})\,d\theta \\ &\leqq (\text{const.}) \cdot \Big[\rho^{2-2\alpha}\int_0^{\theta_n} e^{\rho^2(1-y^2)\theta}\,d\theta \\ &\quad + \frac{1}{\rho}\int_0^{\pi/4} e^{-\frac{1}{2}y^2\rho^2\sin 2\theta}\,d\theta\Big]\,. \end{aligned} \tag{7.17}$$

The last term clearly tends to zero as $\rho = \rho_n{}^* \to \infty$ for any y, and if $y > 1$ so does the first term. If $y < 1$, we have

$$\rho^{2-2\alpha}\int_0^{\theta_n} e^{\rho^2(1-y^2)\theta}\,d\theta = \frac{\rho^{-2\alpha}}{1-y^2}\left(e^{\rho^2\theta_n(1-y^2)} - 1\right). \tag{7.18}$$

From (7.11) and (7.12) since $\rho = \rho_n^*$,

$$\rho^2\theta_n = (\rho_n^*)^2\theta_n \sim \tfrac{1}{2}(2\alpha + 1)\log n \tag{7.19}$$

we see that (7.18) tends to zero if and only if

$$y > (2\alpha + 1)^{-\frac{1}{2}}. \tag{7.20}$$

proving Lemma B.

Lemma B and (5.4) show that (5.1) converges conditionally whenever (7.20) holds and the integral (4.21) converges conditionally. The latter converges for all $y > 0$ except $y = 1$. Thus the statement after (5.1) about conditional convergence of (5.1) is proved.

We next show that (5.1) converges absolutely for

$$y > [3/(2\alpha + 1)]^{\frac{1}{2}}. \tag{7.21}$$

We note that the poles t_n can be written in the form (7.13) where ρ_n and θ_n satisfy (7.10) and (7.11). Thus the modulus of the term of (5.1) containing t_n is, to within a constant multiple,

$$\exp[-\tfrac{1}{2}y^2\rho_n^2 \sin 2\theta]n_{\rho_n} \sim (\text{const.}) \cdot n^{\frac{1}{2}(1-y^2(2\alpha+1))}. \tag{7.22}$$

The power of n on the right of (7.22) must be less than minus one for absolute convergence, which is the case if and only if (7.21) holds. This proves the assertion about absolute convergence of (5.1) and provides an alternate proof of the finiteness of $f(y)$ at $y = 1$ for $1 < \alpha < 2$.

The proof of Lemma B and the convergence of (5.1) in the case $\alpha = 1$ is completely similar and is omitted.

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN I. A. (1964). *Handbook of Mathematical Functions.* National Bureau Standards No. 55. Washington.

[2] DARLING, D. A. (1952). The influence of the maximum term in the addition of independent random variables. *Trans. Amer. Math. Soc.* **73** 95–107.

[3] EFRON, BRADLEY (1969). Students t-test under symmetry conditions. *J. Amer. Statist Assoc.* **64** 1278–1302.

[4] ERDÉLYI *et. al.*, (1954). *Higher Transcendental Functions* **2.** McGraw Hill, New York.

[5] FELLER, WILLIAM (1966). *An Introduction to Probability Theory and Its Applications* **2**, Wiley, New York.

[6] HOTELLING, H. (1961). The behavior of some standard statistical tests under non-standard conditions. *Proc. Fourth Berkeley Symp. Math. Statist.* Prob. **1** 319–360.

[7] MAGNUS, WILHELM and OBERHETTINGER, FRITZ (1954). *Formulas and Theorems for the Functions of Mathematical Physics.* Chelsea, New York.

BELL LABORATORIES
600 MOUNTAIN AVENUE
MURRAY HILL, NEW JERSEY 07974