The Annals of Probability
1974, Vol. 2, No. 3, 486-489

UNIFORM INEQUALITIES FOR CONDITIONAL EXPECTATIONS

BY L. ROGGE

University of Konstanz

The purpose of this note is to show that Neveu's uniform inequality for conditional expectations can be sharpened and extended to arbitrary conditioning sub-σ-fields. An application of this inequality yields that a sequence of conditional expectations given a σ-field $\mathscr{F}_n$ converges uniformly for all test functions to a conditional expectation given a σ-field $\mathscr{F}_\infty$ if and only if the σ-fields $\mathscr{F}_n$ converge to $\mathscr{F}_\infty$ in the usual metric.

Let $(X, \mathscr{F}, P)$ be a probability space. A pseudo-metric can be introduced in the set of all sub-σ-fields of $\mathscr{F}$ by $\delta(\mathscr{A}, \mathscr{B}) \equiv \max\{\sup_{A\in\mathscr{A}} \inf_{B\in\mathscr{B}} P(A \triangle B), \sup_{B\in\mathscr{B}} \inf_{A\in\mathscr{A}} P(A \triangle B)\}$. For complete σ-fields $\mathscr{A}, \mathscr{B} \subset \mathscr{F}$ this is the usual Hausdorffmetric between closed subsets if we endow $\mathscr{F}$ with the pseudometric $\rho(F_1, F_2) \equiv P(F_1 \triangle F_2)$. (See [2] page 33).
If $\mathscr{A} \subset \mathscr{B}$ this is the metric used by Boylan [1] and Neveu [3]. For arbitrary σ-fields the metric above is equivalent to the metric of Boylan but in general somewhat smaller.
If $\mathscr{A}$ is a sub-σ-field of $\mathscr{F}$ and f a P-integrable function then $P^{\mathscr{A}}f$ denotes the $P\,|\,\mathscr{A}$-equivalence class of conditional expectations of f given the σ-field $\mathscr{A}$. For each $q \geqq 1$ denote by $L_q(X, \mathscr{F}, P)$ the system of all $\mathscr{F}$-measurable functions f for which $|f|^q$ is P-integrable and put $||f||_q = (P(|f|^q))^{1/q}$.
At first we prove a preparatory lemma which is the main tool for the results given here.

LEMMA 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be sub-σ-fields of $\mathscr{F}$. Then we obtain for each $\mathscr{B}$-measurable function $f\colon X \to [0, 1]$:*

(i) $||P^{\mathscr{A}}f - f||_1 \leqq 2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))$
(ii) $||P^{\mathscr{A}}f - f||_2 \leqq [\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))]^{\frac{1}{2}}$.

PROOF. (i) At first we prove (i) for $f = 1_B$ with $B \in \mathscr{B}$:

$$\begin{aligned} P(|P^{\mathscr{A}}1_B - 1_B|) &= P(1_{\bar{B}}P^{\mathscr{A}}1_B) + P(1_B(1 - P^{\mathscr{A}}1_B)) = 2P(1_{\bar{B}}P^{\mathscr{A}}1_B) \\ &= 2[P(P^{\mathscr{A}}1_{\bar{B}} \wedge P^{\mathscr{A}}1_B) - P((P^{\mathscr{A}}1_B \wedge P^{\mathscr{A}}1_{\bar{B}})^2)] \\ &\leqq 2[P(P^{\mathscr{A}}1_B \wedge P^{\mathscr{A}}1_{\bar{B}}) - (P(P^{\mathscr{A}}1_B \wedge P^{\mathscr{A}}1_{\bar{B}}))^2] \\ &\leqq 2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))\,; \end{aligned}$$

the third equality follows from:

$$a(1 - a) = a \wedge (1 - a) - (a \wedge (1 - a))^2 \qquad \text{for all } a \in [0, 1]\,,$$

Received April 9, 1973; revised July 20, 1973.
AMS 1970 *subject classifications.* Primary 28A20; Secondary 60645.
Key words and phrases. Conditional expectation, inequality, metric for σ-fields.

the last inequality from:

$$\begin{aligned} P(P^{\mathscr{A}}1_B \wedge P^{\mathscr{A}}1_{\bar{B}}) &= P(|P^{\mathscr{A}}1_B - 1_{\{x:(P_{\mathscr{A}}1_B)(x)>\frac{1}{2}\}}|) \\ &= \inf_{A\in\mathscr{A}} P(|P^{\mathscr{A}}1_B - 1_A|) \\ &= \inf_{A\in\mathscr{A}} P(A \triangle B) \leqq \delta(\mathscr{A}, \mathscr{B}) \quad \text{and} \quad \delta(\mathscr{A}, \mathscr{B}) \leqq \tfrac{1}{2}\,. \end{aligned}$$

Now we prove (i) for a simple function $f = \sum_{i=1}^n \alpha_i 1_{B_i}$ with values in $[0, 1]$. W.l.g. we may assume $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n \leqq 1$ and that the sets B_i, $i = 1, \cdots, n$ are disjoint.
Let $C_k := \sum_{i=k}^n B_i$, $k = 1, \cdots, n$, and $\alpha_0 := 0$. Then $f = \sum_{i=1}^n (\alpha_i - \alpha_{i-1})1_{C_i}$. Hence:

$$\begin{aligned} \|P^{\mathscr{A}}f - f\|_1 &\leqq P(\textstyle\sum_{i=1}^n (\alpha_i - \alpha_{i-1})|P^{\mathscr{A}}1_{C_i} - 1_{C_i}|) \\ &\leqq \textstyle\sum_{i=1}^n (\alpha_i - \alpha_{i-1})2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B})) \\ &\leqq 2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))\,. \end{aligned}$$

Since each $\mathscr{B}$-measurable function $f\colon X \to [0, 1]$ is a limit of an increasing sequence of $\mathscr{B}$-measurable simple functions, we obtain (i).

(ii) We have for each $\mathscr{B}$-measurable function $f\colon X \to [0, 1]$:

$$\begin{aligned} P((P^{\mathscr{A}}f - f)^2) &= P(f \cdot P^{\mathscr{A}}f) - 2P(f \cdot P^{\mathscr{A}}f) + P(f^2) = P(f(f - P^{\mathscr{A}}f)) \\ &\leqq P((f - P^{\mathscr{A}}f)^+) = \tfrac{1}{2}P(|f - P^{\mathscr{A}}f|)\,. \end{aligned}$$

Now (i) implies (ii).

THEOREM 2. *Let $\mathscr{A}$, $\mathscr{B}$ be sub-σ-fields of $\mathscr{F}$ with $\mathscr{A} \subset \mathscr{B}$. Then*

(i) $\sup\{\|P^{\mathscr{A}}f - P^{\mathscr{B}}f\|_1 : f \in \Phi\} \leqq 2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))$
(ii) $\sup\{\|P^{\mathscr{A}}f - P^{\mathscr{B}}f\|_2 : f \in \Phi\} \leqq [\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))]^{\frac{1}{2}}$

where Φ is the system of all $\mathscr{F}$-measurable functions with values in $[0, 1]$.

PROOF. Since $\mathscr{A} \subset \mathscr{B}$ we have $P^{\mathscr{A}}f = P^{\mathscr{A}}P^{\mathscr{B}}f$. As $P^{\mathscr{B}}f$ is $\mathscr{B}$-measurable we obtain the assertions from Lemma 1.

The following trivial example shows that the inequalities in the assertion of the preceding theorem cannot be improved: Let $X = \{0, 1\}$, $\mathscr{A} = \{\phi, X\}$ and $\mathscr{B} = \mathscr{F}$ be the power set of X. Denote for each $\alpha \in [0, \frac{1}{2}]$ by P_α the P-measure defined by $P_\alpha(\{0\}) = \alpha$. Then $\delta(\mathscr{A}, \mathscr{B}) = \alpha$ and

(i) $\sup\{\|P_\alpha{}^{\mathscr{A}}f - P_\alpha{}^{\mathscr{B}}f\|_1 : f \in \Phi\} = 2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))$
(ii) $\sup\{\|P_\alpha{}^{\mathscr{A}}f - P_\alpha{}^{\mathscr{B}}f\|_2 : f \in \Phi\} = [\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))]^{\frac{1}{2}}$.

THEOREM 3. *Let $\mathscr{A}$, $\mathscr{B}$ be arbitrary sub-σ-fields of $\mathscr{F}$. Then*

$$\sup\{\|P^{\mathscr{A}}f - P^{\mathscr{B}}f\|_2 : f \in \Phi\} \leqq [2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))]^{\frac{1}{2}}$$

where Φ is the system of all $\mathscr{F}$-measurable functions with values in $[0, 1]$.

PROOF. Let $f \in \Phi$. We have

$$\begin{aligned} P((P^{\mathscr{A}}f - P^{\mathscr{B}}f)^2) &= P(P^{\mathscr{A}}f(P^{\mathscr{A}}f - P^{\mathscr{B}}f)) + P(P^{\mathscr{B}}f(P^{\mathscr{B}}f - P^{\mathscr{A}}f)) \\ &= P(f(P^{\mathscr{A}}f - P^{\mathscr{B}}P^{\mathscr{A}}f)) + P(f(P^{\mathscr{B}}f - P^{\mathscr{A}}P^{\mathscr{B}}f)) \\ &\leqq P((P^{\mathscr{A}}f - P^{\mathscr{B}}P^{\mathscr{A}}f)^+) + P((P^{\mathscr{B}}f - P^{\mathscr{A}}P^{\mathscr{B}}f)^+) \\ &= \tfrac{1}{2}P(|P^{\mathscr{A}}f - P^{\mathscr{B}}P^{\mathscr{A}}f|) + \tfrac{1}{2}P(|P^{\mathscr{B}}f - P^{\mathscr{A}}P^{\mathscr{B}}f|)\,. \end{aligned}$$

Hence we obtain from Lemma 1 (i)

$$P((P^{\mathscr{A}}f - P^{\mathscr{B}}f)^2) \leqq 2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))$$

which implies the assertion.

From Theorems 2 and 3 we obtain immediately the following Theorem 4.

THEOREM 4. *Let* $H \subset L_q(X, \mathscr{F}, P)$, $a > 0$ *and define*

$$\delta_{H,q}(a) = \sup\{||f1_{\{|f|>a\}}||_q : f \in H\}\,.$$

Then

$$\sup\{||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_q : f \in H\} \leqq C_q a[\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B}))]^{1/q} + 2\delta_{H,q}(a)\,,$$

where

(i) $C_q = 2 \cdot 2^{1/q}$, *if* $\mathscr{A} \subset \mathscr{B}$ *and* $1 \leqq q < 2$,
(ii) $C_q = 2$, *if* $\mathscr{A} \subset \mathscr{B}$ *and* $q \geqq 2$,
(iii) $C_q = 2 \cdot 2^{1/q}$, *if* $\mathscr{A}, \mathscr{B}$ *are arbitrary and* $q \geqq 2$.

PROOF. We have for all $f \in H$:

$$\begin{aligned} ||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_q &\leqq ||P^{\mathscr{A}}(f1_{\{|f|\leqq a\}}) - P^{\mathscr{B}}(f1_{\{|f|\leqq a\}})||_q \\ &\quad + ||P^{\mathscr{A}}(f1_{\{|f|>a\}})||_q + ||P^{\mathscr{B}}(f1_{\{|f|>a\}})||_q \\ &\leqq a||P^{\mathscr{A}}(fa^{-1}1_{\{|f|a^{-1}\leqq 1\}}) - P^{\mathscr{B}}(fa^{-1}1_{\{|f|a^{-1}\leqq 1\}})||_q \\ &\quad + 2\delta_{H,q}(a) \leqq 2a\sup\{||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_q : f \in \Phi\} + 2\delta_{H,q}(a)\,. \end{aligned}$$

Since for each $f \in \Phi$:

$$\begin{aligned} ||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_q^q &\leqq ||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_1\,, && \text{for } q \geqq 1 \\ &\leqq ||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_2^2\,, && \text{for } q \geqq 2\,. \end{aligned}$$

Theorem 2 and Theorem 3 imply the assertion.

For $q = 1$ Theorem 4 (i) yields the inequality of Neveu [3], with somewhat better bounds.

In Theorem 3 of [1] Boylan proved that $\sup\{||P^{\mathscr{F}_n}f - P^{\mathscr{F}_\infty}f||_1 : f \in \Phi\} \to 0$ if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ and $\mathscr{F}_n$ increases or decreases to $\mathscr{F}_\infty$. Since $||f||_1 \leqq ||f||_2$ Theorem 4 (iii) shows that the theorem of Boylan holds true without the assumption that the sequence $\mathscr{F}_n$ decreases or increases to $\mathscr{F}_\infty$. Since for each $B \in \mathscr{B}$

$$\begin{aligned} \inf_{A\in\mathscr{A}} P(A \triangle B) &= \inf_{A\in\mathscr{A}} P(|P^{\mathscr{A}}1_B - 1_A|) \\ &= P(|P^{\mathscr{A}}1_B - 1_{\{x:(P^{\mathscr{A}}1_B(x))>\frac{1}{2}\}}|) \\ &= P(P^{\mathscr{A}}1_B \wedge P^{\mathscr{A}}1_{\bar{B}}) \\ &\leqq P(|P^{\mathscr{A}}1_B - 1_B|) \leqq \sup\{||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_1 : f \in \Phi\} \end{aligned}$$

we obtain that

$$\delta(\mathscr{A}, \mathscr{B}) \leqq \sup \{||P^{\mathscr{A}}f - P^{\mathscr{B}}f||_1 : f \in \Phi\} .$$

Hence $||P^{\mathscr{F}_n}f - P^{\mathscr{F}_\infty}f||_1$ converges to zero uniformly in $f \in \Phi$, if and only if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$.

REFERENCES

[1] BOYLAN, E. S. (1971). Equi-convergence of martingales. *Ann. Math. Statist.* **42** 552-559.

[2] KELLEY, J. L. and NAMIOKA, I. (1963). *Linear Topological Spaces.* Van Nostrand, Princeton.

[3] NEVEU, J. (1972). Note on the tightness of the metric on the set of complete sub-σ-algebras of a probability space. *Ann. Math. Statist.* **43** 1369-1371.

FACHBEREICH STATISTIK
775 KONSTANZ
UNIVERSITÄTS STR. 10
POSTFACH 733