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MIXTURES OF PERFECT PROBABILITY MEASURES

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It is shown that a perfect mixture of perfect probability measures need not be perfect and that a perfect mixture of nonperfect measures can be perfect. Perfect mixtures of discrete measures are shown to be perfect.

1. Introduction and summary. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two measurable spaces and let $\mu(x, B)$ be a transition probability on $X \times \mathscr{B}$, that is, $\mu(x, B)$ is a function defined on $X \times \mathscr{B}$ taking values in $[0, 1]$ such that for every x in X $\mu(x, \cdot)$ is a probability measure on $\mathscr{B}$ and for every B in $\mathscr{B}$ $\mu(\cdot, B)$ is an $\mathscr{A}$-measurable function. Let λ be a probability measure on $\mathscr{A}$. The set function μ on $\mathscr{B}$ defined by $\mu(B) = \int_X \mu(\cdot, B)\, d\lambda$, $B \in \mathscr{B}$ is a probability measure on $\mathscr{B}$. μ is called the λ-mixture of the $\mu(x, \cdot)$'s. λ is called a mixing measure, the $\mu(x, \cdot)$'s are called mixand measures and μ is called a mixture measure. The properties of a mixture measure depend on those of the mixing and mixand measures. We shall study here the role of perfectness of probability measures in the mixture problem.

We call a mixture measure perfect mixture or nonperfect mixture according as the mixing measure is perfect or nonperfect. Rodine (1966) conjectured that perfect mixtures of perfect measures are perfect and also raised the following question. If μ is perfect, does it follow that the $\mu(x, \cdot)$'s are perfect except possibly for x's in a λ-null set? We shall give examples in Section 2 to show that the conjecture is false and that the answer to the above question is in the negative. In Section 3 we show that perfect mixtures of discrete measures are perfect.

All measures considered in this paper are probability measures. $\mathscr{B}_{[0,1]}$ denotes the Borel σ-algebra of $[0, 1]$. If $(X, \mathscr{A}, P)$ is a probability space and if $\{A_n, n = 1, 2, \cdots\}$ is a sequence of sets such that $A_n \in \mathscr{A}$ for every n, then $\sigma(\{A_n\})$ will denote the σ-algebra generated by the collection $\{A_n, n = 1, 2, \cdots\}$. A set $A \in \mathscr{A}$ is said to be an atom of P if for every $\mathscr{A}$-measurable set B contained in A either $P(B) = 0$ or $P(A) = P(B)$. The measure P is said to be discrete if there is a sequence $\{A_n, n = 1, 2, \cdots\}$ of pairwise disjoint atoms of P such that $\sum_{n=1}^{\infty} P(A_n) = 1$.

Let $(X, \mathscr{A}, P)$ be a probability space. P is called perfect if for every $\mathscr{A}$-measurable real-valued function f on X and every subset A of the real line for which $f^{-1}(A)$ belongs to $\mathscr{A}$ there is a linear Borel set B contained in A such that $P(f^{-1}(A)) = P(f^{-1}(B))$. It is known (see Sazonov (1962)):

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(P1) that a measure P on a measurable space $(X, \mathscr{A})$ is perfect if and only if for every $\mathscr{A}$-measurable real-valued function f on X there is a linear Borel set $B(f)$ contained in $f(X)$ such that $P(f^{-1}B(f)) = 1$.

(P2) that a measure is perfect if and only if its restriction to every countably generated sub-σ-algebra is perfect, and

(P3) that the restriction to any sub-σ-algebra of a perfect measure is perfect.

It follows from P1 that a discrete measure is perfect.

Suppose $(X, \mathscr{A})$ is a measurable space with $\mathscr{A}$ countably generated. Let $\{A_n, n = 1, 2, \cdots\}$ be a sequence of sets such that $A_n \in \mathscr{A}$ for every n and $\sigma(\{A_n\}) = \mathscr{A}$. Let $f(x) = \sum_{n=1}^{\infty} (2/3^n) I_{A_n}(x)$ be the characteristic function of $\{A_n\}$. Then $f(x_1) \neq f(x_2)$ if x_1 and x_2 belong to different atoms of $\mathscr{A}$ and $f(\mathscr{A}) = \mathscr{B}_{[0,1]} \cap f(X)$. The following lemma is easily verified using P1.

LEMMA 1. *Suppose μ is a probability measure on $(X, \mathscr{A})$. Then μ is perfect if and only if $(f(X), \mathscr{B}_{[0,1]} \cap f(X), \mu f^{-1})$ is a perfect probability space.*

2. The examples. We shall give three examples in this section. The first one will show that a perfect mixture of perfect measures need not be perfect. The second will show that the mixand measures can all be nonperfect though the mixing and the mixture measures are both perfect. The last will show that perfectness of the mixture and mixand measures does not imply the perfectness of the mixing measure.

EXAMPLE 1. Let X be the unit interval, $\mathscr{A}$ the Borel subsets of X and λ the Lebesgue measure on $\mathscr{A}$. Consider the product space $(X \times X, \mathscr{A} \otimes \mathscr{A}, \lambda \otimes \lambda)$. For our purpose it is enough to consider a set E, contained in $X \times X$, which meets every closed subset of positive $\lambda \otimes \lambda$ measure and no three of whose points are collinear, constructed by Sierpiński (1920). But we use the following lemma due to K. P. S. Bhaskara Rao.

LEMMA 2. *There exists a subset E of $X \times X$ such that*

(a) *E intersects every closed subset of $X \times X$ of positive $\lambda \otimes \lambda$ measure and*

(b) *For every x in X, the set $E_x = \{x' \in X: (x, x') \in E\}$ is either empty or a singleton.*

PROOF. Let ω_c be the first ordinal corresponding to c, the cardinality of the continuum. Let $\{A_\alpha, \alpha < \omega_c\}$ be a well-ordering of all closed subsets of $X \times X$ of positive measure. We shall define a transfinite sequence $\{p_\alpha = (x_\alpha, y_\alpha): \alpha < \beta\}$ as follows. Take $p_1 = (x_1, y_1) \in A_1$. Suppose $\{p_\alpha = (x_\alpha, y_\alpha): \alpha < \beta\}$ have been defined for $\beta < \omega_c$. The set $\{x: \lambda((A_\beta)_x) > 0\}$ is an uncountable Borel set and hence has cardinality c. So we can find x_β in $\{x: \lambda((A_\beta)_x) > 0\} - \{x_\alpha: \alpha < \beta\}$. Take a $y_\beta \in (A_\beta)_{x_\beta}$. Let $p_\beta = (x_\beta, y_\beta)$. Let $E = \{p_\alpha: \alpha < \omega_c\}$. Then E has the required properties.

The set E given by Lemma 2 is such that $(\lambda \otimes \lambda)^*(E) = (\lambda \otimes \lambda)^*(Y) = 1$ where

$Y = X \times X - E$ and $(\lambda \otimes \lambda)^*$ is the outer measure induced by $\lambda \otimes \lambda$. Further, for every x in X, Y_x differs from X by utmost one point. Now let

$$\mathscr{B} = \mathscr{A} \otimes \mathscr{A} \cap Y = \{B \cap Y : B \in \mathscr{A} \otimes \mathscr{A}\}.$$

Since $(\lambda \otimes \lambda)^*(Y) = 1$ if we define $\mu(B \cap Y) = \lambda \otimes \lambda(B)$, $B \in \mathscr{A} \otimes \mathscr{A}$, then μ is a measure on $\mathscr{B}$. Define $\mu(x, B \cap Y)$ on $X \times \mathscr{B}$ by $\mu(x, B \cap Y) = \lambda(B_x \cap Y_x) = \lambda(B_x)$, $B \in \mathscr{A} \otimes \mathscr{A}$, $x \in X$. $\mu(x, B \cap Y)$ is a transition probability on $X \times \mathscr{B}$ and λ and the $\mu(x, \cdot)$'s are perfect. For any $B \in \mathscr{A} \otimes \mathscr{A}$ we have $\int \mu(x, B \cap Y)\, d\lambda = \int \lambda(B_x)\, d\lambda = \lambda \otimes \lambda(B) = \mu(B \cap Y)$. In other words μ is the λ-mixture of $\mu(x, \cdot)$'s. Let g be the restriction to Y of a one-to-one real valued, $\mathscr{A} \otimes \mathscr{A}$-measurable function on $X \times X$. For every linear Borel set B contained in $g(Y)$, $g^{-1}(B)$ is an $\mathscr{A} \otimes \mathscr{A}$ measurable set contained in Y and since $(\lambda \otimes \lambda)^*(E) = 1$, $\mu(g^{-1}(B)) = \lambda \otimes \lambda(g^{-1}(B)) = 0$. Thus by P1 μ is non-perfect.

EXAMPLE 2. Let X be any uncountable set and $\mathscr{A}$ the countable and co-countable σ-algebra on X. For each x in X let $(Y_x, \mathscr{B}_x, P_x)$ be a nonperfect probability space. Let $Y = \prod_{x \in X} Y_x$, the product space and $\mathscr{B} = \bigotimes_{x \in X} \mathscr{B}_x$, the product σ-algebra. Fix f_0 in Y. Let $f_0^x = f_0|_{X-\{x\}}$, the restriction of f_0 to $X - \{x\}$. For every $B \in \mathscr{B}$ define $B_x = f_0^x$th section of $B = \{g(x) \in Y_x : g \in B$ and $g = f_0$ on $X - \{x\}\}$. It can be easily verified that $\{B_x : B \in \mathscr{B}\} = \mathscr{B}_x$. Define $\mu(x, B)$ on $X \times \mathscr{B}$ as $\mu(x, B) = P_x(B_x)$. For each x in X $\mu(x, \cdot)$ is a probability measure on $\mathscr{B}$. P_x on $\mathscr{B}_x$ is nonperfect implies that $\mu(x, \cdot)$ is nonperfect on $\mathscr{B}_0^x = \{B_x \times \prod_{z \in X-\{x\}} Y_z : B_x \in \mathscr{B}_x\}$. Therefore by P3 $\mu(x, \cdot)$ is nonperfect on $\mathscr{B}$ for every x in X. Since every B in $\mathscr{B}$ is a countable dimensional cylinder it follows that either $B_x = \emptyset$ for all but a countable number of x's or $B_x = Y_x$ for all but a countable number of x's. So for each fixed B in $\mathscr{B}$ either $\mu(x, B) = 0$ for all but a countable number of x's or $\mu(x, B) = 1$ for all but a countable number of x's. Hence $\mu(\cdot, B)$ is $\mathscr{A}$-measurable for every B in $\mathscr{B}$. Thus $\mu(x, B)$ is a transition probability on $X \times \mathscr{B}$. Let λ be the 0–1 valued measure on $\mathscr{A}$ defined by $\lambda(A) = 0$ or 1 according as A is countable or cocountable. λ is perfect by P1 because the range of every real valued $\mathscr{A}$-measurable function is a countable set. The mixture μ is such that $\mu(B) = 0$ or 1 according as $f_0 \notin B$ or $f_0 \in B$. For any real valued $\mathscr{B}$-measurable function h on Y, $h(f_0)$, being a singleton, is a linear Borel set contained in $h(Y)$ such that $\mu(h^{-1}(h(f_0))) = 1$. Therefore by P1, μ is perfect as well. By choice $\mu(x, \cdot)$ is nonperfect for every x in X.

The above example is only a modification of the one constructed by K. P. S. Bhaskara Rao and M. Bhaskara Rao (1972) to show that mixture of nonatomic measures need not be nonatomic. The set up in both examples is the same but instead of taking the $\mu(x, \cdot)$'s to be nonatomic, we take them here to be nonperfect.

EXAMPLE 3. In order to show that a nonperfect mixture of perfect measures can be perfect it is sufficient to consider a perfect measure μ on $(Y, \mathscr{B})$, a nonperfect space $(X, \mathscr{A}, \lambda)$ and define $\mu(x, B)$ on $X \times \mathscr{B}$ by taking $\mu(x, B) = \mu(B)$

for all x in X. But in this case the transition probability $\mu(x, B)$ is actually measurable with respect to the trivial sub-σ-algebra $\{X, \phi\}$ of $\mathscr{A}$. Clearly the restriction of λ to this σ-algebra is perfect and thus in this case μ turns out in fact to be a perfect mixture of perfect measures. We give below a non-trivial example of a perfect measure which is nonperfect mixture of perfect measures.

Let Y be the unit interval and $\mathscr{B}$, the Borel subsets of Y. Let X be a subset of the unit interval with outer Lebesgue measure one and inner Lebesgue measure zero. Let $\mathscr{A} = \mathscr{B} \cap X$, the trace of $\mathscr{B}$ on X and let λ on $\mathscr{A}$ be the outer Lebesgue measure. $(X, \mathscr{A}, \lambda)$ is a nonperfect probability space. Define a transition probability $\mu(x, B)$ on $X \times \mathscr{B}$ by $\mu(x, B) = I_B(x)$. Observe that $\mathscr{A}$ is the smallest σ-algebra on X with respect to which the transition function is measurable. Each $\mu(x, \cdot)$ is perfect and the mixture, being a measure on the Borel subsets of the unit interval, is perfect (see [3] Section 2.5) though λ is nonperfect.

3. Perfect mixtures of discrete measures. Let $(X, \mathscr{A}, \lambda)$ be a probability space. Let $(Y, \mathscr{B})$ be a measurable space and let $\mu(x, B)$ be a transition probability on $X \times \mathscr{B}$. Let μ be the λ-mixture of $\mu(x, \cdot)$'s and let P_μ on $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ be the product probability measure defined by

$$P_\mu(C) = \int_X \mu(x, C_x)\, d\lambda\,, \qquad C \in \mathscr{A} \otimes \mathscr{B}.$$

THEOREM 1. *Suppose λ is perfect. Then μ is perfect if and only if P_μ is perfect.*

PROOF. If μ is perfect then both the marginals of P_μ are perfect and hence P_μ is perfect (see [3] Theorem 6). If P_μ is perfect then by P3 the marginal of P_μ on $X \otimes \mathscr{B}$ is perfect and hence μ is perfect.

LEMMA 3. *Suppose $(X, \mathscr{A})$ is a measurable space with $X \in \mathscr{B}_{[0,1]}$* and $\mathscr{A} = \mathscr{B}_{[0,1]} \cap X$. *If $\mu(x, B)$ is a transition probability on $X \times \mathscr{B}_{[0,1]}$ then the set*

$$D = \{(x, y) : \mu(x, \{y\}) > 0\}$$

is a Borel subset of the unit square.

PROOF. First note (starting from indicators of cubes etc.) that for any measurable ϕ from $X \times [0, 1] \times [0, 1]$ to $[0, 1]$ the function ϕ^* from $X \times [0, 1]$ to $[0, 1]$ defined by

$$\phi^*(x, y) = \int \phi(x, y, z)\mu(x, dz)$$

is $\mathscr{A} \otimes \mathscr{B}_{[0,1]}$ measurable. Taking

$$\begin{aligned} \phi(x, y, z) &= 1 \quad \text{if} \quad y = z \\ &= 0 \quad \text{if} \quad y \neq z \end{aligned}$$

we get $\phi^*(x, y) = \mu(x, \{y\})$ is $\mathscr{A} \otimes \mathscr{B}_{[0,1]}$ measurable and hence D is a Borel subset of the unit square.

LEMMA 4. *Suppose* $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ *are two measurable spaces with $X \in \mathscr{B}_{[0,1]}$, Y a subset of* $[0, 1]$, $\mathscr{A} = \mathscr{B}_{[0,1]} \cap X$, *and* $\mathscr{B} = \mathscr{B}_{[0,1]} \cap Y$. *If $\mu(x, B)$ is* *a transition probability on $X \times \mathscr{B}$ then the set*

$$D = \{(x, y) : \mu(x, \{y\}) > 0\}$$

is a Borel subset of the unit square.

PROOF. Define $\mu'(x, B)$ on $X \times \mathscr{B}_{[0,1]}$ by $\mu'(x, B) = \mu(x, B \cap Y)$. It is easy to check that $\mu'(x, B)$ is again a transition probability on $X \times \mathscr{B}_{[0,1]}$. Hence by Lemma 3 the set $D' = \{(x, y) : \mu'(x, \{y\}) > 0\}$ is a Borel subset of the unit square. But $D' = D$.

THEOREM 2. *Let $(X, \mathscr{A}, \lambda)$ be a perfect probability space with $\mathscr{A}$ countably generated. Let $(Y, \mathscr{B})$ be a measurable space with $\mathscr{B}$ countably generated and let $\mu(x, B)$ be a transition probability on $X \times \mathscr{B}$ such that $\mu(x, \cdot)$ is discrete for almost every x in X. Then μ, the λ-mixture of $\mu(x, \cdot)$'s is perfect.*

PROOF. By using P1 and Lemma 1 and by redefining the transition probability suitably, we can, without loss of generality, assume X to be a Borel subset of $[0, 1]$ with $\mathscr{A} = \mathscr{B}_{[0,1]} \cap X$ and Y to be a subset of $[0, 1]$ with $\mathscr{B} = \mathscr{B}_{[0,1]} \cap Y$. By Lemma 4, $D = \{(x, y) : \mu(x, \{y\}) > 0\}$ is a Borel subset of the unit square. Since $\mu(x, \cdot)$ is discrete for almost every x in X $P_\mu(D) = \int \mu(x, D_x)\, d\lambda = 1$. Thus $(D, \mathscr{A} \otimes \mathscr{B} \cap D, P_\mu|_D)$ is a perfect probability space (see [3] Section 2.5) and hence P_μ is perfect. By Theorem 1, μ is perfect.

THEOREM 3. *Perfect mixtures of discrete measures are perfect.*

PROOF. Let $(X, \mathscr{A}, \lambda)$ be a perfect probability space, $(Y, \mathscr{B})$ a measurable space and $\mu(x, B)$ a transition probability on $X \times \mathscr{B}$ such that $\mu(x, \cdot)$ is discrete for almost every x in X. Let μ be the λ-mixture of $\mu(x, \cdot)$'s. Let $\mathscr{B}'$ be a countably generated sub-σ-algebra of $\mathscr{B}$ and let $\mu'(x, B')$ be the restriction of $\mu(x, B)$ on $X \times \mathscr{B}'$. Then $\mu'(x, B')$ is a transition probability on $X \times \mathscr{B}'$ and $\mathscr{A}'$, the smallest sub-σ-algebra of $\mathscr{A}$ with respect to which $\{\mu'(\cdot, B'), B' \in \mathscr{B}'\}$ are all measurable, is countably generated. Thus if $\mu' = \mu|_{\mathscr{B}'}$ and $\lambda' = \lambda|_{\mathscr{A}'}$ then μ' is the λ'-mixture of $\mu'(x, \cdot)$'s where $\mu'(x, \cdot)$ is discrete for almost every x in X. By P2 λ' is perfect. Hence by Theorem 2, μ' is perfect. Again by P2, μ is perfect.

REMARK. The author's original proof of Theorem 2 used notions from the theory of Lebesgue spaces due to Rohlin. The present proof was suggested by B. V. Rao in a private communication.

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