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A NOTE ON MARTINGALE SQUARE FUNCTIONS

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In this note we show that the Littlewood–Paley argument also applies to the martingale square function.

Let $(\Omega, \mathscr{A}, P)$ be a probability space. Let $f = (f_1, f_2, \cdots)$ be a martingale relative to a non-decreasing sequence of sub-σ-fields of $\mathscr{A}$: $\mathscr{A}_1, \mathscr{A}_2, \cdots$ and $\{d_k\}_{k \geqq 1}$ be the difference sequence of f. (i.e., $f_n = \sum_{k=1}^n d_k$, $n \geqq 1$.) The square function of f is $S(f) = (\sum_{k=1}^\infty d_k^2)^{\frac{1}{2}}$ and the maximal function of f is $f^* = \sup_n |f_n|$. Denote $S_n(f) = (\sum_{k=1}^n d_k^2)^{\frac{1}{2}}$ and $f_n^* = \sup_{1 \leqq k \leqq n} |f_k|$, $n \geqq 1$. For $0 < p < \infty$, let $||f||_p = \sup_n ||f_n||_p$ where $||f_n||_p = (E|f_n|^p)^{1/p}$. f is said to be L^p-bounded if $||f||_p$ is finite.

THEOREM. *If f is an L^p-bounded martingale with $1 < p < \infty$, then there exists an $A_p > 0$ such that $||S(f)||_p \leqq A_p ||f||_p$.*

The result was established by Burkholder [1]. (See also [2].) $S(f)$ was referred in Stein [4] as the Littlewood–Paley function of the martingale f. Our purpose is to show that the argument of proving the similar inequality for the Littlewood–Paley function in the Lie group setting ([4]; Chapter II) can be also applied here. It should be noted that the studies in both settings originated in Paley [3]. Our result gives further insight into the subject. Note also that Taibleson [5] applied the same idea in the local field setting where a sort of regular martingale was considered.

We need the following preliminary results:

$$||f^*||_p \leqq p' ||f||_p \quad \text{for} \quad 1 < p < \infty, \quad \text{where} \quad 1/p + 1/p' = 1. \tag{1}$$

$$P(S(f) > \lambda) \leqq 3\lambda^{-1} ||f||_1, \qquad \lambda > 0. \tag{2}$$

(1) is Doob's inequality; (2) follows immediately from the identity lemma of Burkholder [2].

PROOF. From the orthogonality of $\{d_k\}$ it follows that $||S(f)||_2 = ||f||_2$. Applying the Marcinkiewicz interpolation theorem to this and (2), we have, for $1 < p < 2$,

$$||S(f)||_p \leqq B_p ||f||_p \qquad \text{for some } B_p > 0. \tag{3}$$

Now suppose $p > 4$ and let q be the conjugate index of $p/2$. Let $\{e_k\}_{k \geqq 1}$ be the martingale difference sequence of $g \in L^q$ with $||g||_q = 1$, i.e., $g_n = \sum_{k=1}^n e_k$ where $g_n = E_n g$ is the conditional expectation of g relative to $\mathscr{A}_n$. Since $1 <$

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$q < 2$, by (3) we have

$$\|S(g)\|_q \leqq B_q \|g\|_q = B_q . \tag{4}$$

For an L^p-bounded martingale f, since $E|f_{k-1} f_k g_{k-1}| \leqq \|f\|_p^2 \|g\|_q < \infty$, $k > 1$, we have $E(f_{k-1} f_k g_{k-1}) = EE_{k-1}(f_{k-1} f_k g_{k-1}) = E[f_{k-1} g_{k-1} E_{k-1}(f_k)] = E(f_{k-1}^2 g_{k-1})$ and similarly, $E(f_{k-1} d_k g) = E(f_{k-1} d_k g_k)$. Hence, for $k > 1$,

$$\begin{aligned} E(f_{k-1} d_k e_k) &= E[f_{k-1}(f_k g_k - f_{k-1} g_k - f_k g_{k-1} + f_{k-1} g_{k-1})] \\ &= E[f_{k-1} f_k g_k - f_{k-1}^2 g_k] \\ &= E(f_{k-1} d_k g) . \end{aligned}$$

Thus, by repeatedly applying Hölder's inequality with (1) and (4) (and the convention of $f_0 = 0$), we have

$$\begin{aligned} |E[S_n^2(f) g]| &= |[E(f_n^2 - 2 \textstyle\sum_{k=1}^n f_{k-1} d_k) g]| \\ &\leqq |E(f_n^2 g)| + 2|E(\textstyle\sum_{k=1}^n f_{k-1} d_k e_k)| \\ &\leqq \|f_n\|_p^2 \|g\|_q + 2E[f_n^* S_n(f) S_n(g)] \\ &\leqq \|f_n\|_p^2 + 2\|f_n^*\|_p \|S_n(f)\|_p \|S_n(g)\|_q \\ &\leqq \|f_n\|_p^2 + C_p \|f_n\|_p \|S_n(f)\|_p \qquad \text{where } C_p = 2p' B_q . \end{aligned}$$

So

$$\begin{aligned} \|S_n(f)\|_p^2 = \|S_n^2(f)\|_{p/2} &= \sup_{g \in L^q, \|g\|_q = 1} |E[S_n^2(f) g]| \\ &\leqq \|f_n\|_p^2 + C_p \|f_n\|_p \|S_n(f)\|_p . \end{aligned}$$

Hence there exists $A_p > 0$ such that $\|S_n(f)\|_p \leqq A_p \|f_n\|_p$. Therefore, by letting $n \to \infty$, we have

$$\|S(f)\|_p \leqq A_p \|f\|_p \qquad \text{for } p > 4 .$$

Applying the Marcinkiewicz interpolation theorem again, we obtain the desired result.

REMARK. By tracing the constants in the proof, we have $A_p = O(p)$ as $p \to \infty$. This estimate is not as sharp as $A_p = O(p^{\frac{1}{2}})$ obtained in [2].

REFERENCES

[1] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[2] BURKHOLDER, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probability* **1** 19–42.

[3] PALEY, R. E. A. C. (1932). A remarkable series of orthogonal functions. I. *Proc. London Math. Soc.* **34** 241–279.

[4] STEIN, E. M. (1970). Topics in harmonic analysis related to the Littlewood–Paley theory. *Ann. Math. Studies* **63**, Princeton Univ.

[5] TAIBLESON, M. H. (1970). Harmonic analysis on n-dimensional vector spaces over local fields. II. Generalized Gauss kernels and the Littlewood–Paley function. *Math. Ann.* **186** 1–19.

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