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A TAUBERIAN THEOREM OF E. LANDAU AND W. FELLER[1]

BY E. SENETA

Princeton University

A simple proof, with a little-known extension, of a density version of Karamata's Tauberian theorem is presented, and the result applied to limit distributions of the Galton–Watson process.

The theorem given below is of frequent use in probabilistic contexts, in conjunction with a density version of Karamata's Tauberian theorem ([3] page 445), for Laplace transforms. It extends to the case $\rho = 0$ a result given in ([3] page 446), and we shall give an illustration of where this is useful. However, the main purpose of this note is to show that a totally elementary proof of the whole proposition can be given. Feller ([3] page 446, footnote) states "Our proof serves as a new example of how the selection theorem obviates analytical intricacies"; in the following proof a substantially lower level of sophistication suffices. For the historically important but restricted result of Landau, see ([4] pages 44–47); that proof is resembled by ours in approach.

We recall that a function L, defined, finite and positive on $[A, \infty)$ for some $A > 0$, is said to be slowly varying (at infinity) if it is measurable and satisfies $L(\lambda x)/L(x) \to 1$ as $x \to \infty$ for each $\lambda > 0$.

THEOREM. *Suppose U is defined and positive on $[B, \infty)$ for some B sufficiently large, and is given by*

$$U(x) = \int_B^x u(y)\,dy + U(B)$$

where u is nonnegative and ultimately monotone. Then for $\rho \geqq 0$, as $x \to \infty$,

$$U(x) = x^\rho L(x) \Rightarrow xu(x)/U(x) \to \rho \,.$$

PROOF. We shall suppose u is ultimately non-decreasing; in the non-increasing case the argument is analogous. Let $a < b$. Then for sufficiently large x

$$\frac{U(xb) - U(xa)}{U(x)} = \int_{xa}^{xb} \frac{u(y)}{U(x)}\,dy$$

so that

$$\frac{x(b-a)u(xb)}{U(x)} \geqq \frac{U(xb) - U(xa)}{U(x)} \geqq \frac{x(b-a)u(xa)}{U(x)} \tag{1}$$

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by monotonicity of u in the integrand. From the right-hand inequality

$$\frac{b^\rho - a^\rho}{b - a} \geqq \limsup_{x\to\infty} \frac{xu(xa)}{U(x)}$$

and letting $b \to a$ on the left-hand side

$$\rho a^{\rho-1} \geqq \limsup_{x\to\infty} \frac{xu(xa)}{U(x)} .$$

Similarly from the left-hand inequality in (1), letting $a \to b$,

$$\liminf_{x\to\infty} \frac{xu(xb)}{U(x)} \geqq \rho b^{\rho-1} .$$

Thus for any $c > 0$

$$\lim_{x\to\infty} \frac{xu(xc)}{U(x)} = \rho c^{\rho-1}$$

and replacing cx by y, say, completes the proof, since $U(y/c) \sim c^{-\rho}U(y)$. □

In the theory of the Galton–Watson branching process, a result of the author [5], ([2] Section 4) asserts that for a certain limit nonnegative random variable, W, in both the subcritical and supercritical cases,

$$\int_0^x P[W > y]\, dy = L(x) .$$

It follows from the above theorem that $P[W > x] = o(x^{-1}L(x))$, as $x \to \infty$.

(Another proof of the Theorem in the special case $\rho = 0$ was essentially given in ([1] pages 88–89).)

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Department of Statistics
Fine Hall
Princeton University
Princeton, New Jersey 08540