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CONTINUITY OF GAUSSIAN PROCESSES AND RANDOM FOURIER SERIES[1]

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This paper is mainly a survey of results on the problem of finding necessary and sufficient conditions for a Gaussian process to be continuous. The relationship between this problem and the same one for random Fourier series is explored.

Some new results are presented that give continuity conditions for stationary Gaussian processes in terms of the spectrum of the process. Let $X(t)$ be a real-valued stationary Gaussian process; $EX(t) = 0$, $EX^2(t) = 1$. Define F by the equation $EX(t+h)X(t) = \int^\infty \cos \lambda h \, dF(\lambda)$. Assume that $F(\lambda)$ is concave for $\lambda \geqq \lambda_0 > 0$ then $X(t)$ is continuous a.s. if and only if

$$\int^\infty \frac{(1 - F(x))^{\frac{1}{2}}}{x(\log x)^{\frac{1}{2}}} \, dx < \infty .$$

A similar result holds for Fourier series with normal coefficients.

In this paper we present some new results on the continuity of stationary Gaussian processes with concave spectrum and survey recent work on the continuity of Gaussian processes and random Fourier series. Some Gaussian processes are random Fourier series with normal coefficients. However, in Kahane's (1968) treatment of random Fourier series he obtains extensive results on continuity none of which depend upon the series being Gaussian. For Gaussian Fourier series we obtain somewhat sharper results but the proofs involve special properties which are unique to Gaussian processes. This raises some interesting questions about random Fourier series in general.

As a survey this paper is meant to be a sequel to Section 1 of Marcus and Shepp (1971). Many of the important basic results such as the Theorem of Belyaev (1961) and Eaves (1967) that are discussed there are not repeated. Surveys of progress in the study of Gaussian processes have appeared recently in Cramér and Leadbetter (1967), Kahane (1968), Garsia, Rodemich and Rumsey (1970), Fernique (1971), Marcus and Shepp (1971) and Dudley (1972). In this paper we consider only the problem of determining when the sample paths of Gaussian processes are continuous. Random Fourier series are introduced because the two subjects are interrelated and developments in one often lead to results in the other. We specialize in real-valued processes on R^1 although extensions to R^k are generally immediate.

Let $X(t)$ be a real-valued stationary Gaussian process with zero mean and

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$EX(t)^2 = 1$. Define

$$\gamma(h) = EX(t+h)X(t) = \int_0^\infty \cos \lambda h \, dF(\lambda) ; \qquad \sigma^2(h) = 2(1-\gamma(h)) .$$

$F(\lambda)$ is sometimes called the spectrum of $X(t)$. Suppose that there exists a λ_0 and that $F(\lambda)$ is concave for $\lambda_0 \leqq \lambda < \infty$. We obtain the following

THEOREM 1. *Under the above conditions $X(t)$ is* a.s. *continuous if and only if*

$$I(F) = \int^\infty \frac{(1-F(x))^{\frac{1}{2}}}{x(\log x)^{\frac{1}{2}}} dx < \infty . \tag{1}$$

A similar result holds for stationary Gaussian Fourier series, i.e. series of the form

$$\sum_{n=0}^\infty a_n[\xi_n \cos nt + \xi_n' \sin nt] \tag{2}$$

where ξ_n and ξ_n' are independent $N(0, 1)$. As a corollary to Theorem 1, we obtain

THEOREM 2. *Consider $X(t)$ in* (2). *Suppose that $|a_n|$ is eventually non-increasing; then*

$$\sum \frac{(\sum_{k=n}^\infty a_k^2)^{\frac{1}{2}}}{n(\log n)^{\frac{1}{2}}} < \infty \tag{3}$$

is a necessary and sufficient condition for the continuity of $X(t)$.

In Marcus (1972a) we show that (1) is a sufficient condition for continuity of a stationary Gaussian process without additional conditions on the spectrum F.

THEOREM 3. *Let $X(t)$ be a stationary Gaussian process with spectrum F. Then X is continuous if $I(F) < \infty$.*

There are two interesting aspects of Theorem 3. First it extends Hunt's Theorem (1951) to the best possible integral condition for continuity of stationary Gaussian processes involving the spectrum. Hunt showed that

$$\int^\infty [\log(1+\lambda)]^{1+\varepsilon} dF(\lambda) < \infty \tag{4}$$

is a sufficient condition for the continuity of $X(t)$. We show in Appendix (i) that (4) implies (1). It is easy to see that (1) holds for processes for which the integral in (4) is infinite. That (1) is best possible is discussed in Marcus (1972a). Hunt obtained his results using techniques of Paley and Zygmund (1930a), (1930b), (1932) on random Fourier series in which the independent random variables are Bernoulli, i.e. they take on the values ± 1 equally likely. We shall discuss later, in some detail, random Fourier series.

A second interesting aspect of Theorem 3 is that it is a corollary of the Fernique (1964), (1965), Delporte (1964) theorem on a sufficient condition for continuity of Gaussian processes. Their result applies to any Gaussian process X; no stationarity requirements are imposed.

THEOREM 4 (Fernique, Delporte). *If for $0 \leqq s \leqq t \leqq \varepsilon$ there is a function ψ*

for which

(5) $$E(X(s) - X(t))^2 \leqq \psi^2(t-s)$$

where ψ *is non-decreasing on* $[0, \varepsilon]$ *and*

(6) $$J(\psi) = \int_0^\varepsilon \frac{\psi(u)}{u(\log 1/u)^{\frac{1}{2}}}\, du < \infty$$

then X *is continuous* a.s.

The proof of Theorem 3 consists of showing that whenever (1) holds for a stationary Gaussian process X, the increments variance of the process $\sigma^2(h) = 2\int_0^\infty (1 - \cos \lambda h)\, dF(\lambda)$ is majorized by a monotone function $\psi^2(h)$, $h \in [0, \varepsilon]$ for which $J(\psi) < \infty$.

Nisio (1969), using Kahane's work on random Fourier series, obtained the following sufficient condition for the continuity of a stationary Gaussian process X with spectrum F. Define

(7) $$s_n^2 = F(2^{n+1}) - F(2^n)\,; \qquad n = 0, 1, 2, \cdots .$$

She showed that if s_n is non-increasing and $\sum_n s_n < \infty$, X has continuous sample paths a.s. This result is included in Theorem 3. To see this we first present a useful lemma which allows us to relate different results about Gaussian processes.

LEMMA 5 (Copson). (See Hardy, Littlewood and Pólya (1934) Theorem 345). *For any sequence* b_n

$$\sum_n b_n \leqq \sum_n (n^{-1} \sum_{k=n}^\infty b_k^2)^{\frac{1}{2}}$$

and a partial converse Boas (1960), *if the* b_n *are non-increasing and nonnegative*

$$\sum_n (n^{-1} \sum_{k=n}^\infty b_k^2)^{\frac{1}{2}} \leqq 2 \sum_n b_n \,.$$

(For simple proofs of these specialized cases of more general inequalities see Marcus, Shepp (1970) page 389, and Jain, Marcus (1973).)

To apply the lemma define $H(n) = 1 - F(n)$; the integral $I(F)$ is finite if and only if

(8) $$\sum_n \left(\frac{H(2^n)}{n}\right)^{\frac{1}{2}} < \infty \,.$$

Notice that $s_n^2 = H(2^n) - H(2^{n+1})$. The second part of Lemma 5 shows that Theorem 3 contains Nisio's result. (Actually Nisio makes a more general statement involving a decreasing majorant for the s_n. Nevertheless, the same discussion applies in the general case.)

Necessary conditions for the continuity of Gaussian processes and a partial converse to Theorem 4 have been obtained by Marcus and Shepp (1970) for stationary processes and by Jain and Marcus (1973) without requiring stationarity.

THEOREM 6 (Jain, Marcus, Shepp). *If for* $0 \leqq s \leqq t \leqq \varepsilon$, *there is a function* ψ *for which*

$$E(X(s) - X(t))^2 \geqq \psi^2(t-s)$$

where ψ *is non-decreasing on* $[0, \varepsilon]$ *and the integral in* (6)

$$J(\psi) = \infty$$

then X *is discontinuous* a.s.

It is shown (Marcus, Shepp (1970)) that even restricting attention to stationary processes $I(\sigma) < \infty$ is not necessary and sufficient for continuity of X. However, it is necessary and sufficient if we consider only these σ monotone near zero. Theorem 6 has another interesting application. The new results of this paper Theorems 1 and 2 are corollaries of Theorem 6. The novel part of Theorem 1 is the necessary conditions for continuity (sufficiency is covered in Theorem 3). This is obtained by showing that when F is concave and $I(F) = \infty$ there exists a monotone $\psi^2(h) \leqq \sigma^2(h)$ for which $J(\psi) = \infty$. The proof is given in Appendix (ii).

For stationary processes Nisio's extention of Kahane's results gives necessary conditions for continuity not covered by Theorem 6.

THEOREM 7 (Kahane, Nisio). *Let* $X(t)$ *be a stationary Gaussian process,* s_n^2 *as in* (7); *then if* X *is continuous* a.s.

(8a) $$\sum s_n < \infty .$$

(There are no monotonicity conditions on the s_n.)

Actually Nisio included an extraneous condition which is removed by a Theorem of Fernique (1970) and Landau and Shepp (1971). Further discussion of this point can be found in Marcus, Shepp (1971).

In Appendix (iii) we show that Theorem 7 does not imply the necessary part of Theorem 1, i.e. we exhibit an F eventually concave for which $I(F) = \infty$ with $\sum s_n < \infty$. Of course, in this example s_n is not decreasing nor can it be majorized by a decreasing sequence $\{M_n\}$ such that $\sum M_n < \infty$.

Finally we combine Theorem 1, 3 and 7 and Lemma 5 to present conditions on stationary Gaussian processes for which $I(F) < \infty$ gives necessary and sufficient conditions for continuity.

THEOREM 8. *Let* X *be a stationary Gaussian process with spectrum* F *and* s_n *defined as in* (7); *then* $I(F) < \infty$ *is necessary and sufficient for continuity of* X a.s. *if*

(i) $F(\lambda)$ *is concave for* $\lambda_0 \leqq \lambda < \infty$, *or*
(ii) s_n *are non-increasing,* $n \geqq N$.

It is not surprising that $I(F)$ is involved in continuity conditions for X. Under certain smoothness conditions on $1 - F(x)$ it is comparable to $\sigma(1/x)$, see Pitman (1961). Since $1 - F(x)$ is monotone, $\sigma(h)$ has monotonic majorants and minorants of the form Const. $(1 - F(1/h))$. In these cases, Theorems 4 and 6 and a change of variables of integration give continuity conditions for X in terms of $I(F)$. Of course, $I(F) < \infty$ is not a necessary and sufficient condition for the continuity of X in general. It is easy to see this; we shall pursue this point when we look at Gaussian Fourier series.

We shall consider the following examples of random Fourier Series

(9) $$\sum_{n=0}^{\infty} a_n[\eta_n \cos nt + \eta_n' \sin nt]$$

where η_n, η_n' are independent identically distributed random variables $E(\eta_n) = 0$, $E(\eta_n^2) = 1$, and

(10) $$\sum a_n \cos (nt + \Phi_n)$$

where Φ_n are i.i.d. random variables with a uniform distribution on $[0, 2\pi]$. The series (10) are called Steinhaus series.

For these series, (9) and (10)

$$EX(t)X(s) = \sum_{n=0}^{\infty} a_n^2 \cos n(t - s)$$

thus they represent a class of weakly stationary processes all having the same covariance function. They have the same spectrum F with $F(n + 1) - F(n) = a_n^2$; without loss of generality, we assume $\sum a_n^2 = 1$. For these series

(11) $$s_n^2 = \sum_{n=2^n}^{2^{n+1}-1} a_n^2 .$$

When $\eta_n, \eta_n' = N(0, 1)$, (9) is a stationary Gaussian process. Generally speaking continuity properties for weakly stationary processes are very different than for stationary processes but, as we shall see, there are no known continuity results that hold for some values of η_n (satisfying the above moment conditions) and not for others. When $\eta_n = \pm 1$ with equal probability (Bernouli random variables) the series in (9) are called Rademacher series because in earliest treatments (e.g. Paley, Zygmund (1930a)) i.i.d. Bernouli random variables were generated by Rademacher functions. Paley and Zygmund (1930a), (1930b), (1932) considered Steinhaus and Rademacher series and showed that (4) is a sufficient condition for them to be continuous and that (8a) is necessary. Salem and Zygmund (1954) considered Rademacher series and showed that (3) is a sufficient condition for them to be continuous. Kahane (1968) extended their results to all series (9) and (10), in fact, to a much wider class of random Fourier series than these.

THEOREM 9 (Kahane). *Consider the series*

(12) $$\sum_1^{\infty} a_n X_n \cos (nt + \Phi_n)$$

where $X_n e^{i\Phi_n}$ *are independent symmetric random variables,* $EX_n = 0$, $EX_n^2 = 1$. (*The* X_n *and* Φ_n *are not necessarily independent.*) *A sufficient condition for the continuity of* (12) *is*

(13) $$\sum \frac{(\sum_{j=n}^{\infty} a_j^2)^{\frac{1}{2}}}{n(\log n)^{\frac{1}{2}}} < \infty .$$

If

(14) $$EX_n^4 \leqq C(EX_n^2)^2 \quad \textit{for} \quad C \quad \textit{independent of} \quad n$$

then a necessary condition for the continuity of (12) *is*

(15) $$\sum s_n < \infty$$

for s_n *as defined in* (11).

For X_n i.i.d. condition (14) is always satisfied as long as a fourth moment exists. Condition (13) is expressed differently in Kahane ((1968) page 65). It is given as

$$\sum_k 2^{k/2} \left(\sum_{2^k}^{2^{k+1}-1} s_j^2\right)^{\frac{1}{2}} < \infty \tag{16}$$

with the remark that when $s_j \downarrow$ (16) implies $\sum s_j < \infty$. In Appendix (iv) we show that (16) is equivalent to (13). Lemma 5 and (8) show that (13) and $\sum s_j < \infty$ are equivalent for $s_j \downarrow$.

Theorem 2 contains a result for the series (9) when η_n and η_n' are $N(0, 1)$ that is not contained in Theorem 9. Namely that if (13) is infinite and $|a_n| \downarrow$ the series is not continuous. We have already shown in Appendix (iii) that this case is not covered by (15). The proof of Theorem 2 is an immediate consequence of Theorem 1 and the following lemma and its corollary which are themselves consequences of Slepian's (1962) lemma.

LEMMA 10. *Let $X_i(t)$, be stationary Gaussian processes with covariance functions $\Gamma_i(h)$, $\Gamma_i(0) = 1$, $i = 1, 2$. Let $Y(t)$ be a stationary Gaussian process with covariance function $\Gamma_1(h)\Gamma_2(h)$. If $X_i(t)$, $i = 1, 2$ are continuous so is $Y(t)$, and conversely.*

COROLLARY 10a. *Let X be a stationary Gaussian process with spectrum F such that $F(n + 1) - F(n) = a_n^2$ and $F'(\lambda) =$ Const., $n < \lambda < n + 1$ where $n = 0, 1, \cdots$. Consider the Gaussian Fourier series (2) with these values a_n^2. Then the two processes are mutually continuous or discontinuous.*

Lemma 10 and Corollary 10a are proved in Appendix (v).

THEOREM 11. *Consider the stationary Gaussian Fourier series (2); then (13) is necessary and sufficient for continuity of these series if*

(i) *$|a_n|$ are non-increasing, $n \geqq N$, or*
(ii) *s_n are non-increasing, $n \geqq N$.*

The proofs of many major results on Gaussian processes depend on two facts that are unavailable when studying series of the form (9), (10) and (12). For instance, Theorem 4 requires knowing the probability distribution of $X(t) - X(s)$, a simple matter for Gaussian processes but probably impossible for the series (9). Theorem 6 from which Theorems 1 and 2 are derived has as its critical component Slepian's lemma, or, to be more precise a corollary of it.

LEMMA 12 (Marcus, Shepp (1971)). *Let X and Y be Gaussian processes for which*

$$E(Y(s) - Y(t))^2 \leqq E(Y(s) - X(t))^2 \tag{17}$$

for $0 \leqq s \leqq t \leqq \delta$. Then if X is continuous on $[0, \delta]$, so is Y.

We can show that the only series of the form (9) that satisfies Slepians' lemma are the ones with $\eta_k, \eta_k' = N(0, 1)$. However, it is possible that for others a result such as Lemma 12 is true.

QUESTION 1. For what stochastic processes with second moments can we obtain Lemma 12? Does Lemma 12 hold for any of the processes of the form (9) besides the Gaussian process?

For any of the series (9) for which the answer to Question 1 is affirmative, Theorem 2 will be true.

The method of proof employed by Kahane for Theorem 9 follows lines established by Paley, Salem and Zygmund which utilize the fact that they are dealing with trigonometric polynomials. Bernstein's Theorem on the behavior of the maximum of a trigonometric polynomial of degree n plays a critical role in their proofs. It is interesting to note that the elegant proof of Theorem 4 by Garsia, Rodemich and Rumsey (1970) employs a lemma that Garsia (1970) describes as an analogue of Bernstein's inequality for linear combinations of eigen-functions. Kahane also uses measure theoretic arguments to extend continuity properties first obtained for special examples of random Fourier series to all the series (12). It would be interesting to know if the following general statement is true.

QUESTION 2. Are all the series (9) and (10) or even (12) mutually continuous or discontinuous depending only on the values of a_n? Or, a more restricted question, is Theorem 2 true for all the series (9) and (10) or even (12)?

A somewhat related question is

QUESTION 3. Suppose that (9) represents a continuous process $X(t)$. Let $X_k(t)$ be independent copies of $X(t)$ and $\{c_k\} \in l^2$. Is $\sum c_k X_k(t)$ continuous? It is if X is Gaussian but what about the other cases? (It is not necessarily if the $X_k(t)$ are not all independent copies of the same process.) This question can be asked for any continuous process $Y(t)$ with $EY = 0$, $EY^2 = 1$.

Random lacunary Fourier series satisfy continuity conditions that are different than those for series with "smoother" spectrums. (In Marcus (1972b) this same phenomenon is described for the moduli of continuity). We shall use these series to show that $I(F) < \infty$ is not a necessary and sufficient condition for continuity of Gaussian Fourier series. Consider the lacunary Fourier series

$$\sum_{k=0}^{\infty} b_k[\eta_k \cos 2^k t + \eta_k' \sin 2^k t] \tag{18}$$

where as in (9), η_k, η_k' are i.i.d. random variables $E(\eta_k) = 0$, $E(\eta_k^2) = 1$, $\sum b_k^2 = 1$. It follows from Szidon's Theorem (see Zygmund (1959) Section 6.4) that the series (18) are continuous if and only if $\sum |b_k| < \infty$. Refer to (11), $s_n = b_n$; therefore

THEOREM 13. *The series* (18) *are continuous* a.s. *if and only if* $\sum s_n < \infty$.

It is easy to see that $\sum s_n$ can be finite and $I(F) = \infty$; take $s_{2^k} = 1/k^2$, $k = 1, 2, \cdots$; all other $s_n = 0$ and use (8). (The reason the example in Appendix (iii) is more delicate is that the a_n (see (2)) are non-increasing.)

Clearly we cannot find necessary and sufficient conditions for continuity involving the s_n or integral conditions on F, since $I(F) < \infty$ is a condition for one class of processes and $\sum s_n < \infty$ for another. For further progress one must consider the relations between $|a_n|$ as n varies. Imposing the condition $|a_n| \downarrow$ eliminates gaps in the series; it is the gaps that cause difficulties.

Dudley's (1967), (1972) sufficient condition for the continuity of Gaussian processes based on the metric entropy of compact subsets of a Hilbert space is a candidate for a necessary condition for the continuity of a stationary Gaussian process on R^n. His approach to the study of Gaussian processes is to consider a single process defined as a linear norm preserving map from compact subsets of a Hilbert space into normal random variables. His version of Theorem 4 involves the number of sets of diameter 2ε necessary to cover certain compact subsets in the domain of the map. The apparent strength of this approach is for processes with infinite dimensional time parameter; however, his result is also stronger than Theorem 4 for processes on R^n.

We shall formulate Dudley's Theorem for real-valued Gaussian processes on $[0, 1]$. Let $X(t, \omega)$ be such a process, $\omega \in \Omega$, $EX(t) = 0$, $EX(t)^2 = 1$. The process $\{X(t, \omega) : t \in [0, 1]\}$ is a compact subset of $L^2(\Omega)$. A neighborhood of diameter 2ε in $L^2(\Omega)$ about the element $X(t, \omega)$ are the elements $X(s, \omega)$ for which $E(X(s) - X(t))^2 \leqq \varepsilon$. For any ε, $\{X(t, \omega) : t \in [0, 1]\}$ can be covered by a finite number of neighborhoods of diameter 2ε.

Define $N(\varepsilon)$ as the smallest number of sets of diameter 2ε which cover $\{X(t, \omega) : t \in [0, 1]\}$ and $H(\varepsilon) = \log N(\varepsilon)$. $H(\varepsilon)$ is called the metric entropy of $\{X(t) : t \in [0, 1]\}$.

THEOREM 14 (Dudley). *Let $X(t)$ and $H(\varepsilon)$ be defined as above. A sufficient condition for $X(t)$ to have continuous sample paths is*

$$\int_0 H^{\frac{1}{2}}(\varepsilon)\, d\varepsilon < \infty . \tag{19}$$

Dudley (1967) shows that Theorem 14, stated in a broader context, implies Theorem 4. We will show this for $X(t)$ stationary with increments variance $\sigma^2(h)$. As in (5), assume $\sigma^2(h) \leqq \psi^2(h)$ with $\psi \uparrow$ for $h \in [0, \delta]$. Then $N(\varepsilon)$ is less than $1/\psi^{-1}(\varepsilon)$. Therefore $H(\varepsilon) = (-\log \psi^{-1}(\varepsilon))^{\frac{1}{2}}$. Substituting this in (19), changing variables, and integrating by parts, we get (6).

An interesting aspect of Theorem 14 in the context of the study of stationary Gaussian processes is shown by the following theorem (Marcus (1972a)):

THEOREM 15. *Consider the series* (2). *If $\sum |a_n| < \infty$, $\int_0 H^{\frac{1}{2}}(\varepsilon)\, d\varepsilon < \infty$.*

Theorem 15 does not reveal anything new about the continuity properties of (2). When $\sum |a_n| < \infty$, (2) can be shown to be continuous by the three series theorem. What is significant is that for the series (18), (19) implies continuity when $\sum b_k < \infty$. In other words these examples which show that $I(F) < \infty$ is not a necessary and sufficient condition for continuity of stationary Gaussian processes do not apply to (19). As far as we know the following question is open.

QUESTION 4. Let $X(t)$ be a stationary Gaussian process. Is (19) necessary and sufficient for $X(t)$ to be continuous? If the condition of stationarity is removed the answer is no (Dudley (1967)).

In some cases smoothing the increments variance or spectrum of a stationary

Gaussian process that is continuous gives processes which are no longer continuous. Fernique (in conversation) raised the question: Suppose $\varphi^2(h)$ is the increments variance of a continuous stationary Gaussian process;

$$\sigma^2(h) = h^{-1} \int_0^h \varphi^2(u)\, du \tag{20}$$

is also the increments variance of a stationary Gaussian process. Is the process corresponding to $\sigma^2(h)$ also continuous? The answer is no, not necessarily. In fact answering this question led to Theorems 1 and 2. The next lemma ties the two together.

LEMMA 16. *Let $X(t)$ be a stationary Gaussian process with increments variance $\varphi^2(h)$. Define $\sigma^2(h)$ as in* (20) *and let $Y(t)$ be the stationary Gaussian process corresponding to $\sigma^2(h)$. Then the spectrum of $Y(t)$ is concave.*

The proof of this Lemma is contained in Lemma A.1 (ii) of the Appendix.

Consider the continuous sationary Gaussian process

$$\sum k^{-\frac{3}{2}}[\xi_k \cos 2^{2^k} t + \xi_k' \sin 2^{2^k} t] \tag{21}$$

where ξ_k, ξ_k' are i.i.d. $N(0, 1)$. Let $\varphi^2(h)$ be the increments variance of (21) and $\sigma^2(h)$ defined as in (20). Let G be the spectrum of (21) and F the spectrum corresponding to $\sigma^2(h)$. By Lemma 16 the spectrum F is concave with

$$F'(\lambda) = \sum_{j=k}^{\infty} (k^3 2^{2^k})^{-1}, \qquad 2^{2^k} \leqq \lambda < 2^{2^{k+1}}.$$

This F is almost the same as in the example of Appendix (iii). By a minor variation in the argument of Appendix (iii) and using Theorem 1, we see that the process with increments variance $\sigma^2(h)$ is not continuous. The process (21) with increments variance $\varphi^2(h)$ is continuous.

Smoothing the spectrum can turn a continuous process into a discontinuous process. Take the spectrum G of the continuous process (21) and polygonalize it, i.e. form F such that $F(2^{2^{k+1}}) - F(2^{2^k}) = G(2^{2^{k+1}}) - G(2^{2^k})$ and such that $F'(\lambda)$ is constant for $2^{2^k} < \lambda < 2^{2^{k+1}}$. By the same argument as above, the process with spectrum F is not continuous.

We have stated that many of the results that have been given can be extended to processes on R^k. As an example here is a sufficient condition for continuity of random multiple Fourier series which are also stationary Gaussian processes. Consider

$$\begin{aligned}\sum_{n_1,\cdots,n_k} a_{n_1,\cdots,n_k}[\xi_{n_1,\cdots,n_k} &\cos(n_1 t_1 + \cdots + n_k t_k) \\ &+ \xi'_{n_1,\cdots,n_k} \sin(n_1 t_1 + \cdots + n_k t_k)]\end{aligned} \tag{22}$$

where $(t_1, \cdots, t_k)$ is a point in the k-dimensional torus, $n_1, \cdots, n_k$ are integers greater than or equal to zero, $\xi_{n_1,\cdots,n_k}$ and $\xi'_{n_1,\cdots,n_k}$ are i.i.d. $N(0, 1)$ and $\sum a^2_{n_1,\cdots,n_k} = 1$. Define

$$S(N) = \{(n_1, \cdots, n_k): \text{at least one } n_i \geqq N,\ i = 1, \cdots, k\}$$

and

$$T(N) = \sum_{S(N)} a^2_{n_1,\cdots,n_k}.$$

THEOREM 17. *Consider the series* (22) *and* $T(N)$ *as defined above.*

$$\sum_N \frac{T(N)^{\frac{1}{2}}}{N(\log N)^{\frac{1}{2}}} < \infty$$

is a sufficient condition for the continuity of (22).

A related result holds for $(t_1, t_2, \cdots)$ in an infinite-dimensional torus. Theorem 17 can be obtained from Theorem 4. It probably holds in greater generality similar to Theorem 9. Kahane has outlined an approach ((1968) page 70, Exercise 3) for multiple Rademacher series. Perhaps the use of Lemma 5 will allow some simplification. In any case, multiple random Fourier series awaits systematic study.

Finally, we refer the reader to Fernique (1971) for an approach to the continuity problem for Gaussian processes that is quite different from any of those discussed in this paper.

APPENDIX

(i) It follows from the monotonicity of $\log \lambda$ and $F(\lambda)$ that (4) is equivalent to

$$\int^{\infty} \frac{(1 - F(\lambda))(\log \lambda)^{\varepsilon}}{\lambda} d\lambda < \infty .$$

Applying Schwarz's inequality to $I(F)$ written in the form

$$\int^{\infty} \frac{(1 - F(\lambda))^{\frac{1}{2}}(\log \lambda)^{\varepsilon/2}}{\lambda(\log \lambda)^{(1+\varepsilon)/2}} d\lambda$$

we obtain the desired result.

(ii) PROOF OF THEOREM 1. We begin with two lemmas.

LEMMA A.1. *Let* $\sigma^2(h) = 1/h \int_0^h \varphi^2(u)\, du$ *where*

$$\varphi^2(u) = 2 \int_0^\infty (1 - \cos \lambda u)\, dG(\lambda) ;$$ *then*

(a) $\sigma^2(h) = 2 \int_0^\infty (1 - \cos \lambda h) f(\lambda)\, d\lambda$ *where* $f(\lambda) = \int_\lambda^\infty (1/u)\, dG(u)$ *and* $\int_0^\infty f(\lambda)\, d\lambda = 1$.
(b) $\sigma^2(h) \geqq C(1 - G(1/h))$, $h > 0$, *C a constant.*

PROOF.

$$\sigma^2(h) = 2/h \int_0^h \int_0^\infty (1 - \cos \lambda u)\, dG(\lambda)\, du$$

$$(A.1) \qquad = 2 \int_0^\infty \left(1 - \frac{\sin \lambda h}{\lambda h}\right) dG(\lambda)$$

$$= 2\left(1 - \int_0^\infty \frac{\sin uh}{uh} dG(u)\right) .$$

Also,

$$\int_0^\infty \frac{\sin uh}{uh} dG(u) = \int_0^\infty \frac{1}{u} \int_0^u \cos \lambda h\, d\lambda\, dG(u)$$

$$= \int_0^\infty \cos \lambda h \int_\lambda^\infty \frac{1}{u}\, dG(u)\, d\lambda$$

$$= \int_0^\infty \cos \lambda h f(\lambda)\, d\lambda .$$

Since

$$\int_0^\infty f(\lambda)\,d\lambda = \int_0^\infty \int_\lambda^\infty \frac{1}{u}\,dG(u)\,d\lambda = \int_0^\infty \int_0^u \frac{1}{u}\,d\lambda\,dG(u) = 1$$

we obtain (a). For (b) note that (A.1)

$$\geqq 2 \int_{1/h}^\infty \left(1 - \frac{\sin \lambda h}{\lambda h}\right) dG(\lambda) > C(1 - G(1/h))\,.$$

LEMMA A.2. *Let F be a concave function $F(0) = 0$, $F(\infty) = 1$ and $f(\lambda) = F'(\lambda) = 1$ for $\lambda \in [0, 1/100]$. There exists a distribution G, $G(\infty) = 1$ such that*

$$f(\lambda) = \int_\lambda^\infty \frac{1}{u}\,dG(u)\,. \tag{A.2}$$

PROOF. First we consider the case where f decreases only in jumps. Let $f(s) = f_k$, $a_k < s \leqq a_{k+1}$, $a_0 = 0$. The function G which satisfies (A.2) is a discrete distribution with jumps of magnitude $a_{k+1}(f_k - f_{k+1})$ at a_{k+1}. G is a distribution function since

$$\begin{aligned} \sum_{k=0}^\infty a_{k+1}(f_k - f_{k+1}) &= \sum_{k=0}^\infty f_k(a_{k+1} - a_k) \\ &= \sum_{k=0}^\infty \int_k^{k+1} f(s)\,ds \\ &= \int_0^\infty f(s)\,ds = 1\,. \end{aligned}$$

Let $f(\lambda)$ be any decreasing function with $f(\lambda) = 1$, $\lambda \in [0, 1/100]$ and $f_n(\lambda)$ an increasing sequence of step functions with $f(\lambda)$ as its limit. Let $G_n(\lambda)$ be the corresponding distribution functions which as we have just shown satisfy (A.2). Since $f_n(\lambda) \uparrow$, $G_n(\lambda) \uparrow$; also $G_n(\lambda) \leqq 1$. Therefore (A.2) holds for f and G the limits of f_n and G_n for all $\lambda > 0$. We do not have to worry about the limit at $\lambda = 0$ since the hypothesis $f(\lambda) = 1$, $\lambda \in [0, 1/100]$ implies that $G(\lambda) \equiv 0$ in this interval.

PROOF OF THEOREM 1. Let x be a stationary Gaussian process with spectrum $F(\lambda)$ such that $F(\lambda)$ is concave for $\lambda \geqq \lambda_0$. We can find a function $F_1(\lambda)$ that satisfies the hypotheses of Lemma A.2 and such that $F_1(\lambda) = F(\lambda)$ for $\lambda > \lambda_0' \geqq \lambda_0$. By considering the representation of stationary Gaussian processes by stochastic integrals we know that the processes with spectrums F and F_1 are mutually continuous or discontinuous. Therefore, with no loss of generality we will assume that F satisfies the hypotheses of Lemma A.2.

By this lemma, the increments variance of X is

$$\sigma^2(h) = 2 \int_0^\infty (1 - \cos \lambda h) f(\lambda)\,d\lambda$$

where $f(\lambda)$ satisfies (A.2) for some distribution G. By Lemma A.1, $\sigma^2(h) \geqq C(1 - G(1/h))$ for $h \geqq 0$. Therefore if $I(G) = \infty$ a change of variables gives $J(\psi) = \infty$ where $\psi^2(h) = C(1 - G(1/h))$ is a monotone minorant for $\sigma^2(h)$. By Theorem 6, X is discontinuous.

We now show $I(G) = \infty$. The hypothesis of Theorem 1 states $I(F) = \infty$.

$$\begin{aligned} 1 &- F(x) \\ &= \int_x^\infty f(\lambda)\,d\lambda = \int_x^\infty \int_\lambda^\infty \frac{1}{u}\,dG(u)\,d\lambda = \int_x^\infty \int_x^u \frac{1}{u}\,d\lambda\,dG(u) = \int_x^\infty \frac{u - x}{u}\,dG(u)\,, \end{aligned}$$

Therefore $I(F) = \infty$ implies

$$\int^{\infty} \frac{\left(\int_x^{\infty} \frac{u-x}{u}\, dG(u)\right)^{\frac{1}{2}}}{x(\log x)^{\frac{1}{2}}}\, dx = \infty$$

which implies $I(G) = \infty$ since

$$\int_x^{\infty} \frac{u-x}{u}\, dG(u) \leqq 1 - G(x)\, .$$

(iii) In this example $F(\lambda)$ is chosen to be absolutely continuous with $F'(\lambda) = (k^3 2^{2^{k+1}})^{-\frac{1}{2}}$ for $2^{2^k} \leqq \lambda < 2^{2^{k+1}}$, $k \geqq K > 2$. Recall $s_j^2 = F(2^{j+1}) - F(2^j)$; therefore for $2^k \leqq j < 2^{k+1}$, $s_j = 2^{j/2}(k^3 2^{2^{k+1}})^{-\frac{1}{2}}$ and

$$\sum_{j=2^N}^{\infty} s_j = \sum_{k=N}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} (k^3 2^{2^{k+1}})^{-\frac{1}{2}} 2^{j/2} \leqq \text{Const.} \sum_{k=N}^{\infty} k^{-\frac{3}{2}} < \infty\, .$$

By (8), $I(F) = \infty$ is equivalent to $\sum_n (H(2^n)/n)^{\frac{1}{2}} = \infty$. We have,

$$\begin{aligned}\sum_{j=2^N}^{\infty} \left(\frac{H(2^j)}{j}\right)^{\frac{1}{2}} &= \sum_{k=N}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} \left(\frac{H(2^j)}{j}\right)^{\frac{1}{2}} \\ &\geqq \sum_{k=N}^{\infty} H(2^{2^{k+1}})^{\frac{1}{2}} \sum_{j=2^k}^{2^{k+1}-1} j^{-\frac{1}{2}} \\ &\geqq \text{Const.} \sum_{k=N}^{\infty} \left(\frac{1}{k+1}\right)^{\frac{3}{2}} 2^{(k+1)/2} = \infty\, .\end{aligned}$$

(iv) We first show that $\sum 2^{k/2}(\sum_{j=2^k}^{2^{k+1}-1} s_j^2)^{\frac{1}{2}} < \infty \Rightarrow \sum 2^{k/2}(\sum_{j=2^k}^{\infty} s_j^2)^{\frac{1}{2}} < \infty$.

$$\begin{aligned}\sum_{k=1}^{\infty} 2^{k/2}\left(\sum_{j=2^k}^{\infty} s_j^2\right)^{\frac{1}{2}} &= \sum_{k=1}^{\infty} 2^{k/2}\left(\sum_{l=k}^{\infty} \sum_{j=2^l}^{2^{l+1}-1} s_j^2\right)^{\frac{1}{2}} \\ &\leqq \sum_{k=1}^{\infty} 2^{k/2} \sum_{l=k}^{\infty} \left(\sum_{j=2^l}^{2^{l+1}-1} s_j^2\right)^{\frac{1}{2}} \\ &= \sum_{l=1}^{\infty} \sum_{k=1}^{l} 2^{k/2}\left(\sum_{j=2^l}^{2^{l+1}-1} s_j^2\right)^{\frac{1}{2}} \\ &\leqq C \sum_{l=1}^{\infty} 2^{l/2}\left(\sum_{j=2^l}^{2^{l+1}-1} s_j^2\right)^{\frac{1}{2}}\, .\end{aligned}$$

Next we show that $\sum 2^{k/2}(\sum_{j=2^k}^{\infty} s_j^2)^{\frac{1}{2}} < \infty \Rightarrow \sum_{n=1}^{\infty}((1/n)\sum_{j=n}^{\infty} s_j^2)^{\frac{1}{2}} < \infty$.

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=n}^{\infty} s_j^2\right)^{\frac{1}{2}} &= \sum_{l=0}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} \left(\frac{1}{n} \sum_{j=n}^{\infty} s_j^2\right)^{\frac{1}{2}} \\ &\leqq \sum_{l=0}^{\infty} 2^l \left(\frac{1}{2^l} \sum_{j=2^l}^{\infty} s_j^2\right)^{\frac{1}{2}} \\ &= \sum_{l=0}^{\infty} 2^{l/2}\left(\sum_{j=2^l}^{\infty} s_j^2\right)^{\frac{1}{2}}\, .\end{aligned}$$

We have already observed that condition (13) is equivalent to condition (8), $(H(2^n) = \sum_{j=n}^{\infty} s_j^2)$. What we have above is that (16) implies (8) and consequently (13).

Finally we show that (8) and equivalently (13) implies (16)

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=n}^{\infty} s_j^2\right)^{\frac{1}{2}} &= \sum_{l=0}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} \left(\frac{1}{n} \sum_{j=n}^{\infty} s_j^2\right)^{\frac{1}{2}} \\ &\geqq \frac{1}{2} \sum_{l=1}^{\infty} 2^{l/2}\left(\sum_{j=2^l}^{2^{l+1}-1} s_j^2\right)^{\frac{1}{2}}\, .\end{aligned}$$

(v) PROOF OF LEMMA 10. Lemma 10 follows from Lemma 12 of this paper. Note that 15 can be replaced by

$$E(Y(s) - Y(t))^2 \leqq kE(X(s) - X(t))^2$$

where k is some constant, since we could prove first that $(k^{-\frac{1}{2}})Y(t)$ is continuous and then infer that $Y(t)$ is continuous.

In the hypothesis of Lemma 10 define

$$\sigma_i^2(h) = 2(1 - \Gamma_i(h)), \qquad i = 1, 2.$$

$$E(Y(t + h) - Y(t))^2 = \frac{\sigma_1^2(h)}{2} + \frac{\sigma_2^2(h)}{2} - \frac{\sigma_1^2(h)\sigma_2^2(h)}{4}.$$

Define $Z(t) = X_1(t) + X_2(t)$ where the processes X_1 and X_2 are constructed so that they are independent of each other.

$$E(Z(t + h) - Z(t))^2 = \sigma_1^2(h) + \sigma_2^2(h).$$

Since $Z(t)$ is continuous, Lemma 12 implies that $Y(t)$ is continuous. The converse follows similarly since for h sufficiently small

$$E(Y(t + h) - Y(t))^2 > \tfrac{1}{4} \max [\sigma_1^2(h), \sigma_2^2(h)].$$

PROOF OF COROLLARY 10 AND THEOREM 2. Consider

$$\sum_{k=0}^{\infty} a_k(\eta_k \cos kt + \eta_k' \sin kt) \tag{A.3}$$

where η_k, η_k' are independent $N(0, 1)$, $\sum a_k^2 = 1$. The spectrum of this process $H(\lambda)$ has jumps a_k^2 at $\lambda = k$. Let

$$\begin{aligned} G(\lambda) &= 0 & \lambda &< 0 \\ &= \lambda & 0 \leqq \lambda &\leqq 1 \\ &= 1 & \lambda &> 1. \end{aligned}$$

Define $F(\lambda) = \int_0^\infty G(\lambda - u)\, dH(u)$. F is a distribution function which increases linearly between $F(k)$ and $F(k + 1)$ and $F(k + 1) - F(k) = a_k^2$.

Let $X_1(t)$ be a stationary Gaussian process with spectrum $G(\lambda)$. By Theorem 3, X_1 has continuous sample paths. Therefore by Lemma 10, (A.3) and the process with spectrum F are either both continuous or both discontinuous. Therefore Theorem 2 follows from Theorem 1.

REFERENCES

BELYAEV, YU. K. (1961). Continuity and Hölder conditions for sample functions of stationary Gaussian processes. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **2** Univ. of California Press, 23–33.

BOAS, R. P. Jr. (1960). Inequalities for monotonic series. *J. Math. Anal. Appl.* **1** 121–126.

CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes*. Wiley, New York.

DELPORTE, L. (1964). Fonctions aléatoires presque sûrement continues sur un intervalle fermé, *Ann. Inst. Henri Poincaré*, Sec. B **1** 111–215.

DUDLEY, R. M. (1967). The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Functional Analysis* **1** 290–330.

DUDLEY, R. M. (1973). Sample functions of the Gaussian process. *Ann. Probability* **1** 66–103.

EAVES, D. M. (1967). Sample functions of Gaussian random homogeneous fields are either continuous or very irregular. *Ann. Math. Statist.* **38** 1579–1582.

FERNIQUE, X. (1964). Continuité des processus Gaussiens. *C. R. Acad. Sci. Paris* **258** 6058–6060.

FERNIQUE, X. (1965). Continuité des certains processus Gaussiens. *Sem. R. Fortet, Institute Henri Poincaré*. Paris.

FERNIQUE, X. (1970). Intégrabilité des vecteurs Gaussiens. *C. R. Acad. Sci. Paris Ser. A*, **270** 1698–1699.

FERNIQUE, X. (1971). Régularité de processus Gaussiens. *Invent. Math.* **12** 304–320.

GARSIA, A. (1970). On the smoothness of functions satisfying certain integral inequalities. Monterey Conference in Functional Analysis.

GARSIA, A. RODEMICH, E. and RUMSEY, H. Jr. (1970). A real variable lemma and the continuity of paths of some Gaussian Processes. *Indiana Univ. Math. J.* **20** 565–578.

HARDY, G. H., LITTLEWOOD, J. E. and PÓLYA, G. (1934). *Inequalities*. Cambridge Univ. Press.

HUNT, G. A. (1951). Random Fourier transforms. *Trans. Amer. Math. Soc.* **71** 38–69.

JAIN, N. and MARCUS, M. B. (1973). A new proof of a sufficient condition for discontinuity of Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*. To appear.

KAHANE, J. P. (1968). *Some Random Series of Functions*. D. C. Heath Lexington.

LANDAU, H. J. and SHEPP, L. A. (1971). On the supremum of a Gaussian process. *Sankhya Ser. A* **32** 369–378.

MARCUS, M. B. (1972a). A comparison of continuity conditions for Gaussian processes. *Ann. Probability* **1** 123–130.

MARCUS, M. B. (1972b). Gaussian lacunary series and the modulus of continuity for Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*. **22** 301–322.

MARCUS, M. B. and SHEPP, L. A. (1970). Continuity of Gaussian processes. *Trans. Amer. Math. Soc.* **151** 377–392.

MARCUS, M. B. and SHEPP, L. A. (1972). Sample behavior of Gaussian processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **2** 423–441.

NISIO, M. (1969). On the continuity of stationary Gaussian processes. *Nagoya Math. J.* **34** 89–104.

PALEY, R. E. A. C. and ZYGMUND, A. (1930a), (1930b), (1932). On some series of functions (1), (2), (3), *Proc. Cambridge Philos. Soc.* **26** 337–357, 458–475 and **28** 190–205.

PITMAN, E. T. G. (1961). Some theorems on characteristic functions of probability distributions. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **2** 393–402.

SALEM, R. and ZYGMUND, A. (1954). Some properties of trigonometric series whose terms have random signs. *Acta Math.* **91** 245–301.

SLEPIAN, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell. System Tech. J.* **41** 463–501.

ZYGMUND, A. (1959). *Trigonometric Series*. Cambridge Univ. Press.

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