DISCUSSION ON PROFESSOR KINGMAN'S PAPER

PROFESSOR D. L. BURKHOLDER (*University of Illinois*). The key to the pointwise ergodic theorem for subadditive stochastic processes is the decomposition

$$x_{st} = y_{st} + z_{st}\,. \tag{1}$$

Here $\mathbf{y}$ is an additive process satisfying $Ey_{01} = \gamma(\mathbf{x})$ and $\mathbf{z}$ is a nonnegative subadditive process with $\gamma(\mathbf{z}) = 0$. Kingman's elegant proof of the existence of such a decomposition for any subadditive process $\mathbf{x}$ is the most difficult part of his paper [2] so a slightly different proof, one which is more probabilistic in its orientation, may be of some interest.

The main novelty in the following proof is the use of Komlós's theorem [3]: *If $X_1, X_2, \cdots$ is an L^1-bounded random variable sequence* ($\sup_n E|X_n| < \infty$), *then there is a sequence $n_1 < n_2 < \cdots$ of positive integers and an integrable random variable Y such that*

$$j^{-1} \sum_{i=1}^{j} X_{n_i} \to Y$$

almost everywhere as $j \to \infty$. This theorem could be avoided if the sequence $\mathbf{f}_0 = (f_{0n})$ defined below could be shown to converge almost everywhere. However, quite apart from this possibility, Komlós's theorem gives at once enough information to carry through the proof of (1); it is enough to know that the sequence of Cesàro means of some subsequence of $\mathbf{f}_0$ converges almost everywhere.

The first steps leading to Komlós's remarkable theorem were made by Steinhaus, Austin, Rényi, and Révész; see [3]. Recent contributions have been made by Chatterji; for example, see [1].

Now let $\mathbf{x} = (x_{st})$ be a subadditive process and $\gamma = \gamma(\mathbf{x})$. The desired decomposition (1) may be deduced easily from the following fact.

LEMMA. *There is a stationary random variable sequence $f_0, f_1, \cdots$ such that $Ef_0 = \gamma$ and*

$$\sum_{k=s}^{t-1} f_k \leqq x_{st}\,, \qquad 0 \leqq s < t\,. \tag{2}$$

Given this, let $y_{st} = \sum_{k=s}^{t-1} f_k$ and $z_{st} = x_{st} - y_{st}$. Then $\mathbf{y}$ is an additive process with $Ey_{01} = \gamma$ and $\mathbf{z}$ is a nonnegative subadditive process with $\gamma(\mathbf{z}) = 0$:

$$\begin{aligned} t^{-1}Ez_{0t} &= t^{-1}E(x_{0t} - y_{0t}) \\ &= t^{-1}g_t - \gamma \to 0 \end{aligned}$$

as $t \to \infty$. This proves (1).

PROOF OF LEMMA. Let

$$f_{kn} = n^{-1} \sum_{r=1}^{n} (x_{k,k+r} - x_{k+1,k+r})\,. \tag{3}$$

Since $(x_{s+1,t+1})$ has the same distribution as (x_{st}), it is clear that $\mathbf{f}_0 = (f_{0n})$, $\mathbf{f}_1 = (f_{1n}), \cdots$ is a stationary sequence.

Let $s \leqq k \leqq t-1$ and $n > t$. By subadditivity, $x_{kr} - x_{k+1,r} \leqq x_{k,k+1}$, $r \geqq k+1$, so that

$$nf_{kn} = \sum_{r=k+1}^{k+n} (x_{kr} - x_{k+1,r}) \leqq \sum_{r=t+1}^{n} (x_{kr} - x_{k+1,r}) + tx_{k,k+1} .$$

Therefore,

$$n \sum_{k=s}^{t-1} f_{kn} \leqq \sum_{r=t+1}^{n} (x_{sr} - x_{tr}) + t \sum_{k=s}^{t-1} x_{k,k+1} .$$

By subadditivity, the first sum on the right-hand side is dominated by $\sum_{r=t+1}^{n} x_{st} = (n-t)x_{st}$. Accordingly,

$$\sum_{k=s}^{t-1} f_{kn} \leqq x_{st} + n^{-1}w_{st} \tag{4}$$

where $w_{st} = t[\sum_{k=s}^{t-1} x_{k,k+1} - x_{st}]$.

In particular, $f_{0n} \leqq x_{01}$ and, by (3),

$$Ef_{0n} = n^{-1} \sum_{r=1}^{n} (g_r - g_{r-1}) = n^{-1}g_n \geqq \gamma .$$

Therefore,

$$E|f_{0n}| \leqq E|x_{01}| + E(x_{01} - f_{0n}) \leqq E|x_{01}| + g_1 - \gamma$$

and $\mathbf{f}_0$ is L^1-bounded. Hence, by the theorem of Komlós, there is a sequence $n_1 < n_2 < \cdots$ of positive intergers and an integrable function f_0 such that

$$A_{0j} = j^{-1} \sum_{i=1}^{j} f_{0n_i} \to f_0$$

almost everywhere as $j \to \infty$. Since $\mathbf{f}_0, \mathbf{f}_1, \cdots$ is stationary,

$$A_{kj} = j^{-1} \sum_{i=1}^{j} f_{kn_i}$$

also converges almost everywhere, say to f_k, and $f_0, f_1, \cdots$ is a stationary sequence. Let $\delta_j = j^{-1} \sum_{i=1}^{j} n_i^{-1}$. Then, by (4),

$$\sum_{k=s}^{t-1} A_{kj} \leqq x_{st} + \delta_j w_{st} . \tag{5}$$

Now let $j \to \infty$ to obtain (2).

It remains to show that $Ef_0 = \gamma$. By (2) with $s = 0$,

$$Ef_0 = t^{-1} \sum_{k=0}^{t-1} Ef_k \leqq t^{-1}Ex_{0t} = t^{-1}g_t \to \gamma$$

as $t \to \infty$ so that $Ef_0 \leqq \gamma$. To show $Ef_0 \geqq \gamma$, recall that $Ef_{0n} = n^{-1}g_n \to \gamma$, which implies that $EA_{0j} \to \gamma$. Hence, by (5), which implies that $x_{01} - A_{0j} \geqq 0$, and Fatou's lemma,

$$g_1 - Ef_0 = E(x_{01} - f_0) \leqq \liminf_{j\to\infty} E(x_{01} - A_{0j}) = g_1 - \gamma .$$

Therefore, $Ef_0 \geqq \gamma$ and the proof of the lemma is complete.

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PROFESSOR DARYL DALEY (*Australian National University, Canberra*). I have much enjoyed reading Professor Kingman's paper, and have been prompted by it to think again about the two-dimensional Poisson growth process of Morgan and Welsh (1965). In that model, as with several other physical applications, subadditive processes arise as first-passage time random variable decompositions, and typically x_{st} is then a non-decreasing nonnegative random variable. Morgan and Welsh's model is on the integer lattice in the positive quadrant and hence the subadditive process to which it leads is a discrete parameter one as in Subsection 1.1. But equally one could ask whether any further simplification or properties may come out of studying continuous parameter subadditive processes with nonnegative increments. Certainly $\mathbf{S}_3$ then implies the boundedness condition at (1.4.7). And the local sample path behavior will presumably be related to the finiteness or otherwise of the dominating additive process

$$Y_{st} = \sup \textstyle\sum_{i=1}^{n} x_{t_{i-1},t_i}$$

where the supremum is taken over all finite partitions $t_0 = s < t_1 < \cdots < t_n = t$ of the interval (s, t) (in other words, Y_{st} is the Burkill integral of $\mathbf{x}$ on (s, t)). Now

$$\mu \equiv E(Y_{01}) = E(Y_{0h})/h \geqq \sup_{h>0} E(x_{0h})/h ,$$

so the finiteness of μ, entailing the a.s. finiteness of Y_{0t} for all t, then gives

$$P\{x_{0h} > u\} \leqq \mu h/u .$$

Presumably some other local properties may be deducible also (as for example from the joint distributions, when available, of $\mathbf{y}$).

The other feature which occurred to me in thinking about Morgan and Welsh's model is that the process studied there has some features of a branching process in the small, but that "crowded living conditions" prevent exponential growth in the large, producing instead a linear growth rate (cf. also Kendall's comments re the growth of forest fires in Bartlett (1957)). There arises intuitively the suggestion that this linear growth rate is due to there being asymptotically a stationary additve process, the subadditivity of x_{st} being simply due to the initial transient phase of the process. In other words, the joint distributions of

$$\{x_{0t_1} - x_{0t_0}, x_{0t_2} - x_{0t_1}, \cdots, x_{0t_n} - x_{0t_{n-1}}\}$$

for fixed $T_j = t_j - t_{j-1}$, $j = 1, \cdots, n$, converge in the limit $t_0 \to \infty$ to those of a stationary additive process with nonnegative increments, this distribution being the same as that of **y** at (1.2.6). But I can not see that this intuition helps much, for it seems to underlie the proof in [8] of the decomposition at (1.2.6), and it brings us little closer to constructing from the local properties of the process **x** the process **y**.

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PROFESSOR H. KESTEN (*Cornell University*). I found Professor Kingman's article most interesting and was impressed by his elegant derivation of the result of Furstenberg and myself on the growth of products of random matrices. Even though subadditive processes seem a simple tool, the applications show that they are quite powerful. As added evidence I would like to mention D. Richardson's recent beautiful proof ([1]) that "animals" growing in the plane according to certain stochastic rules have an asymptotic shape. (In particular this is true in Eden's growth model [2]). Richardson's work also suggests the lemma below, which can be used to show that

$$\lim_{n\to\infty} n^{-\frac{1}{2}} l(\pi_n) = c \quad \text{w.p. } 1$$

whenever the π_n are defined on a common probability space in such a way that $l(\pi_n)$ is non-decreasing (see Sub-section 2.4).

LEMMA. *Let X_s, $s \geqq 1$. and $Y_{s,t}$, $1 \leqq s < t$, be random variables (s, t integral) satisfying the following conditions*:

$$(1) \qquad P\{X_{s+t} \leqq x\} \geqq P\{X_s + X_t' + Y_{s,t} \leqq x\}$$

for all real x and $s, t \geqq 0$, where X_t' has the same distribution as X_t but is independent of all X_s,

$$(2) \qquad EY_{s,t}^2 \leqq C\,,$$

$$(3) \qquad E|X_1|^2 < \infty \qquad \textit{and}$$

$$(4) \qquad E(X_s^-)^2 < C$$

for all $s, t \geqq 1$ and some C (independent of s and t). Then, there exists a $0 \leqq \gamma < \infty$ such that

$$\sum_{k=0}^{\infty} \left\{\left|\frac{X_{m2^k}}{m2^k} - \gamma\right| > \varepsilon\right\} < \infty$$

for all $m \geqq 1$, $\varepsilon > 0$.

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Professor P. Ney (*University of Wisconsin*). It has been a pleasure to read a preprint of Professor Kingman's stimulating lecture on subadditive ergodic theory. I also took this opportunity to go back to the 1965 paper of Hammersley and Welsh, in which subadditive processes were introduced as a tool for treating percolation processes. In that paper and in the present lecture a number of interesting models and open problems are discussed.

Associated with some of these models are generalizations of the classical renewal process. I would like to briefly point out two of these, one still an open problem, and the other recently solved.

The first deals with a special case of the percolation process discussed in Sub-section 2.1 of Professor Kingman's paper, and was introduced in the Hammersley, Welsh paper. The graph under consideration in this problem is the square lattice in the (x, y) plane. Roughly speaking, to obtain the new renewal function one looks at the minimum time needed to get from the origin to the line $x = n$ over graphs lying in the cylinder $0 \leqq x \leqq n$, with the times between nodes being i.i.d. random variables. One then takes the maximum value of n for which the above minimum is $\leqq t$. The result is a random variable x_t, whose expectation $R(t)$ is the desired renewal function. It reduces to the ordinary renewal function when the plane is replaced by the line. Hammersley and Welsh proved an analogue of the *weak* renewal theorem, namely that $R(t)/t \to \lambda$ (say), They conjectured the strong renewal theorem, that $R(t + h) - R(t) \to \lambda h$, but as far as I know this result is still open. The constant λ depends on the random travel time between nodes of the graph, and its determination is not completely known.

The second model deals with the products of random matrices discussed in Sub-section 2.2 of the present lecture. Here $X_n = Y_1 Y_2 \cdots Y_n$, where $\{Y_n\}$ is a stationary sequence of positive random matrices. By identifying a suitable subadditive process, the author gives an elegant new proof of the Furstenberg–Kesten theorem that if $E \log^+ ||Y_1|| < \infty$, then $\lim_{n\to\infty} n^{-1} \log ||X_n|| = \alpha$ exists w.p. 1. ($||\ ||$ is the max. norm). Kesten has recently shown that if the Y_n's are independent and identically distributed and if $\alpha > 0$, then for $h \geqq 1$, and fixed row vector v,

$$\lim_{t\to\infty} E\#\{n : t \leqq |vY_1 \cdots Y_n| \leqq th\} = \frac{\log h}{\alpha}.$$

After a logarithmic transformation we can see that this impressive result is a strong renewal theorem for the subadditive process in Professor Kingman's paper. (Professor Kesten's result is in a recent preprint, and I appreciate his making it available to me).

Professor Frank Spitzer (*Cornell University*). I have enjoyed reading this paper, to see in particular how beautifully it solves the problem of products of random matrices. I observe also that an old problem of mine is a corollary to

Professor Kingman's theorem: Let x_t be a spatially homogeneous Markov process with values in R^ν. (For example, x_t could be ν-dimensional Brownian motion.) Let A be a compact subset of R^ν. Attach x_t to A and consider the volume, V_t, swept out by A during $[0, t]$; that is, $V_t = V_{0,t}$ where

$$V_{s,t} = |\bigcup_{s \leq \tau \leq t} (x_\tau + A)| ,$$

$|\cdot|$ denotes volume and $x_t + A$ denotes the random set $\{y \in R^\nu : y - x_t \in A\}$. Then $V_{s,t}$ is subadditive, and hence

$$\lim_{t\to\infty} t^{-1} V_t = C(A) \quad \text{exists a.s.}$$

An easy calculation now shows that

$$C(A) = \lim_{t\to\infty} t^{-1} E(V_t) = \text{(electrostatic) capacity of} \quad A ,$$

which is the capacity of Markov process theory.

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PROFESSOR J. M. HAMMERSLEY (*Institute of Economics and Statistics, Oxford*). Kingman employs postulates which are, naturally enough, appropriate for the theorems he wishes to prove. On the other hand subadditive processes, which do not satisfy these postulates, can arise in certain practical situations. An example is the first-death problem for an age-dependent branching process, defined as follows:

(i) The family tree originates from a single progenitor who dies at time $T=0$.

(ii) Each person in the tree has a (birth to death) lifetime U distributed with cumulative distribution function $G(u)$ and mean lifetime $\bar{u} = EU < \infty$. We suppose that $G(u) = 0$ for $u < 0$, and that $G(\infty) = 1$.

(iii) When anyone dies, he is replaced immediately by j newly-born offspring with probability p_j $(j = 0, 1, \cdots)$. We write $P(z) = \sum_{j=0}^{\infty} p_j z^j$; and we suppose $P(1) = 1$ and $p_0 < 1 < P'(1) = q$, say. Hence there is a positive chance $\rho > 0$ that the tree will propagate indefinitely, where $P(1 - \rho) = 1 - \rho$.

(iv) All members of the tree are independent, both in lifetimes and in the numbers of their offspring.

(v) The rth generation $(r = 1, 2, \cdots)$ consists of the offspring of the $(r - 1)$th generation, the zeroth generation being just the single progenitor.

The first-death problem for this branching process is to discuss at what moment of time a death first occurs to some member of the rth generation. We can set up a subadditive process x_{rs} for integers $0 \leqq r < s$ as follows:

(a) $x_{rs} = \infty$ if the subsequent recipe fails at any juncture through non-existence (for example, if the rth generation is empty).

(b) Let T_1 be the earliest time at which some person in the rth generation dies. Consider all members of the sth generation who are descendants of *this* particular person in the rth generation. Let T_2 be the earliest time at which one of *these* descendants dies. Define $x_{rs} = T_2 - T_1$.

In particular, x_{0r} is the first-death time for the rth generation.

In this problem, postulate $\mathbf{S}_1$ holds and so does $\mathbf{S}_2'$. But $\mathbf{S}_2$ does not, because the joint distribution of (x_{02}, x_{13}) differs from that of (x_{13}, x_{24}). We can, nevertheless, obtain a few results for this first-death problem from the following theorems:

THEOREM 1. *If a sequence of distributions F_r in the convolutive semigroup* $\mathbf{D}$ *is superconvolutive, namely if*

$$F_{r+s}(x) \geqq (F_r * F_s)(x) \qquad \text{for all } x, \tag{1}$$

then the function

$$K_r(\theta) = \log \int_{-\infty}^{\infty} e^{-\theta x}\, dF_r(x) \tag{2}$$

is a superadditive function of r for each fixed $\theta \geqq 0$, and $\log F_r(rx)$ *is a superadditive function of r for each fixed x. Also the limits*

$$\psi(x) = \lim_{r\to\infty} r^{-1} \log F_r(rx)\,, \qquad K(\theta) = \lim_{r\to\infty} r^{-1} K_r(\theta)\,, \qquad (\theta \geqq 0) \tag{3}$$

satisfy the reciprocal relations

$$\psi(x) = \inf_{\theta\geqq 0}\,[K(\theta) + \theta x]\,, \qquad K(\theta) = \sup_x\,[\psi(x) - \theta x]\,, \qquad (\theta \geqq 0)\,. \tag{4}$$

If, in addition, the distributions F_r are all proper, there exists a constant γ such that

$$\begin{aligned} F_r(rx) &\to 1 \quad (x > \gamma) \\ &\to 0 \quad (x < \gamma) \qquad \text{as } r \to \infty\,. \end{aligned} \tag{5}$$

THEOREM 2. *If $Q(z)$ is a concave non-decreasing function defined on $0 \leqq z \leqq 1$ such that for some ρ satisfying $0 < \rho \leqq 1$*

$$Q(\rho) = \rho \qquad \text{and,} \qquad q = Q'(0) > 1\,, \tag{6}$$

and if the sequence of distributions F_r is given by

$$F_{r+1}(x) = Q[(F_r * G)(x)]\,, \qquad (r = 0, 1, 2, \cdots) \tag{7}$$

where $F_0(x) = 0$ or 1 according as $x < 0$ or $x > 0$, and where G is a proper distribution of a nonnegative random variable with a mean $\bar{u}$, then the sequence F_r is superconvolutive and

$$\begin{aligned} F_r(rx) &\to \rho \quad (x > \gamma) \\ &\to 0 \quad (x < \gamma) \qquad \text{as } r \to \infty\,, \end{aligned} \tag{8}$$

where γ is the unique root of

$$\inf_{\theta \geqq 0} \int_{0-}^{\infty} e^{\theta(\gamma - u)}\, dG(u) = 1/q\,, \qquad \gamma < \bar{u}\,. \tag{9}$$

Moreover, if x_r is a random variable with distribution F_r, then conditionally on $x_r < \infty$ the random variable x_r/r converges in probability to γ as $r \to \infty$, and

$$\gamma = \lim_{r\to\infty} E(x_r/r \mid x_r < \infty)\,. \tag{10}$$

THEOREM 3. *An independent subadditive process is characterized by a superconvolutive sequence of distributions, but the converse is false in general.*

Theorem 1 is the superconvolutive generalization of two familiar convolutive theorems, namely the weak law of large numbers and Chernoff's theorem [1] on the extreme tails of a distribution. Theorem 2 yields an explicit equation (9) for the time constant γ of the subadditive process for the first-death problem in an age-dependent branching process, when we take $Q(z) = 1 - P(1 - z)$.

I have submitted a paper for publication which proves these theorems, and which discusses some other applications of superconvolutive sequences and subadditive processes.

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I am most grateful to those who have taken the trouble to comment so constructively on the paper; their contributions go to show that there is yet a good deal to be discovered about subadditive processes and their applications.

Professor Burkholder is of course quite right to see the weight of the proof of the pointwise ergodic theorem in the demonstration that there is an additive process **y**, lying below the subadditive process **x**, but having the same value of γ. My argument used the most obvious of all compactness principles, and it is pleasant to see how smoothly the more sophisticated approach by Komlós's theorem reaches the desired result. Is it possible that the existence of **y** lies less deep than either of these arguments would suggest, and that a process **y** with the required properties might be constructed more directly?

Dr. Daley's remarks, as well as suggesting more further applications, raise the possibility of using the Burkill integral to study continuous-parameter subadditive processes. This is a tool which has not often appeared in the statistical literature, though recent work by G. S. Goodman (to appear in the *Annals of Mathematics*) shows its relevance to some rather different stochastic problems.

I am most impressed by Professor Kesten's proof of almost sure convergence in Ulam's problem, and by the technique which may well be applicable to similar problems. An interesting feature of his lemma, and one which makes it potentially very powerful, is that when $Y = 0$ its hypotheses bear only on the individual distributions of the X_t, and not on their joint distributions.

Professor Ney points out that I have seized on just one aspect, the Law of Large Numbers, of the work of Hammersley and Welsh, and that they raise a number of other fascinating questions. It would indeed be interesting to establish an analogue for subadditive processes of Blackwell's renewal theorem, particularly if this were powerful enough to contain and to generalize the result of Kesten which he quotes.

Professor Spitzer's example shows how subadditive processes arise in a natural way in yet another important area of probability theory. What is unusual and significant about his result is the explicit computation of the constant γ; perhaps there is a lesson here for the other subadditive problems for which such an identification is not yet possible.

Dr. Hammersley's remarks, which reached me after I had written the preceding paragraphs, are characteristically profound and challenging. I acknowledge his counter-example to my assertion that processes satisfying $\mathbf{S}_2'$ but not $\mathbf{S}_2$ are "highly artificial." But it is worth noting that his example does satisfy a weakened form of $\mathbf{S}_2$:

$\mathbf{S}_2''$: For each $k \geqq 1$, the sequence $(x_{(n-1)k,nk}; n \geqq 1)$ is stationary.

Many of the proofs of [8] go through with $\mathbf{S}_2$ replaced by $\mathbf{S}_2''$, and Theorem 1 is still true, except that convergence with probability one must be replaced by

$$\mathbf{P}\{\limsup_{t\to\infty} x_{0t}/t = \xi\} = 1 .$$

However, in Hammersley's example there is no difficulty in replacing "lim sup" by "lim" by a simple argument (essentially van Dantzig's idea of "collective marks") which gives a lower bound for x_{0t}.

An alternative way to reach the same conclusion is to apply Kesten's lemma (with $Y = 0$). Professor Kesten has unwittingly but significantly contributed to the Hammersley theory of superconvolutive sequences.

There remains the difficulty of the infinite values which x_{st} may take with positive probability. Say that an individual is *fecund* if he has infinitely many descendants. Then the fecund individuals form an age-dependent branching process with the same lifetime distribution G but a new family size distribution

$$p_0^* = 0 , \qquad p_j^* = \rho^{-1} \sum_{k=j}^{\infty} p_k \binom{k}{j} \rho^j (1-\rho)^{k-j} , \qquad (j \geqq 1) .$$

With the obvious notation, $q^* = q$, $\gamma^* = \gamma$, $x_{st}^* < \infty$ and if the original tree propagates indefinitely $x_{0n} \leqq x_{0n}^*$. From this it is very easy to deduce the strong law for x_{0n} from that for x_{0n}^*.

The really interesting aspect of Dr. Hammersley's contribution, however, is the technique for determining the actual value of γ. As it stands, it is rather specific to the particular example, but as the basis of a general method for finding

γ, when some element of independence is present in a subadditive process, it is potentially of great practical and theoretical importance.

Perhaps I might reply by recommending the interested reader to the Hammersley–Welsh paper which started this branch of random process theory, and which contains a good deal not touched upon in the present account.

J. F. C. KINGMAN