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A SIMPLE PROOF OF A KNOWN RESULT IN RANDOM WALK THEORY[1]

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Let $\{X_n, n \geqq 1\}$ be a stationary independent sequence of real random variables, $S_n = X_1 + \cdots + X_n$, and α_A the hitting time of the set A by the process $\{S_n, n \geqq 1\}$, where A is one of the half-lines $(0, \infty)$, $[0, \infty)$, $(-\infty, 0]$ or $(-\infty, 0)$. This note provides a simple proof of a known result in random walk theory on necessary and sufficient conditions for $E\{\alpha_A\}$ to be finite. The method requires neither generating functions nor moment conditions on X_1.

Let $\{X_n, n \geqq 1\}$ be a stationary independent process of real random variables defined on some probability space $(\Omega, \mathscr{F}, P)$, $S_0 \equiv 0$, and $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$. Set $\alpha_A = n$ if $n = \inf\{k : k \geqq 1 \text{ and } S_k \in A\}$ and $\alpha_A = +\infty$ if no such n exists, where A is one of $(0, \infty)$, $[0, \infty)$, $(-\infty, 0]$, or $(-\infty, 0)$; that is, α_A is the hitting time of the set A by the process $\{S_n, n \geqq 1\}$. This note provides a simple proof of a known result in random walk theory on the finiteness of $E\{\alpha_A\}$.

Assume that $P\{X_1 = 0\} < 1$. It follows, without recourse to moment conditions on the distribution of X_1 (see Theorem 8.2.5 in Chung (1968)), that there are three mutually exclusive possibilities for the random walk $\{S_n, n \geqq 1\}$, each occurring with probability one:

(i) $\lim_{n\to\infty} S_n = -\infty$,
(ii) $\lim_{n\to\infty} S_n = +\infty$, or
(iii) $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = +\infty$.

With these conditions there is the following known result from random walk theory; see Section 8.4 of [1] or the second section of Chapter 12 in Feller (1971).

THEOREM. *If A is $(-\infty, 0]$ or $(-\infty, 0)$ then $E\{\alpha_A\} < +\infty$ if and only if* (i) *holds. If A is $[0, \infty)$ or $(0, \infty)$ then $E\{\alpha_A\} < +\infty$ if and only if* (ii) *holds. If* (iii) *holds then $E\{\alpha_A\} = +\infty$ for each A.*

Here is our proof of this standard result.

The second statement of the theorem follows from the first (consider the random walk generated by $\{-X_n, n \geqq 1\}$), and the third from the first and the second. It suffices, therefore, to prove the first statement.

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Suppose condition (i) holds. For $j \geqq 0$ let $M_j = \max(S_0, \cdots, S_j)$, $m_j = P\{M_j = 0\}$, $M = \sup_{j\geqq 0} M_j$, $m = P\{M = 0\}$, $f_j = P\{\alpha_{(-\infty,0]} > j\}$, $W_0 \equiv 0$, $W_{j+1} = \max(W_j + X_{j+1}, 0)$ and $L(j) = \max\{i : i \leqq j \text{ and } W_i = 0\}$. The random variables M_j and W_j are identically distributed for each j, whence

$$
\begin{aligned}
P\{M_{n+1} > 0\} &= \textstyle\sum_{j=0}^{n} P\{W_{n+1} > 0, L(n) = j\} \\
(*) \qquad &= \textstyle\sum_{j=0}^{n} P\{W_j = 0, X_{j+1} > 0, \cdots, X_{j+1} + \cdots + X_{n+1} > 0\} \\
&= \textstyle\sum_{j=0}^{n} m_j f_{n+1-j}
\end{aligned}
$$

for each $n \geqq 0$. Now $m = P\{\alpha_{(0,\infty)} = +\infty\}$ and since (i) holds it follows that $m > 0$; see Theorem 8.2.4 in [1]. The above equation gives $1 \geqq m_0 f_{n+1} + \cdots + m_n f_1$, which in turn yields $f_1 + \cdots + f_{n+1} \leqq m^{-1}$ because $m_j \geqq m$ for all j. We conclude that $E\{\alpha_{(-\infty,0]}\}$ is finite, and it then follows from $(*)$ that $\alpha_{(-\infty,0]}$ has mean m^{-1}. For $\alpha_{(-\infty,0)}$, let $\beta_0 \equiv 0$, $\beta_k = \inf\{n > \beta_{k-1} : S_n \leqq S_{\beta_{k-1}}\}$ and $Y_k = S_{\beta_k} - S_{\beta_{k-1}}$ when $k \geqq 1$. We have $\beta_1 = \alpha_{(-\infty,0]}$, and by virtue of (i) each of $\{\beta_k - \beta_{k-1}, k \geqq 1\}$ and $\{Y_k \geqq 1\}$ is a stationary independent sequence. Moreover, $P\{Y_1 < 0\} \geqq P\{X_1 < 0\} > 0$. Denoting by t the first index k for which $Y_k < 0$, we observe that $\alpha_{(-\infty,0)} = \beta_t$ and that t is geometrically distributed with parameter $P\{Y_1 < 0\}$. Wald's Lemma then gives $E\{\alpha_{(-\infty,0)}\} = E\{\alpha_{(-\infty,0]}\} \cdot E\{t\} < +\infty$.

To show the condition is necessary suppose that (i) does not hold. Then (ii) or (iii) holds, and either dictates that $m_{n+1} \to 0$, whence $(*)$ yields $m_0 f_{n+1} + \cdots + m_n f_1 \to 1$. Under these circumstances $\sum_{j\geqq 1} f_j$ must diverge, that is, $E\{\alpha_{(-\infty,0]}\} = +\infty$. Moreover, $\alpha_{(-\infty,0]} \leqq \alpha_{(-\infty,0)}$. This completes the proof.

REFERENCES

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