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THE LAW OF THE ITERATED LOGARITHM FOR GAUSSIAN PROCESSES

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Strassen's law of the iterated logarithm for Brownian motion is extended to a class of Gaussian processes. Let $\{X(t), t \geqq 0\}$ be a real continuous Gaussian process with $X(0) = 0$, mean zero and continuous covariance kernel $R(s, t)$. Define a random sequence $\{f_n(t, \omega)\}$ in $C[0, 1]$ by $f_n(t, \omega) = X(nt, \omega)/(2R(n, n) \log \log n)^{\frac{1}{2}}$. Under certain conditions on R it is shown that with probability one $\{f_n(t, \omega)\}$ is equicontinuous and the set of its limit points is the unit ball of the reproducing kernel Hilbert space with reproducing kernel Γ determined by R. The result generalizes the author's earlier result (1972).

1. Introduction. Let $X = \{X(t, \omega), 0 \leqq t < \infty\}$ be a real, sample continuous Gaussian process with $X(0) = 0$, mean zero and continuous covariance function $R(s, t)$. Define, for each ω, a sequence of functions $\{f_n(t, \omega)\}$ in the space $C[0, 1]$ of all continuous functions vanishing at the origin, with the sup norm $||\cdot||_C$, by

$$f_n(t, \omega) = (2\sigma^2(n) \log_2 n)^{-\frac{1}{2}} X(nt, \omega), \qquad 0 \leqq t \leqq 1,\ n = 3, 4, \cdots, \tag{1}$$

where $\sigma^2(n) = R(n, n)$ and $\log_2 n = \log \log n$.

Strassen (1964) proved that if X is a Brownian motion, then, for almost every (a.e.) ω, the set of limit points of the sequence of functions $\{f_n(t, \omega)\}$ coincides with the set K_B of all absolutely continuous functions h in $C[0, 1]$ such that $\int_0^1 (dh/dt)^2\, dt \leqq 1$. An extension of the above theorem of Strassen to a class of Gaussian processes is given in Oodaira (1972), and the set of limit points of $\{f_n(t, \omega)\}$ is characterized as a bounded set (the unit ball, if R is normalized) of the reproducing kernel Hilbert space (RKHS) $H(R)$ with reproducing kernel (r.k.) $R(s, t)$, $0 \leqq s, t \leqq 1$.

One of the assumptions in Oodaira (1972) is the following: there is a positive function $v(r)$, $r \geqq 0$, such that $v(r) \uparrow \infty$ and

$$R(rs, rt) = v(r)R(s, t) \qquad \text{for all } r, s, t \geqq 0\,. \tag{2}$$

Note that if such a function $v(r)$ exists, it must be of the form r^ρ with $\rho > 0$. Gaussian processes satisfying the condition (2) are called semi-stable and studied by Lamperti (1962). The purpose of this paper is to generalize the result in Oodaira (1972) by replacing condition (2) by the following asymptotic one: there are a positive function $v(r)$ and a covariance kernel $\Gamma(s, t)$ such that

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$v^{-1}(r)R(rs, rt) \to \Gamma(s, t)$ as $r \to \infty$ uniformly in $0 \leqq s, t \leqq 1$. Under this condition, together with other conditions which will be stated in Section 2, the set of limit points of $\{f_n(t, \omega)\}$ is characterized as the unit ball of the RKHS $H(\Gamma)$ with r.k. $\Gamma(s, t)$, $0 \leqq s, t \leqq 1$.

Consider, for example, the process X defined by

$$X(t) = \int_0^t Y(u)\, du\,, \tag{3}$$

where $\{Y(u)\}$ is an Ornstein–Uhlenbeck process with covariance function $e^{-|h|}$. This process X is not semi-stable. However, it is easy to see that $(2r)^{-1}R(rs, rt)$ tends to the covariance kernel of standard Brownian motion, and we may conclude that the set of limit points of $\{f_n(t, \omega)\}$ for the process (3) is the unit ball of the RKHS associated with Brownian motion, i.e., the set K_B. The details of this example and other examples will be given in Section 8.

Although the class of semi-stable Gaussian processes is relatively narrow, it should be noted that under our conditions the limit kernel Γ satisfies condition (2).

The results will be stated in Section 2. In Section 3 several known propositions that are needed in the proofs are stated as lemmas without proof. The proofs given in Sections 4–7 differ from those of Oodaira (1972) in that a different approximation to a subsequence of $\{f_n(t, \omega)\}$ is used instead of "partial sums" of the norm convergent orthogonal expansion of X. Section 8 contains some examples and remarks.

2. Results. Let $X = \{X(t, \omega), 0 \leqq t < \infty\}$ be a real separable measurable Gaussian process defined on a probability space $(\Omega, \mathscr{F}, P)$, satisfying the conditions stated in the beginning of the introduction. Further we assume the following Condition A.

CONDITION A. There exist a positive function $v(r) \to \infty$, a covariance kernel $\Gamma(s, t)$, $0 \leqq s, t \leqq 1$, and a positive non-decreasing function $g(x)$, $x \geqq 0$, such that

(A-1) $\sup_{0 \leq s, t \leq 1} |v^{-1}(r)R(rs, rt) - \Gamma(s, t)| \to \infty$ as $r \to \infty$,

(A-2) $|R(rs, rs) - 2R(rs, rt) + R(rt, rt)| \leqq v(r)g(|s - t|)$ for all $r, s, t \geqq 0$, and

$$\int_1^\infty g^{\frac{1}{2}}(e^{-u^2})\, du < \infty\,,$$

(A-3) $\Gamma(s, t)$ is strictly positive definite and $\Gamma(t, t)$ is strictly monotone increasing.

REMARK 1. If $v^{-1}(r)R(rs, rt)$ has a finite nonzero limit as $r \to \infty$, for $0 \leqq s$, $t \leqq 1$, then the limit $\Gamma(s, t)$ is a covariance kernel. Without loss of generality we may and do assume

(A-4) $\Gamma(1, 1) = 1$,

and then the kernel $\Gamma(s, t)$ is uniquely determined.

REMARK 2. It follows from (A-1) and (A-3) that $v(r)$ is a regularly varying function with exponent $\rho > 0$, i.e., $v(r) = r^\rho L(r)$, where $L(r)$ is slowly varying (see Section 8).

Let $H(\Gamma)$ be the RKHS with r.k. $\Gamma(s, t)$, $0 \leqq s, t \leqq 1$. For the theory of reproducing kernels we refer to Aronszajn (1950) and Parzen (1959). Let K denote the unit ball of $H(\Gamma)$, i.e.,

$$K = \{h \in H(\Gamma) \mid ||h||_H \leqq 1\} ,$$

where $||\cdot||_H$ is the norm of $H(\Gamma)$. Note that $H(\Gamma) \subset C[0, 1]$, since Γ is continuous by (A-1). Define, for each ω, a sequence of functions $\{f_n(t, \omega)\}$ in $C[0, 1]$ by (1).

THEOREM 1. *If conditions* (A-1), (A-2) *and* (A-4) *are satisfied, then, for* a.e. ω, *the sequence of functions* $\{f_n(t, \omega)\}$ *is equicontinuous.*

THEOREM 2. *If Condition* A *is satisfied, then, for* a.e. ω, *the set of limit points of* $\{f_n(t, \omega)\}$ *is contained in* K.

To prove the inverse inclusion relation we assume the following stronger Condition B. Let $L(X, t_0)$ denote the closed linear manifold spanned by $\{X(t), 0 \leqq t \leqq t_0\}$ and let

$$L^*(X, r\delta) = \textstyle\bigcap_{h>0} L(X, r\delta + h) \qquad \text{for } 0 \leqq \delta < 1 ,$$

$$L'(X, r\delta) = L(X, r) \ominus L^*(X, r\delta) \qquad \text{(orthogonal complement)} .$$

Let $X^*_{r\delta}(t)$ and $X'_{r\delta}(t)$ be the projections of $X(t)$, $0 \leqq t \leqq r$, on $L^*(X, r\delta)$ and $L'(X, r\delta)$, respectively. Set

$$R^*_{r\delta}(s, t) = EX^*_{r\delta}(s)X^*_{r\delta}(t) , \qquad 0 \leqq s, t \leqq r ,$$

$$R'_{r\delta}(s, t) = EX'_{r\delta}(s)X'_{r\delta}(t) , \qquad 0 \leqq s, t \leqq r .$$

Clearly we have $R(s, t) = R^*_{r\delta}(s, t) + R'_{r\delta}(s, t)$, $0 \leqq s, t \leqq r$, $0 \leqq \delta < 1$, and $R'_{r\delta}(s, t) = 0$ if $\min(s, t) \leqq r\delta$.

CONDITION B. For each $0 \leqq \delta < 1$ there exist covariance kernels $\Gamma_\delta{}^*(s, t)$ and $\Gamma_\delta{}'(s, t)$, $0 \leqq s, t \leqq 1$, such that

(B-1) $$\sup_{0\leqq s,t\leqq 1} |v^{-1}(r)R^*_{r\delta}(rs, rt) - \Gamma_\delta{}^*(s, t)| = o((\log r)^{-1}) ,$$
$$\sup_{0\leqq s,t\leqq 1} |v^{-1}(r)R'_{r\delta}(rs, rt) - \Gamma_\delta{}'(s, t)| = o((\log r)^{-1}) ,$$

(B-2) $H(\Gamma) = H(\Gamma_\delta{}^*) \oplus H(\Gamma_\delta{}')$, where $\Gamma(s, t) = \Gamma_\delta{}^*(s, t) + \Gamma_\delta{}'(s, t)$,

(B-3) $\Gamma_\delta{}^*(t, t) \to 0$ as $\delta \to 0$, uniformly in $0 \leqq t \leqq 1$,

(B-4) $\Gamma_\delta{}'(s, t)$, $\delta \leqq s, t \leqq 1$. is strictly positive definite and $\Gamma_\delta{}'(t, t)$, $\delta \leqq t \leqq 1$, is strictly monotone increasing.

REMARK 3. $\Gamma_\delta{}^*(\cdot, t)$ and $\Gamma_\delta{}'(\cdot, t)$ belong to $H(\Gamma)$ for all $0 \leqq t \leqq 1$, and (B-2) means that $H(\Gamma_\delta{}^*)$ and $H(\Gamma_\delta{}')$ are isometrically isomorphic to the orthogonal subspaces spanned by $\{\Gamma_\delta{}^*(\cdot, t), 0 \leqq t \leqq 1\}$ and $\{\Gamma_\delta{}'(\cdot, t), 0 \leqq t \leqq 1\}$, respectively.

THEOREM 3. *If Conditions* (A-2), (A-4) *and* B *are satisfied, then, for* a.e. ω, *the set of limit points of* $\{f_n(t, \omega)\}$ *contains* K.

Combining Theorems 2 and 3, we have

THEOREM 4. *If Conditions* A *and* B *are fulfilled, then, for* a.e. ω, *the set of limit points of* $\{f_n(t, \omega)\}$ *coincides with* K.

Suppose that Condition (2) holds, i.e., X is semi-stable. Then Conditions (A-1) and (B-1) are trivially satisfied, and we have the following corollary (see also Oodaira (1972)).

COROLLARY. *If Conditions* (2), (A-2) (A-4) *and* (B-2)—(B-4) *are fulfilled, then, for* a.e. ω, *the sequence of functions* $\{f_n(t, \omega)\}$ *is equicontinuous and the set of its limit points is the unit ball of the* RKHS $H(R)$ *with* r.k. $R(s, t)$, $0 \leqq s, t \leqq 1$.

It is easy to see that all conditions of the corollary are satisfied for Brownian motion, and hence the above corollary generalizes a theorem of Strassen (1964).

3. Preliminaries. In this section we list several known propositions which will be used in the proofs given in Sections 4–7.

We need an upper bound for the "tail" distribution of the supremum of X, and the following bound due to Fernique (1964) will be used. A similar result obtained by Marcus (1970) may also be used.

LEMMA 1 (Fernique). *Let* $Y = \{Y(t), 0 \leqq t \leqq 1\}$ *be a sample continuous, separable, real Gaussian process with mean zero and continuous covariance* $R(s, t)$. *Suppose that* $E\{Y(s) - Y(t)\}^2 \leqq \psi(|s - t|)$ *and* $\psi(h)$, $h \geqq 0$, *is positive and increasing. Then, for all integers* p *and all* $x \geqq (1 + 4 \log p)^{\frac{1}{2}}$, *we have*

$$P\{||Y||_C \geqq x(||R||_C^{\frac{1}{2}} + 4 \textstyle\int_1^\infty \psi(p^{-u^2})\, du)\} \leqq 4p^2 \textstyle\int_x^\infty e^{-u^2/2}\, du\,,$$

where $||\cdot||_C$ *is the supremum.*

Next, in the proofs of Lemmas 5 and 6 and Theorem 3 we shall approximate a geometric subsequence of $\{f_n(t, \omega)\}$ by a sequence of functions belonging to $H(\Gamma)$. The following proposition due to Parzen (1959) is useful for our purposes.

LEMMA 2 (Parzen). *Let* T *be a separable metric space and let* Γ *be a continuous strictly positive definite kernel defined on* $T \times T$. *Let* $T_n = \{t_1, t_2, \cdots, t_{N(n)}\}$, $n = 1, 2, \cdots$, *be a monotone increasing sequence of finite subsets of* T *such that* $\bigcup_{n=1}^\infty T_n$ *is dense in* T. *Let* C *denote the class of all continuous functions on* T. *Define, for any* $f \in C$ *and for* $n = 1, 2, \cdots$, *the set* $\{c_1(f), \cdots, c_{N(n)}(f)\}$ *by*

$$f(t_j) = \textstyle\sum_{k=1}^{N(n)} c_k(f)\Gamma(t_j, t_k)\,, \qquad j = 1, 2, \cdots, N(n)\,,$$

and, for any $f, g \in C$, *define*

$$(f, g)_n = \textstyle\sum_{k=1}^{N(n)} c_k(f)g(t_k) = \sum_{k=1}^{N(n)} c_k(g)f(t_k)\,. \tag{4}$$

Then,

(i) *for any* $f \in C$, $(f, f)_n$ *is monotone increasing,*
(ii) $f \in C$ *and* $\lim_n (f, f)_n < \infty$ *if and only if* $f \in H(\Gamma)$, *and*

(iii) *if* $f, g \in H(\Gamma)$, *then* $\lim_n (f, g)_n = (f, g)_H$, *where* $(\cdot, \cdot)_H$ *is the inner product of* $H(\Gamma)$.

Applying Lemma 2, it is readily shown that K is closed in $C[0; 1]$, and since K is also relatively compact, we have (see Oodaira (1972))

LEMMA 3. *K is compact in $C[0, 1]$.*

Finally the following Lemma 4 will be used in the proof of Theorem 2. A real matrix $A = (a_{ij})$ with positive diagonal elements is said to be a matrix with dominant principal diagonal if the sum of absolute values of all the off diagonal elements in each row is less than the diagonal element in that row. For such a matrix we define $\alpha_{ij} = |a_{ij}|$, $i, j = 1, 2, \cdots, n$, and $s_i = \sum_{j=1}^n \alpha_{ij} - \alpha_{ii}$, $i = 1, 2, \cdots, n$. Then the following lower bound for $\det |A|$ is obtained (see Ostrowski (1952)):

$$\det |A| \geqq \prod_{i=1}^n (\alpha_{ii} - s_i) .$$

This implies the following (see Marcus (1968))

LEMMA 4. *A matrix with dominant principal diagonal is strictly positive definite.*

4. Proof of Theorem 1. The proof is quite similar to that of Oodaira (1972) (see also Chover (1967)).

It suffices to prove that for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for a.e. ω and for some integer $N = N(\varepsilon, \omega) \geqq 3$,

$$|f_n(s, \omega) - f_n(t, \omega)| < \varepsilon \tag{5}$$

if $|s - t| < \delta$ and $n \geqq N$. Let $q = q(\varepsilon)$ be an integer, which will be specified later, and put $\delta(\varepsilon) = 2^{-q}$. By (1), (5) can be written as

$$|X(ns, \omega) - X(nt, \omega)| < \varepsilon(2\sigma^2(n) \log_2 n)^{\frac{1}{2}} ,$$

where $|s - t| < \delta$ and $0 \leqq s, t \leqq 1$. Let

$$A(n) = \{\omega \mid \sup_{|s-t|<2^{-q}, 0\leqq s,t\leqq 1} |X(ns) - X(nt)| \geqq \varepsilon(2\sigma^2(n) \log_2 n)^{\frac{1}{2}}\} .$$

We wish to prove that $P(\limsup_n A(n)) = 0$.

Put $n_k = 2^k$, $k \geqq \max(q, 3)$, and let

$$B(k) = \{\omega \mid \max_{2^k\leqq n<2^{k+1}} \sup_{|s-t|<2^{-q}, 0\leqq s,t\leqq 1} |X(ns) - X(nt)| \geqq \varepsilon(2\sigma^2(2^k) \log_2 2^k)^{\frac{1}{2}}\} .$$

It suffices to show that $P(\limsup_k B(k)) = 0$. Let

$$C(k) = \{\omega \mid \sup_{0\leqq h\leqq 2^{k+1-q}, 0\leqq t<t+h\leqq 2^{k+1}} |X(t + h) - X(t)| \geqq \varepsilon(2\sigma^2(2^k) \log_2 2^k)^{\frac{1}{2}}\}$$

and

$$C(k, \nu) = \{\omega \mid \sup_{t,t+h \in I(k,\nu)} |X(t + h) - X(t)| \geqq \varepsilon(2\sigma^2(2^k) \log_2 2^k)^{\frac{1}{2}}\} ,$$

where $I(k, \nu) = [(\nu - 1)2^{k-q+1}, (\nu + 1)2^{k-q+1}]$, $\nu = 1, 2, \cdots, 2^q - 1$. Since $B(k) \subset C(k) \subset \bigcup_{\nu=1}^{2^q-1} C(k, \nu)$, it is enough to show that for each ν

$$P(\limsup_k C(k, \nu)) = 0 .$$

Let

$$D(k, \nu) = \{\omega \mid \sup_{t \in I(k,\nu)} |X(t) - X(t_\nu)| \geqq (\varepsilon/2)(2\sigma^2(2^k) \log_2 2^k)^{\frac{1}{2}}\} ,$$

where $t_\nu = (\nu - 1)2^{k-q+1}$. Then $P(C(k, \nu)) \leqq 2P(D(k, \nu))$.

Define

$$Y(s; k, \nu) = X(s2^{k-q+2} + t_\nu) - X(t_\nu) ,$$
$$0 \leqq s \leqq 1, \nu = 1, 2, \cdots, 2^q - 1 .$$

By Condition (A-2), we have, for $0 \leqq s, t \leqq 1$,

$$E\{Y(s; k, \nu) - Y(t; k, \nu)\}^2 \leqq v(2^{k-q+2})g(|s - t|)$$

and

$$|EY(s; k, \nu)Y(t; k, \nu)| \leqq v(2^{k-q+2})g(1) .$$

Thus Lemma 1 can be applied to obtain

$$P(D(k, \nu)) \leqq 4p^2 \int_{y(k)}^\infty e^{-u^2/2}\, du ,$$

where

$$y(k) = (\varepsilon/2)(2 \log_2 2^k)^{\frac{1}{2}}\{\sigma(2^k)/v^{\frac{1}{2}}(2^{k-q+2})\}\{g^{\frac{1}{2}}(1) + 4 \int_1^\infty g^{\frac{1}{2}}(p^{-u^2})\, du\}^{-1} .$$

Since $\sigma^2(r) \sim v(r) = r^\rho L(r)$, $\rho > 0$, we have

$$\sigma(2^k)/v^{\frac{1}{2}}(2^{k-q+2}) > C2^{(q-2)\rho/2}$$

with some $C > 0$ for all sufficiently large k, and so we can choose q sufficiently large so that

$$\varepsilon' = (\varepsilon/2)\sigma(2^k)/v^{\frac{1}{2}}(2^{k-q+2})C' > 1$$

for all sufficiently large k, where

$$g^{\frac{1}{2}}(1) + 4 \int_1^\infty g^{\frac{1}{2}}(p^{-u^2})\, du \leqq C' < \infty .$$

Then we have

$$P(C(k, \nu)) \leqq 8p^2 \int_{y(k)}^\infty e^{-u^2/2}\, du \leqq C''(\log 2^k)^{-\varepsilon'} \leqq C'''k^{-\varepsilon'} ,$$

and, by the Borel–Cantelli lemma, $P(\limsup_k C(k, \nu)) = 0$. This completes the proof.

The following corollary enables us to work with a subsequence of $\{f_n(t, \omega)\}$ for the proof of Theorem 2 (see Chover (1967)).

COROLLARY. *For any* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon)$ *such that for* a.e. ω *and for some integer* $N = N(\varepsilon, \omega)$, *we have* $\|f_m - f_n\|_C < \varepsilon$ *for all* $m, n \geqq N$ *with* $|1 - (m/n)| < \delta$.

PROOF. For $n \geqq m \geqq 3$,

$$f_n((m/n)t) = (\sigma^2(m) \log_2 m/\sigma^2(n) \log_2 n)^{\frac{1}{2}} f_m(t) ,$$

and hence

$$|f_m(t) - f_n(t)| \leqq |1 - (\sigma^2(m) \log_2 m/\sigma^2(n) \log_2 n)^{\frac{1}{2}}||f_m(t)| + |f_n((m/n)t) - f_n(t)| .$$

Note that $\log_2 m/\log_2 n \to 1$ as $m/n \to 1$, and $|\sigma^2(m)/\sigma^2(n) - 1| \to 0$ as $m, n \to \infty$ and $m/n \to 1$. Since $f_m(0) = 0$, we have, for a.e. ω, $|f_m(t)| \leqq C$ for all sufficiently large m. Therefore, if m/n is sufficiently close to 1, then, for a.e. ω, the first term is less than $\varepsilon/2$ for all sufficiently large m and n. By Theorem 1, for a.e. ω, the second term is less than $\varepsilon/2$ for sufficiently large n, if m/n is sufficiently close to 1. The corollary is proved.

5. Lemmas. We shall show that any geometric subsequence of $\{f_n(t, \omega)\}$ can be approximated by a sequence of functions in $H(\Gamma)$.

Define the random variables $\xi_j(\omega; m, n)$, $j=1, 2, \cdots, 2^m$, for $m, n=1, 2, \cdots$, by

$$\sum_{j=1}^{2^m} \xi_j(\omega; m, n)\Gamma(t_i, t_j) = v^{-\frac{1}{2}}(n)X(nt_i, \omega) ,$$

where $t_i = i/2^m$, $t_j = j/2^m$, $i, j = 1, 2, \cdots, 2^m$. Note that $\xi_j(\omega; m, n)$ are Gaussian and that, for each ω, the functions

$$\sum_{j=1}^{2^m} \xi_j(\omega; m, n)\Gamma(t, t_j) , \qquad 0 \leqq t \leqq 1,\ m, n = 1, 2, \cdots$$

belong to $H(\Gamma)$.

LEMMA 5. *For any geometric subsequence of indices* $\{n_k = [c^k], c > 1\}$ *and for any* $\varepsilon > 0$, *there exist, for* a.e. ω, *some integers* $m = m(\varepsilon)$ *and* $k_0 = k_0(\varepsilon, \omega, m)$ *such that*

$$\sup_{0\leq t\leq 1} |f_{n_k}(t, \omega) - v^{\frac{1}{2}}(n_k)(2\sigma^2(n_k) \log_2 n_k)^{-\frac{1}{2}} \sum_{j=1}^{2^m} \xi_j(\omega; m, n_k)\Gamma(t, t_j)| < \varepsilon$$

for all $k \geqq k_0$.

PROOF. Let

$$\begin{aligned} A(m, k) &= \{\omega \mid \sup_{0\leq t\leq 1} |f_{n_k}(t, \omega) - v^{\frac{1}{2}}(n_k)(2\sigma^2(n_k) \log_2 n_k)^{-\frac{1}{2}} \\ &\qquad \times \textstyle\sum_{j=1}^{2^m} \xi_j(\omega; m, n_k)\Gamma(t, t_j)| \geqq \varepsilon\} \\ &= \{\omega \mid \sup_{0\leq t\leq 1} |v^{-\frac{1}{2}}(n_k)X(n_k t, \omega) - \textstyle\sum_{j=1}^{2^m} \xi_j(\omega; m, n_k)\Gamma(t, t_j)| \\ &\qquad \geqq \varepsilon(\sigma(n_k)v^{-\frac{1}{2}}(n_k))(2 \log_2 n_k)^{\frac{1}{2}}\} . \end{aligned}$$

It suffices to prove that $P(\limsup_k A(m, k)) = 0$ for sufficiently large m.

Let

$$Y(t; m, k) = v^{-\frac{1}{2}}(n_k)X(n_k t) - \sum_{j=1}^{2^m} \xi_j(m, n_k)\Gamma(t, t_j) , \qquad 0 \leqq t \leqq 1 ,$$

$$\alpha_k(s, t) = v^{-1}(n_k)R(n_k s, n_k t) - \Gamma(s, t) , \qquad 0 \leqq s, t \leqq 1$$

and

$$\alpha_k = \sup_{0\leq s, t\leq 1} |\alpha_k(s, t)| .$$

Then

$$\begin{aligned} E\{Y(s; m, k) - Y(t; m, k)\}^2 &\leqq 2[E\{v^{-\frac{1}{2}}(n_k)(X(n_k s) - X(n_k t))\}^2 \\ &\qquad + E\{\textstyle\sum_{j=1}^{2^m} \xi_j(m, n_k)(\Gamma(s, t_j) - \Gamma(t, t_j))\}^2] . \end{aligned}$$

The first term in the bracket is $\leqq g(|s - t|)$, by (A-2). The second term in the bracket can be written as

$$||\Gamma(\cdot, s) - \Gamma(\cdot, t)||_m^2 + (\Gamma(\cdot, s) - \Gamma(\cdot, t), (\alpha_k(\cdot, *), \Gamma(*, s) - \Gamma(*, t))_m)_m ,$$

where $(\cdot, *)_m$ is defined by (4) with $t_j = j/2^m$, $j = 1, 2, \cdots, 2^m$, and $||\cdot||_m^2 = (\cdot, \cdot)_m$. Since

$$\begin{aligned} &|(\Gamma(\cdot, s) - \Gamma(\cdot, t), (\alpha_k(\cdot, *), \Gamma(*, s) - \Gamma(*, t))_m)_m| \\ &\qquad \leqq C\alpha_k||\Gamma(\cdot, s) - \Gamma(\cdot, t)||_m^2 \to 0 \qquad \text{as } k \to \infty \end{aligned}$$

and $||\Gamma(\cdot, s) - \Gamma(\cdot, t)||_m^2 \leqq ||\Gamma(\cdot, s) - \Gamma(\cdot, t)||_H^2$, the second term is $\leqq C'||\Gamma(\cdot, s) - \Gamma(\cdot, t)||_m^2 \leqq C'g(|s - t|)$ for all sufficiently large k. Hence

$$E\{Y(s; m, k) - Y(t; m, k)\}^2 \leqq C^* g(|s - t|)$$

for all sufficiently large k. We have also

$$\begin{aligned}
&\sup_{0 \leqq s, t \leqq 1} |EY(s; m, k)Y(t; m, k)| \\
&\qquad \leqq \sup_{0 \leqq t \leqq 1} E(Y(t; m, k))^2 \\
&\qquad \leqq \sup_{0 \leqq t \leqq 1} |v^{-1}(n_k)R(n_k t, n_k t) - \Gamma(t, t)| \\
&\qquad\qquad + \sup_{0 \leqq t \leqq 1} |\Gamma(t, t) - ||\Gamma(\cdot, t)||_m^2| \\
&\qquad\qquad + \sup_{0 \leqq t \leqq 1} 2|(\Gamma(\cdot, t), \alpha_k(\cdot, t))_m| \\
&\qquad\qquad + \sup_{0 \leqq t \leqq 1} |(\Gamma(\cdot, t), (\alpha_k(\cdot, *), \Gamma(*, t))_m)_m| \\
&\qquad \leqq \sup_{0 \leqq t \leqq 1} |\Gamma(t, t) - ||\Gamma(\cdot, t)||_m^2| + C''\alpha_k .
\end{aligned} \tag{6}$$

Since $||\Gamma(\cdot, t)||_m^2$ converges to $\Gamma(t, t)$ uniformly in t by Lemma 2 and Dini's theorem, (6) can be made arbitrarily small for all sufficiently large k by choosing m sufficiently large.

Thus Lemma 1 can be applied to obtain

$$P(A(m, k)) \leqq 4p^2 \int_{y(m,k)}^{\infty} e^{-u^2}\, du ,$$

where

$$\begin{aligned}
y(m, k) = \varepsilon(\sigma(n_k)v^{-\frac{1}{2}}(n_k))(2 \log_2 n_k)^{\frac{1}{2}}\{[\sup_{0 \leqq s, t \leqq 1} |EY(s; m, k)Y(t; m, k)|]^{\frac{1}{2}} \\
+ 4(C^*)^{\frac{1}{2}} \int_1^{\infty} g^{\frac{1}{2}}(p^{-u^2})\, du\}^{-1} .
\end{aligned}$$

Note that $\int_1^{\infty} g^{\frac{1}{2}}(p^{-u^2})\, du \to 0$ as $p \to \infty$ and $\sigma^2(n_k)/v(n_k) \to 1$ as $k \to \infty$. Hence we can choose first m and then p sufficiently large so that

$$\begin{aligned}
\varepsilon' &= \tfrac{1}{2}\varepsilon^2\sigma^2(n_k)v^{-1}(n_k)\{[\sup_{0 \leqq s, t \leqq 1} |EY(s; m, k)Y(t; m, k)|]^{\frac{1}{2}} \\
&\qquad + 4(C^*)^{\frac{1}{2}} \int_1^{\infty} g^{\frac{1}{2}}(p^{-u^2})\, du\}^{-2} \\
&> 1
\end{aligned}$$

for all sufficiently large k. Then

$$P(A(m, k)) \leqq C_1(\log c^k)^{-\varepsilon'} = C_2 k^{-\varepsilon'}$$

and the Borel–Cantelli lemma gives $P(\limsup_k A(m, k)) = 0$. The proof is complete.

For the proof of Theorem 3 we need to modify Lemma 5 as follows. Define the random variables $\xi_j(\omega; \delta, m, n)$, $j = 1, 2, \cdots, 2^m$, for $m, n = 1, 2, \cdots$, by

$$\sum_{j=1}^{2^m} \xi_j(\omega; \delta, m, n)\Gamma_\delta'(t_i, t_j) = v^{-\frac{1}{2}}(n)X'_{n\delta}(nt_i, \omega) ,$$

where $t_i = \delta + (1 - \delta)i/2^m$, $t_j = \delta + (1 - \delta)j/2^m$, $i, j = 1, 2, \cdots, 2^m$. Then

LEMMA 6. *For any geometric subsequence of indices* $\{n_k = [c^k], c > 1\}$ *and for any* $\varepsilon > 0$, *there exist, for* a.e. ω, *a* $\delta = \delta(\varepsilon) > 0$ *and some integers* $m = m(\varepsilon)$ *and*

$k_0 = k_0(\varepsilon, \delta, m, \omega)$ *such that*

$$\sup_{0\leqq t\leqq 1} |f_{n_k}(t, \omega) - v^{\frac{1}{2}}(n_k)(2\sigma^2(n_k)\log_2 n_k)^{-\frac{1}{2}} \sum_{j=1}^{2m} \xi_j(\omega; \delta, m, n_k)\Gamma_\delta'(t, t_j)| < \varepsilon$$

for all $k \geqq k_0$.

Proof. The proof proceeds in the same manner as that of Lemma 5, replacing Γ by Γ_δ'. Let

$$Y(t; \delta, m, k) = v^{-\frac{1}{2}}(n_k)X'_{n_k\delta}(n_k t) - \sum_{j=1}^{2m} \xi_j(\delta, m, n_k)\Gamma_\delta'(t, t_j),$$
$$\beta_k(s, t) = v^{-1}(n_k)R'_{n_k\delta}(n_k s, n_k t) - \Gamma_\delta'(s, t),$$
$$\beta_k = \sup_{0\leqq s,t\leqq 1} |\beta_k(s, t)|.$$

Then, corresponding to (6), we have

$$\sup_{0\leqq s,t\leqq 1} |EY(s; \delta, m, k)Y(t; \delta, m, k)| \leqq \sup_{0\leqq t\leqq 1} |\Gamma_\delta^*(t, t)| + \sup_{0\leqq t\leqq 1} |\Gamma_\delta'(t, t) - ||\Gamma_\delta'(\cdot, t)||_m^2| + C\beta_k \cdot$$

By Conditions (B-1) and (B-3), this can be made arbitrarily small for all sufficiently large k by choosing δ sufficiently small and then m sufficiently large. Applying Lemma 1 just as before, we obtain the conclusion.

6. Proof of Theorem 2. Let $K(\varepsilon)$ be the ε-neighborhood of K. It suffices to show that for arbitrary $\varepsilon > 0$ the sequence $\{f_n(t, \omega)\}$ ultimately lies in $K(3\varepsilon)$. Let $n_k = [c^k]$ with $c > 1$. Choosing $c = c(\varepsilon)$ sufficiently close to 1, $|1 - (n/n_k)|$ can be made arbitrarily small for $n_k \leqq n < n_{k+1}$. By Corollary to Theorem 1, it is hence sufficient to show that the sequence $\{f_{n_k}(t, \omega)\}$ ultimately lies in $K(2\varepsilon)$. Now, by Lemma 5, it is enough to prove that

$$Z(t, \omega; m, n_k) = v^{\frac{1}{2}}(n_k)(2\sigma^2(n_k)\log_2 n_k)^{-\frac{1}{2}} \sum_{j=1}^{2m} \xi_j(\omega; m, n_k)\Gamma(t, t_j)$$

with a sufficiently large m lies ultimately in $K(\varepsilon)$. In the following lemma we shall prove that $(1+\varepsilon)^{-1}Z \in K$. Then

$$\begin{aligned} ||Z - (1+\varepsilon)^{-1}Z||_C &= \varepsilon||(1+\varepsilon)^{-1}Z||_C \\ &\leqq \varepsilon||(1+\varepsilon)^{-1}Z||_H \sup_{0\leqq t\leqq 1} (\Gamma(t, t))^{\frac{1}{2}} \\ &\leqq \varepsilon, \end{aligned}$$

and we have $Z \in K(\varepsilon)$. Thus it remains to prove the following

Lemma 7. $||Z||_H \leqq 1 + \varepsilon$ *ultimately for* a.e. ω.

Proof. Let

$$\begin{aligned} A(k) &= \{\omega \mid ||Z||_H^2 > (1+\varepsilon)^2\} \\ &= \{\omega \mid \textstyle\sum_{i,j=1}^{2m} \xi_i(\omega; m, n_k)\xi_j(\omega; m, n_k)\Gamma(t_i, t_j) \\ &\qquad > (1+\varepsilon)^2(\sigma^2(n_k)/v(n_k))(2\log_2 n_k)\}. \end{aligned}$$

We shall show that $P(\limsup_k A(k)) = 0$.

Let (Γ) and $(\xi)_k$ denote, respectively, the $2^m \times 2^m$ matrix $(\Gamma(t_i, t_j))$ and the 2^m-column vector $(\xi_j(\omega; m, n_k))$. Let S be the nonsingular matrix such that

$S'(\Gamma)S = I$, where the prime denotes the transpose, and put $(\eta)_k = S^{-1}(\xi)_k$. Then $(\xi)_k'(\Gamma)(\xi)_k = (\eta)_k'(\eta)_k$, and, if $(R/v)_k$ and $(\alpha)_k$ denote respectively the $2^m \times 2^m$-matrices $(R(n_k t_i, n_k t_j)/v(n_k))$ and $(\alpha_k(t_i, t_j))$, then the covariance matrix of $(\eta)_k$ is given by

$$\Lambda_k = E(\eta)_k(\eta)_k' = S'(R/v)_k S = S'((\Gamma) + (\alpha)_k)S = I + S'(\alpha)_k S .$$

Since $\alpha_k \to 0$ as $k \to \infty$, $(1 + \varepsilon')I - \Lambda_k$ with $\varepsilon' > 0$ is a matrix with dominant principal diagonal for all sufficiently large k. By Lemma 4 it is strictly positive definite, and, since Λ_k is also strictly positive definite, so is the matrix $(1 + \varepsilon')\Lambda_k^{-1} - I$. Let T_k be the nonsingular matrices such that $T_k'\Lambda_k^{-1}T_k = I$ and let $(\zeta)_k = T_k^{-1}(\eta)_k$. Then $(\zeta)_k = (\zeta_j(\omega; k))$ are independent Gaussian random variables with mean zero and variance one. Since $(\eta)_k'(\eta)_k \leqq (1 + \varepsilon')(\eta)_k'\Lambda_k^{-1}(\eta)_k \leqq (1 + \varepsilon')(\zeta)_k'(\zeta)_k$, we have, for any $\varepsilon' > 0$,

$$\sum_{i,j=1}^{2m} \xi_i(\omega; m, n_k)\xi_j(\omega; m, n_k)\Gamma(t_i, t_j) \leqq (1 + \varepsilon') \sum_{j=1}^{2m} \zeta_j^2(\omega; k)$$

for all sufficiently large k.

Let ε' be small enough so that $(1 + \varepsilon)^2/(1 + \varepsilon') > 1$. Then for ε'' such that $(1 + \varepsilon)^2/(1 + \varepsilon') > \varepsilon'' > 1$,

$$\begin{aligned} P(A(k)) &\leqq P\{\textstyle\sum_{j=1}^{2m} \zeta_j^2(\omega; k) > ((1 + \varepsilon)^2/(1 + \varepsilon'))(\sigma^2(n_k)/v(n_k))(2 \log_2 n_k)\} \\ &\leqq P\{\textstyle\sum_{j=1}^{2m} \zeta_j^2(\omega; k) > \varepsilon''(2 \log_2 n_k)\} \end{aligned}$$

for all sufficiently large k, because $\sigma^2(n_k)/v(n_k) \to 1$ as $k \to \infty$. Since $\sum_{j=1}^{2m} \zeta_j^2(\omega; k)$ has the χ^2-distribution with 2^m degrees of freedom, we have $P(A(k)) \leqq Ck^{-\varepsilon''}$, and, so, by the Borel–Cantelli lemma, $P(\limsup_k A(k)) = 0$. The proof is complete.

7. **Proof of Theorem** 3. Since K is compact (Lemma 3), it suffices to show that, for any $h \in K$ and for any $\varepsilon > 0$, there are, for a.e. ω, infinitely many $f_{n_k}(t, \omega)$ in some subsequence $\{f_{n_k}(t, \omega)\}$ such that $||f_{n_k} - h||_C < 3\varepsilon$. By Lemma 6, choosing δ sufficiently small and m sufficiently large, we have, for a.e. ω,

$$(7) \qquad ||f_{n_k}(\cdot, \omega) - v^{\frac{1}{2}}(n_k)(2\sigma^2(n_k) \log_2 n_k)^{-\frac{1}{2}} \sum_{j=1}^{2m} \xi_j(\omega; \delta, m, n_k)\Gamma_\delta'(\cdot, t_j)||_C < \varepsilon .$$

h can be approximated as follows. Let

$$h_\delta(t) = (h(\cdot), \Gamma_\delta'(\cdot, t))_H , \qquad 0 \leqq t \leqq 1 .$$

Then $h_\delta \in H(\Gamma_\delta')$ and, by (B-3),

$$||h - h_\delta||_C \leqq ||h||_H \sup_{0\leqq t\leqq 1} \{\Gamma_\delta^*(t, t)\}^{\frac{1}{2}} \to 0 \qquad \text{as } \delta \to 0 .$$

Define h_j, $j = 1, 2, \cdots, 2^m$, by

$$\sum_{j=1}^{2m} h_j \Gamma_\delta'(t_i, t_j) = h_\delta(t_i) ,$$

where $t_i = \delta + (1 - \delta)i/2^m$, $t_j = \delta + (1 - \delta)j/2^m$, $i, j = 1, 2, \cdots, 2^m$, and let

$$h_{\delta m}(t) = \sum_{j=1}^{2m} h_j \Gamma_\delta'(t, t_j) , \qquad 0 \leqq t \leqq 1 .$$

Then $h_{\delta m} \in H(\Gamma_\delta')$ and, by Lemma 2,

$$||h_\delta - h_{\delta m}||_C^2 \leqq ||h_\delta - h_{\delta m}||_\delta^2 = ||h_\delta||_\delta^2 - ||h_\delta||_m^2 \to 0 \qquad \text{as } m \to \infty ,$$

where $||\cdot||_\delta$ is the norm of $H(\Gamma_\delta')$. Hence, for any $\varepsilon > 0$, we can make, by choosing δ sufficiently small and then m sufficiently large,

$$(8) \qquad ||h - h_{\delta m}||_C < \varepsilon .$$

Let δ and m be so chosen that (7) and (8) hold, and let $n_k = [(2/\delta)^k]$. It is then sufficient to show that $P(\limsup_k A(k)) = 1$, where

$$\begin{aligned} A(k) &= \{\omega \mid ||v^{\frac{1}{2}}(n_k)(2\sigma^2(n_k)\log_2 n_k)^{-\frac{1}{2}} \\ &\quad \times \textstyle\sum_{j=1}^{2m} \xi_j(\omega; \delta, m, n_k)\Gamma_\delta'(\cdot, t_j) - h_{\delta m}(\cdot)||_C < \varepsilon\} \\ &= \{\omega \mid ||\textstyle\sum_{j=1}^{2m} \{\xi_j(\omega; \delta, m, n_k) - (2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)h_j\} \\ &\quad \times \Gamma_\delta'(\cdot, t_j)||_C < \varepsilon(2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)\} . \end{aligned}$$

Let

$$\begin{aligned} B(k) &= \{\omega \mid ||\textstyle\sum_{j=1}^{2m} \{\xi_j(\omega; \delta, m, n_k) - (2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)h_j\} \\ &\quad \times \Gamma_\delta'(\cdot, t_j)||_\delta < \varepsilon(2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)\} . \end{aligned}$$

Then $A(k) \supset B(k)$, since $||\cdot||_C \leqq ||\cdot||_\delta$. Let (Γ_δ') denote the matrix $(\Gamma_\delta'(t_i, t_j))$ and let S be the nonsingular matrix such that $S'(\Gamma_\delta')S = I$. Put

$$\eta_j(\omega; k) = \textstyle\sum_{p=1}^{2m} s_{jp}\xi_p(\omega; \delta, m, n_k) ,$$
$$h_j^* = \textstyle\sum_{p=1}^{2m} s_{jp}h_p ,$$

where s_{jp} are the elements of S^{-1}. Then we have

$$\begin{aligned} &||\textstyle\sum_{j=1}^{2m} \{\xi_j(\delta, m, n_k) - (2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)h_j\}\Gamma_\delta'(\cdot, t_j)||_\delta^2 \\ &\qquad = \textstyle\sum_{j=1}^{2m} \{\eta_j(k) - (2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)h_j^*\}^2 . \end{aligned}$$

Let

$$\begin{aligned} C(j, k) &= \{\omega \mid |\eta_j(k) - (2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)h_j^*| \\ &\quad < 2^{-m/2}\varepsilon(2\sigma^2(n_k)\log_2 n_k)^{\frac{1}{2}} v^{-\frac{1}{2}}(n_k)\} , \qquad j = 1, 2, \cdots, 2^m . \end{aligned}$$

Then $B(k) \supset \bigcap_{j=1}^{2m} C(j, k)$, and hence it suffices to show that, for each j, $P(\limsup_k C(j, k)) = 1$.

The random variables $\eta_j(k)$ are normally distributed with mean zero, and the covariance matrix of $\eta_j(k)$, $j = 1, 2, \cdots, 2^m$, is $I + S'(\beta)_k S$, where $(\beta)_k$ is the $2^m \times 2^m$-matrix $(\beta_k(t_i, t_j))$. Further, if $k' < k$, then $E\eta_j(k)\eta_j(k') = 0$, because $\eta_j(k') \in L^*(X, n_k\delta)$ and $\eta_j(k) \in L'(X, n_k\delta)$. Therefore, for each j, $\{\eta_j(k)\}$ is a sequence of independent Gaussian random variables with mean zero and variance $1 + w_{jk}$, where $|w_{jk}| \leqq C\beta_k$ with some constant C.

Let Φ be the distribution function of the standard normal distribution. Then

$$\begin{aligned} P(C(j, k)) &= \Phi((h_j^* + 2^{-m/2}\varepsilon)(1 + w_{jk})^{-\frac{1}{2}}\sigma(n_k)v^{-\frac{1}{2}}(n_k)(2\log_2 n_k)^{\frac{1}{2}}) \\ &\quad - \Phi((h_j^* - 2^{-m/2}\varepsilon)(1 + w_{jk})^{-\frac{1}{2}}\sigma(n_k)v^{-\frac{1}{2}}(n_k)(2\log_2 n_k)^{\frac{1}{2}}) \\ &\geqq \Phi((|h_j^*| + 2^{1-m/2}\varepsilon)(1 + w_{jk})^{-\frac{1}{2}}\sigma(n_k)v^{-\frac{1}{2}}(n_k)(2\log_2 n_k)^{\frac{1}{2}}) \\ &\quad - \Phi(|h_j^*|(1 + w_{jk})^{-\frac{1}{2}}\sigma(n_k)(v^{-\frac{1}{2}}(n_k)(2\log_2 n_k)^{\frac{1}{2}}) \\ &\geqq C'(1 + w_{jk})^{\frac{1}{2}}\sigma^{-1}(n_k)v^{\frac{1}{2}}(n_k)(\log_2 n_k)^{-\frac{1}{2}} \\ &\quad \times \exp(-|h_j^*|^2(1 + w_{jk})^{-1}\sigma^2(n_k)v^{-1}(n_k)\log_2 n_k) . \end{aligned}$$

Note that

$$\sum_{j=1}^{2m} (h_j{}^*)^2 = \sum_{i,j=1}^{2m} h_i h_j \Gamma_\delta{}'(t_i, t_j) = \|h_\delta\|_m{}^2 \leqq \|h_\delta\|_\delta{}^2 \leqq \|h\|_H{}^2 \leqq 1 ,$$

and

$$(1 + w_{jk})^{-1}\sigma^2(n_k)v^{-1}(n_k) = 1 + o((\log n_k)^{-1}) = 1 + o(k^{-1}) .$$

Hence we have

$$\begin{aligned} P(C(j, k)) &\geqq C''(\log_2 n_k)^{-\frac{1}{2}} \exp(-(1 + o(k^{-1})) \log_2 n_k) \\ &\geqq C'''(k^{1+o(k^{-1})} \log k)^{-1} \end{aligned}$$

and $\sum_k P(C(j, k)) = \infty$. Since $C(j, k)$ are independent, we obtain

$$P(\limsup_k C(j, k)) = 1$$

by the Borel–Cantelli lemma. This completes the proof.

8. Examples and remarks. (a) The process X defined by (3) in the introduction has the covariance function

$$R(s, t) = s + t - |s - t| - 1 + e^{-s} + e^{-t} - e^{-|s-t|} .$$

It immediately follows that Condition A is satisfied with $g(x) = x$, $v(r) = 2r$, and

$$\Gamma(s, t) = \tfrac{1}{2}(s + t - |s - t|) ,$$

which is the covariance function of Brownian motion. Since

$$\begin{aligned} X_{r\delta}^*(t) &= \textstyle\int_0^t Y(u)\, du && \text{for } 0 \leqq t \leqq r\delta \\ &= \textstyle\int_0^{r\delta} Y(u)\, du + Y(r\delta) \int_{r\delta}^t e^{-u+r\delta}\, du && \text{for } r\delta \leqq t \leqq r , \end{aligned}$$

we have

$$\begin{aligned} R_{r\delta}^*(s, t) &= R(s, t) && \text{for } \min(s, t) \leqq r\delta \\ &= 2(r\delta - 1 - e^{-r\delta}) + (e^{r\delta} - 1)(2e^{-r\delta} - e^{-s} - e^{-t}) \\ &\quad + e^{2r\delta}(e^{-r\delta} - e^{-s})(e^{-r\delta} - e^{-t}) && \text{for } \min(s, t) \geqq r\delta . \end{aligned}$$

Put

$$\begin{aligned} \Gamma_\delta{}^*(s, t) &= \textstyle\int_0^\delta \chi(s, u)\chi(t, u)\, du , \\ \Gamma_\delta{}'(s, t) &= \textstyle\int_\delta^1 \chi(s, u)\chi(t, u)\, du , \end{aligned}$$

where $\chi(t, u) = 1$ for $u \leqq t$ and 0 for $u > t$. It is now easy to check Condition B, and hence the set of limit points of $\{f_n(t, \omega)\}$ for the process X is, for a.e. ω, the unit ball K_B of the RKHS associated with Brownian motion.

Suppose now that $\{Y(u)\}$ is a real stationary Gaussian process with spectral density $f(\lambda) = C/(\lambda^2 + \alpha)^2$ (or $f(\lambda) = C/(\lambda^4 + \alpha^4)$), and let $X(t) = \int_0^t Y(u)\, du$. Then, by similar computations, it can be shown that the set of limit points of $\{f_n(t, \omega)\}$ for this process is also the set K_B.

(b) Let ξ_k, $k = 0, 1, \cdots$ be independent identically distributed Gaussian random variables with mean zero and variance one, and set

$$S_0 = 0 , \qquad S_n = \sum_{k=0}^{n-1} (n - k)^\alpha \xi_k , \qquad 0 \leqq \alpha < \tfrac{1}{2},\ n = 1, 2, \cdots .$$

Define a Gaussian process X by

$$X(t) = S_n \qquad \text{for } t = n$$
$$= \text{linearly interpolated} \qquad \text{for } n \leqq t \leqq n+1 .$$

Then the covariance function $R(s, t)$ of $X(t)$ satisfies the following relation:

$$\sup_{0 \leqq s,t \leqq 1} |r^{-(2\alpha+1)}R(rs, rt) - \int_0^{s \wedge t} (s-\lambda)^\alpha (t-\lambda)^\alpha \, d\lambda| = O(r^{-1}) .$$

Conditions A and B are easily checked, and so we may conclude that the set of limit points of $\{f_n(t, \omega)\}$ for the above X is the unit ball of the RKHS with r.k.

$$\Gamma(s, t) = (2\alpha + 1) \int_0^{s \wedge t} (s-\lambda)^\alpha (t-\lambda)^\alpha \, d\lambda .$$

Similarly, if

$$S_n = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^\alpha \xi_k , \qquad \alpha \geqq 0 ,$$

it can be shown that the set of limit points of $\{f_n(t, \omega)\}$ is the unit ball of the RKHS with r.k.

$$\Gamma(s, t) = (2\alpha + 1) \int_0^{s \wedge t} \left(1 - \frac{\lambda}{s}\right)^\alpha \left(1 - \frac{\lambda}{t}\right)^\alpha d\lambda .$$

Some extensions of these examples will be published elsewhere.

(c) We show that if Conditions (A-1) and (A-3) hold, then $v(r) = r^\rho L(r)$ with some $\rho > 0$, where $L(r)$ is slowly varying, and $\Gamma(xs, xt) = x^\rho \Gamma(s, t)$ for all $0 \leqq x, s, t \leqq 1$.

Let $0 \leqq x \leqq 1$. Then, for $0 \leqq s, t \leqq 1$,

$$v(rx)/v(r) = (R(rxs, rxt)/v(r))/(R(rxs, rxt)/v(rx))$$
$$\to \Gamma(xs, xt)/\Gamma(s, t) = \phi(x) < \infty \qquad \text{as } r \to \infty .$$

Then, by Lemma 1, pages 268–9, and Problem 25, page 279, of Feller (1966), we have $\phi(x) = \Gamma(xs, xt)/\Gamma(s, t) = x^\rho$ with $-\infty < \rho < \infty$ and $v(r) = r^\rho L(r)$. Since $\Gamma(t, t)$ is strictly increasing, ρ is positive.

Note that if $\lim_r R(rs, rt)/v(r) = \Gamma(s, t)$ for $0 \leqq s, t \leqq 1$, then $\lim_r R(rs, rt)/v(r)$ exists for any $s, t \geqq 0$ and is equal to $\Gamma(qs, qt)/q^\rho$ for any q such that $0 \leqq qs, qt \leqq 1$. Indeed,

$$R(rs,rt)/v(r) = (R((r/q)qs, (r/q)qt)/v(r/q))/(v(rq)/v(r))$$
$$\to \Gamma(qs, qt)/q^\rho \qquad \text{as } r \to \infty .$$

Thus, if we define $\Gamma(s, t) = \lim_r R(rs, rt)/v(r)$ for $s, t \geqq 0$, then we have $\Gamma(rs, rt) = r^\rho \Gamma(s, t)$ for all $r, s, t \geqq 0$, i.e., Γ is the covariance kernel of a semi-stable Gaussian process.

(d) The set K_B has a simple representation and it could be effectively used in the proof, provided that the set of limit points of $\{f_n(t, \omega)\}$ is K_B. However, a set different from K_B can appear as the set of limit points. Consider, for example, the following class of (semi-stable) covariance kernels:

$$\Gamma_\alpha(s, t) = (2\alpha + 1) \int_0^{s \wedge t} (s-\lambda)^\alpha (t-\lambda)^\alpha \, d\lambda , \qquad \alpha > -\tfrac{1}{2},\ 0 \leqq s, t \leqq 1$$

(see Oodaira (1972)). Let K_α be the unit ball of the RKHS $H(\Gamma_\alpha)$. It is readily seen that for each $t_0 \in [0, 1]$

$$\sup_{h \in K_\alpha} |h(t_0)| = (\Gamma_\alpha(t_0, t_0))^{\frac{1}{2}} = t_0^{\alpha+\frac{1}{2}}$$

and $h_\alpha^*(\cdot) = \Gamma_\alpha(\cdot, t_0)/(\Gamma_\alpha(t_0, t_0))^{\frac{1}{2}} \in K_\alpha$ attains the supremum. It then follows that $K_\alpha \not\subset K_\beta$ for $\alpha < \beta$. For, if $K_\alpha \subset K_\beta$, then $h_\alpha^* \in K_\beta$ and we would have $h_\alpha^*(t_0) = t_0^{\alpha+\frac{1}{2}} \leqq (\Gamma_\beta(t_0, t_0)^{\frac{1}{2}} = t_0^{\beta+\frac{1}{2}}$ for $0 < t_0 < 1$. Note that $K_0 = K_B$.

The characterization of the set of limit points of $\{f_n(t, \omega)\}$ for Gaussian processes in terms of RKHS seems quite natural in view of the role of RKHS in the theory of Gaussian processes and will be probably valid under conditions weaker than Conditions A and B. It may also be used for some other processes which converge weakly to Gaussian processes other than Brownian motion. Recently Finkelstein (1971) has shown that an analogue of Strassen's law of the iterated logarithm holds for empirical distribution functions. It is of interest to note that the set of limit points characterized there is the unit ball of the RKHS associated with the Brownian bridge.

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