The Annals of Probability
1973, Vol. 1, No. 6, 1068–1070

A BOUND ON TAIL PROBABILITIES FOR QUADRATIC FORMS IN INDEPENDENT RANDOM VARIABLES WHOSE DISTRIBUTIONS ARE NOT NECESSARILY SYMMETRIC[1]

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Let $\{X_i\}_{i=-\infty}^{\infty}$ be a sequence of independent random variables with zero means and let $P[|X_i| \geqq x] \leqq M \int_x^\infty \exp\{-\gamma t^2\}\, dt$ for all $x \geqq 0$ where M and γ are positive constants. Let $((a_{ij}))_{i,j=-\infty}^{\infty}$ be an infinite matrix of real numbers with $a_{ij} = a_{ji}$ for all i, j and $\Lambda^2 = \sum_{i,j} a_{ij}^2 < \infty$. Let $||A||$ be the norm of $A = ((|a_{ij}|))$ considered as an operator on l_2 and set $S = \sum_{i,j} a_{ij}(X_i X_j - E(X_i X_j))$. In this note it is shown that there exist positive constants C_1 and C_2 depending only on M and γ such that $P[S \geqq \varepsilon] \leqq \exp\{-\min(C_1 \varepsilon/||A||, C_2 \varepsilon^2/\Lambda^2)\}$ for all $\varepsilon > 0$. This result has previously been established in the literature for sequences of random variables which have symmetric distributions.

Let $\{X_i\}_{i=-\infty}^{\infty}$ be a sequence of independent random variables with zero means and suppose there exist positive constants M and γ such that

$$P[|X_i| \geqq x] \leqq M \int_x^\infty \exp\{-\gamma t^2\}\, dt$$

for all i and $x \geqq 0$. Let a_{ij} be real numbers for $i, j = 0, \pm 1, \pm 2, \cdots$ with $a_{ij} = a_{ji}$ for all i, j and $\Lambda^2 = \sum_{i,j} a_{ij}^2 < \infty$. We will denote by $||A||$ the norm of $A = ((|a_{ij}|))_{i,j=-\infty}^{\infty}$ considered as an operator on l_2. Set

$$S_N = \sum_{i,j=-N}^{N} a_{ij}(X_i X_j - E(X_i X_j))$$

and let S be the limit of S_N as $N \to \infty$ where it exists.

The main purpose of this note is to prove the following theorem which shows that the bound established by Hanson and Wright [3] for tail probabilities for quadratic forms in independent symmetric random variables also holds when the random variables are not symmetric.

THEOREM. *With the above assumptions, S exists as an almost sure and a quadratic mean limit of the sequence $\{S_N\}$ and there exist constants C_1 and C_2 depending only on M and γ such that for every $\varepsilon > 0$*

$$P[S \geqq \varepsilon] \leqq \exp\{-\min(C_1 \varepsilon/||A||, C_2 \varepsilon^2/\Lambda^2)\}. \tag{1}$$

PROOF. The proof given in [3] for the almost sure and quadratic mean convergence of S_N does not require that the random variables have symmetric distributions. Clearly, $\Lambda_N^2 = \sum_{i,j=-N}^{N} a_{ij}^2 \to \Lambda^2$. On page 1083 of [3] it is shown

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Received March 5, 1973.

[1] This research was supported by the National Science Foundation under Grant GP-30966XX.

AMS 1970 *subject classifications.* Primary 60F05; Secondary 60E05.

Key words and phrases. Quadratic forms, tail probabilities.

that $||A_N||$, the maximum of the absolute values of the eigenvalues of $A_N = ((|a_{ij}|))_{i,j=-N}^N$, converges to $||A||$. So for the second conclusion we will show that for each N,

$$P[S_N > \varepsilon] \leqq \exp\{-\min(C_1\varepsilon/||A_N||, C_2\varepsilon^2/\Lambda_N^2)\}.$$

Since S_N converges to S it follows that $P[S > \varepsilon]$ is bounded above by the right-hand side of (1), and then (1) itself follows because the upper bound given there is a continuous function of ε.

For each i, let X_i', X_i'', and X_i''' all have the same distribution as X_i and let the collection $\{X_i, X_i', X_i'', X_i''': i = 0, \pm 1, \pm 2, \cdots\}$ be independent. (These random variables can all be defined on an enlarged probability space if necessary.) For each i, let Y_i denote the symmetric random variable $X_i - X_i'$ and set $S_N' = \sum_{i,j=-N}^N a_{ij}(X_i'X_j' - E(X_i'X_j'))$, $S_N^{(s)} = \sum_{i,j=-N}^N a_{ij}(Y_i Y_j - E(Y_i Y_j))$ and $T_N = 2\sum_{i,j=-N}^N a_{ij} X_i X_j'$. For $\theta \geqq 0$, $P[S_N > \varepsilon] \leqq e^{-\varepsilon\theta}E(e^{\theta S_N})$. Since $E(S_N') = 0$, Jensen's inequality yields $E(e^{\theta S_N'}) \geqq 1$ for all θ and so $E(e^{\theta S_N}) \leqq E(e^{\theta(S_N+S_N')}) \leqq [E(e^{2\theta(S_N+S_N'-T_N)})E(e^{2\theta T_N})]^{\frac{1}{2}} = [E(e^{2\theta S_N^{(s)}})E(e^{2\theta T_N})]^{\frac{1}{2}}$. Next we observe that for $x \geqq 0$, $P[|Y_i| \geqq x] \leqq 2P[|X_i| \geqq x/2] \leqq M\int_x^\infty \exp\{-\gamma t^2/4\}\,dt$ and so (using the argument given in [3] to establish expression (13)) it can be shown that

$$(2) \qquad e^{-\varepsilon\theta}[E(e^{2\theta S_N^{(s)}})]^{\frac{1}{2}} \leqq \exp\{-\varepsilon\theta + 2C\theta^2\lambda^4\Lambda_N^2\}$$

for $0 \leqq 2\theta\lambda^2||A_N|| \leqq \tau$, where C and τ are positive constants which do not depend on $\{X_i\}$ or $\{a_{ij}\}$, and where λ depends only on M and γ.

To complete the proof we need to bound $E(e^{\theta T_N})$. In what follows we will make repeated use of the following result: if ϕ is a Borel function, if $X_1, \cdots, X_n$ are independent, if $E|\phi(X_1, X_2, \cdots, X_n)| < \infty$ and if $1 \leqq k \leqq n$, then

$$E(\phi(X_1, \cdots, X_n) \mid X_1, \cdots, X_k) = \psi(X_1, \cdots, X_k)$$

almost surely where $\psi(x_1, \cdots, x_k) = E(\phi(x_1, \cdots, x_k, X_{k+1}, \cdots, X_n))$. Conditioning on $X_{-N}, \cdots, X_N$, we obtain

$$E(e^{\theta T_N}) = E(E(\exp\{2\theta \textstyle\sum_{i,j=-N}^N a_{ij} X_i X_j'\}) \mid X_{-N}, X_{-N+1}, \cdots, X_N)$$

and since $E(\exp\{-2\theta \sum_{i,j=-N}^N a_{ij} x_i X_j''\}) \geqq 1$,

$$(3) \qquad \begin{aligned} E(e^{\theta T_N}) &\leqq E(E(\exp\{2\theta \textstyle\sum_{i,j=-N}^N a_{ij} X_i(X_j' - X_j'')\}) \mid X_{-N}, X_{-N+1}, \cdots, X_N) \\ &= E(\exp\{2\theta \textstyle\sum_{i,j=-N}^N a_{ij} X_i(X_j' - X_j'')\}). \end{aligned}$$

Repeating this argument conditioning this time on $X'_{-N} - X''_{-N}, X'_{-N+1} - X''_{-N+1}, \cdots, X_N' - X_N''$, it can be shown that (3) is bounded above by $E(\exp\{2\theta \sum_{i,j=-N}^N a_{ij}(X_i - X_i''')(X_j' - X_j'')\})$. With $U_i = X_i - X_i'''$ and $V_j = X_j' - X_j''$, we observe that $E(e^{\theta T_N}) \leqq \sum_{k=0}^\infty (2\theta)^k E(\sum_{i,j=-N}^N a_{ij} U_i V_j)^k/k!$ for $|\theta|$ less than the radius of convergence of this series. Modifying slightly the argument from (8) to (10) on page 1082 of [3], it can be shown that this series is bounded above by

$$(4) \qquad E(\exp\{2\theta\lambda^2 \textstyle\sum_{i,j=-N}^N |a_{ij}| Z_i Z_j'\})$$

for $0 \leqq \theta \leqq \tau_N$ where τ_N is any number chosen so that expression (4) is finite for $|\theta| \leqq \tau_N$ and $Z_{-N}, Z'_{-N}, \cdots, Z_N, Z_N'$ are i.i.d. $\mathscr{N}(0, 1)$ variables. Denote by Z and Z' the vectors $(Z_{-N}, \cdots, Z_N)$ and $(Z'_{-N}, \cdots, Z_N')$, respectively, and choose an orthogonal matrix D such that $D^T A_N D = B = ((b_{ij}))$ is diagonal. Now $b_i = b_{ii}$ for $i = -N, \cdots, N$ are the eigenvalues of A_N. If we let $W = ZD$ and $W' = Z'D$ then $\sum_{i,j=-N}^N |a_{ij}| Z_i Z_j' = Z A_N Z'^T = ZD(D^T A_N D)D^T Z'^T = \sum_{i=-N}^N b_i W_i W_i'$. Since D is orthogonal, $W_{-N}, W'_{-N}, \cdots, W_N, W_N'$ are i.i.d. $\mathscr{N}(0, 1)$ variables.

Using Theorem 2.5 of Griffiths, Platt and Wright [1] and then Lemma 3 of Hanson [2], we can find constants C_1 and τ_1 not depending on $\{X_i\}$ or $\{a_{ij}\}$ such that $E(e^{\theta W_i W_i'}) \leqq e^{C_1 \theta^2}$ for $|\theta| \leqq \tau_1$. Hence

$$[E(e^{2\theta T_N})]^{\frac{1}{2}} \leqq [E(\exp\{4\theta\lambda^2 \textstyle\sum_{i=-N}^N b_i W_i W_i'\})]^{\frac{1}{2}} \leqq \exp\{8C_1\lambda^4\theta^2 \textstyle\sum_{i=-N}^N b_i^2\}$$

for $0 \leqq 4\theta\lambda^2 ||A_N|| \leqq \tau_1$. In [3] it was shown that $\sum_{i=-N}^N b_i^2 = \Lambda_N^2$ and so, combining the above result with (2), we see that $P[S_N > \varepsilon] \leqq \exp\{-\varepsilon\theta + C^*\theta^2\Lambda_N^2\}$ for $0 \leqq \theta \leqq \tau^*/||A_N||$ where $C^* = 10\lambda^4 \max(C, C_1)$ and $\tau^* = \min(\tau/2, \tau_1/4)/\lambda^2$. Choosing $\theta_0 = \min(\varepsilon/2C^*\Lambda_N^2, \tau^*/||A_N||)$, we see that

$$\begin{aligned} P[S_N > \varepsilon] &\leqq \exp\{-\theta_0(\varepsilon - C^*\theta_0\Lambda_N^2)\} \leqq \exp\{-\theta_0\varepsilon/2\} \\ &= \exp\{-\min(\varepsilon^2/4C^*\Lambda_N^2, \varepsilon\tau^*/2||A_N||)\}. \end{aligned}$$

The proof is completed upon observing that $C_1 = \tau^*/2$ and $C_2 = 1/4C^*$ do not depend on $\{a_{ij}\}$ and depend on $\{X_i\}$ only through M and γ.

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