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SOME RESULTS ABOUT MULTIDIMENSIONAL BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS[1]

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A multidimensional branching process with random environments is considered. Two results are proven about this process. The first proves that all nonzero states of the process are transient. Since the process in question is not Markov, the proof of this result is more involved than in the classical case. Our second result deals with the extinction of the process when we are in the critical case. We prove as in the classical theory that extinction occurs w.p. 1.

1. Introduction. Let $\{Z_n(\bar{\zeta}) = (Z_n^1(\bar{\zeta}), \cdots, Z_n^p(\bar{\zeta}))\}_{n\geqq 0}$ be a p-dimensional ($p \geqq 2$) branching process with random environments (MBPRE). The process can be described as follows. Assume given a stationary ergodic sequence $\{\zeta_n\}_{n\geqq 0}$ of "environmental" random mappings. For a.e. realization of this process is associated a sequence $\{\phi_{\zeta_n}(s) = (\phi_{\zeta_n}^1(s), \cdots, \phi_{\zeta_n}^p(s))\}_{n\geqq 0}$ of vectors of p-dimensional probability generating functions (pgf vectors). When conditioned on the $\{\zeta_n\}$ process, the $\{Z_n\}$ behave as a p-dimensional temporally nonhomogeneous branching process where the number of offspring of type j produced by an individual of type i in the nth generation is governed by $\phi_{\zeta_n}^i(s)$.

This process was first introduced by Athreya and Karlin [1], and the reader is referred to their paper for a detailed discussion of the construction of the process for the case $p = 1$.

The purpose of this paper is twofold. First we will show that all nonzero states are transient. Using this result, together with a result of Athreya and Karlin [1], we will give necessary and sufficient conditions for the extinction of a MBPRE.

2. Preliminaries and statements of results. We start by introducing the following notation:

X = set of p-tuples of nonnegative integers

$e_j = (\delta_{1j}, \delta_{2j}, \cdots, \delta_{pj})$, $1 \leqq j \leqq p$, δ_{ij} the usual delta function

$C = [0, 1]^p$; $C^0 = [0, 1)^p$

$Rs = (s, \cdots, s)$; $s \in [0, 1]$

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[We will write **0** for $R0$ and **1** for $R1$.]

$$s^j = \prod_{i=1}^{p} s_i^{j_i}\,; \qquad s \in C, j \in X .$$

An explicit construction of the environmental process can be done in a manner identical to that in [1], with the obvious modifications being made for the increased dimension. The details will be omitted. We will therefore, assume that there exists some probability space $(\Omega, \mathscr{F}, P)$ and a discrete time stationary ergodic stochastic process $\{\zeta_t\}_{t\geq 0}$ defined on it such that we can associate to each realization $\bar{\zeta}$ of the ζ_t process a sequence of pgf vectors $\{\phi_{\zeta_n}(s)\}$ where

$$\phi_{\zeta_n}(s) = (\phi^1_{\zeta_n}(s), \cdots, \phi^p_{\zeta_n}(s))$$

and

$$\phi^i_{\zeta_n}(s) = \sum_{j\in X} b_j{}^i(\zeta_n)s^j \qquad s \in C .$$

Furthermore, we will assume that w.p. 1 the pgf vector, $\phi_\zeta(s)$, has components $\phi_\zeta{}^i(s)$ whose coefficients satisfy

(i) $0 \leqq b_0{}^i(\zeta) + \sum_{j=1}^{p} b^i_{e_j}(\zeta) < 1$, and

(ii) $\partial\phi_\zeta{}^i(s)/\partial s_j|_{s=1-} < \infty$, $1 \leqq j \leqq p$.

It is convenient at this point to introduce several σ-algebras. (If $\mathscr{G}$ is a collection of random variables, then $\sigma(\mathscr{G})$ is the smallest σ-algebra generated by $\mathscr{G}$.) Let

$$F(\bar{\zeta}) = \sigma(\zeta_0, \zeta_1, \cdots)$$
$$F_n(\bar{\zeta}) = \sigma(Z_0, Z_1, \cdots, Z_n, \zeta_0, \zeta_1, \cdots) \qquad n \geqq 1 .$$

It follows from the description in Section 1, that w.p. 1

$$E\{s^{Z_{n+1}} | F_n(\bar{\zeta})\} = \phi_{\zeta_n}(s)^{Z_n} \qquad s \in C, n \geqq 1 . \tag{2.1}$$

An immediate consequence of (2.1) is

LEMMA 2.1. *With probability* 1,

$$E\{s^{Z_{n+1}} | Z_0 = e_i; F(\bar{\zeta})\} = \phi^i_{\zeta_0}(\phi_{\zeta_1}(\cdots \phi_{\zeta_n}(s)))\,; \qquad s \in C . \tag{2.2}$$

A direct implication of Lemma 2.1 is that when conditioning on the entire environment, $\bar{\zeta} = (\zeta_0, \zeta_1, \cdots)$, the process behaves like a temporally nonhomogeneous branching process and therefore, the lines of descent are independent subject to conditioning.

The first objective of this paper is to prove the transcience of all nonzero states. We introduce the following notation:

$$D_j\phi^i_{\zeta_n}(s) = \frac{\partial}{\partial s_j}\phi^i_{\zeta_n}(s)\,; \qquad D_{jk}\phi^i_{\zeta_n}(s) = \frac{\partial^2}{\partial s_j\,\partial s_k}\phi^i_{\zeta_n}(s)\,,$$
$$s \in C,\ 1 \leqq i, j, k \leqq p .$$

Define the matrices:

$$M_{\zeta_n}(s) = (D_j\phi^i_{\zeta_n}(s))_{1\leqq i,j\leqq p}\,; \qquad M_{\zeta_n}(\mathbf{1}) = M_{\zeta_n} .$$
$$Y_n(s, \bar{\zeta}) = \prod_{i=0}^{n-1} M_{\zeta_i}(s)\,; \qquad Y_n(\mathbf{1}, \bar{\zeta}) = Y_n(\bar{\zeta})$$

for $n \geqq 0$, $s \in C$.

Let $A = (a_{ij})_{1 \leqq i,j \leqq p}$ be any matrix, A^t is its transpose; $||A|| = \max_{1 \leqq i \leqq p} \sum_{j=1}^{p} |a_{ij}|$

$$\max[A] = \max_{1 \leqq i,j \leqq p} |a_{ij}| ; \qquad \min[A] = \min_{1 \leqq i,j \leqq p} |a_{ij}|$$

$B[A] = \max[A]/\min[A]$.

For u any row vector, we write as usual $u > 0$ if all components are positive and $u \geqq 0$ if all components are nonnegative. Similarly for two vectors u and v we write $u < v$ if $v - u > 0$ and so on. Similar comments hold for matrices. If u and v are two vectors then $(u, v) = \sum_{i=1}^{p} u_i v_i$. We now state Theorem 1.

THEOREM 1. *Let* $\{Z_n\}_{n \geqq 0}$ *be an* MBPRE. *Assume that there exist constants* $C, D > 0$ *such that*

$$0 < C \leqq \min[M_{\zeta_0}] \leqq \max[M_{\zeta_0}] \leqq D < \infty$$

and

$$0 \leqq \max_{1 \leqq i,j,k \leqq p} D_{ij} \phi^k_{\zeta_0}(\mathbf{1}) \leqq D \qquad \text{w.p. 1.}$$

Then,

$$P\{\lim_{n\to\infty} (Z_n, \mathbf{1}) = 0 \,|\, |Z_0| = 1; F(\bar{\zeta})\} + P\{\lim_{n\to\infty} (Z_n, \mathbf{1}) = \infty \,|\, |Z_0| = 1; F(\bar{\zeta})\} = 1 \qquad \text{w.p. 1.}$$

Athreya and Karlin [1] proved Theorem 1 for the case $p = 1$. In the course of their proof they needed a result of Church [3]. For our purposes we need to prove the following generalization of Church's result.

THEOREM A. *Let* $\{f_n(s)\}_{n \geqq 0}$ *be a sequence of* pgf *vectors. Assume there exist constants* $C, D > 0$ *such that for all* $n \geqq 0$

$$C \leqq D_j f_n{}^i(\mathbf{1}) \leqq D \qquad \text{and} \qquad 0 \leqq D_{k,j} f_n{}^i(\mathbf{1}) \leqq D \qquad 1 \leqq i, j, k \leqq p .$$

Let $h_n(s) = f_0(f_1(\cdots f_n(s)))$. *Then*

(a) $\lim_{n\to\infty} h_n(Rs) = g(s)$ *exists for all* $s \in [0, 1)$

(b) *Either* $g(s) = g(0)$ *for all* $s \in [0, 1)$ *or* $g(s)$ *is strictly increasing (componentwise) in* $[0, 1)$.

A proof of Theorem A can be found in [7].

Theorem 1 is proved in Section 3.

We now turn our attention to the problem of extinction. To fascilitate the statements of the results, we introduce the following notation. Let,

$$B = \{\omega : Z_n(\omega) = \mathbf{0} \text{ for some } n\} ,$$
$$q^i(\bar{\zeta}) = P\{B \,|\, Z_0 = e_i; F(\bar{\zeta})\} ,$$
$$q(\bar{\zeta}) = (q^1(\bar{\zeta}), \cdots, q^p(\bar{\zeta})) .$$

B is referred to as the set of extinction and $q(\bar{\zeta})$ as the vector of extinction probabilities.

It follows from Lemma 2.1 that

$$q^i(\bar{\zeta}) = \lim_{n\to\infty} \phi^i_{\zeta_0}(\phi_{\zeta_1}(\cdots \phi_{\zeta_n}(\mathbf{0}))) \qquad 1 \leqq i \leqq p . \tag{2.3}$$

An immediate consequence of (2.3) is

$$q(\bar{\zeta}) = \phi_{\zeta_0}(q(T\bar{\zeta})) \qquad \text{w.p. 1.} \tag{2.4}$$

with T denoting the shift operation:

$$T\bar{\zeta} = T(\zeta_0, \zeta_1, \cdots) = (\zeta_1, \zeta_2, \cdots) .$$

If we assume that the environmental process satisfies the conditions of Theorem 1, then we can conclude from (2.4) that

PROPOSITION 2.1. *The sets*

$$\{\bar{\zeta} : q(\bar{\zeta}) = \mathbf{1}\} \quad \text{and} \quad \{\bar{\zeta} : q(T\bar{\zeta}) = \mathbf{1}\}$$

coincide modulo a set of measure 0. *The same is true of the sets*

$$\{\bar{\zeta} : q(\bar{\zeta}) < \mathbf{1}\} \quad \text{and} \quad \{\bar{\zeta} : q(T\bar{\zeta}) < \mathbf{1}\} .$$

It follows from the ergodicity of T that

$$P\{\bar{\zeta} : q(\bar{\zeta}) = \mathbf{1}\} = 0 \ \text{ or } \ 1 \quad \text{and} \quad P\{\bar{\zeta} : q(\bar{\zeta}) < \mathbf{1}\} = 0 \ \text{ or } \ 1 . \tag{2.5}$$

The conditions of Therom 1 also imply that if $q^i(\bar{\zeta}) < 1$ for some i, then $q^i(\bar{\zeta}) < 1$ for all i. It therefore follows that

$$P\{\bar{\zeta} : q(\bar{\zeta}) = \mathbf{1}\} \quad \text{or} \quad P\{\bar{\zeta} : q(\bar{\zeta}) < \mathbf{1}\} = 1 . \tag{2.6}$$

In order to determine conditions under which of the two cases of (2.6) are valid we must first discuss some facts about products of matrices. Furstenberg and Kesten [4] proved the following theorem.

THEOREM B. *Let* $\{X_n\}_{n \geq 0}$ *be a stationary ergodic sequence of random mappings with values in the set of* $p \times p$ *matrices. Assume* $E\{\log^+ ||X_1||\} < \infty$, (*for any positive number* a, $\log^+ a = \min(0, \log a)$). *Then,*

$$\lim_{n\to\infty} \frac{1}{n} \log ||\textstyle\prod_{i=0}^{n-1} X_i|| = \pi \quad \textit{exists} \qquad \text{w.p. 1.}$$

and

$$\lim_{n\to\infty} \frac{1}{n} E\{\log ||\textstyle\prod_{i=0}^{n-1} X_i||\} = \pi .$$

Consider the matrices $\{M_{\zeta_n}\}_{n \geq 0}$. If we assume the conditions of Theorem 1, then we may apply Theorem B and conclude

$$\lim_{n\to\infty} \frac{1}{n} \log ||Y_n(\bar{\zeta})|| = \pi \quad \text{exists} \qquad \text{w.p. 1.}$$

Athreya and Karlin [1] proved the following result which gives sufficient conditions for when $P\{\bar{\zeta} : q(\bar{\zeta}) < 1\} = 0$ or 1.

THEOREM C. *Let* $\{Z_n\}_{n \geq 0}$ *be an* MBPRE *with the associated environmental process satisfying the conditions of Theorem* 1. *Also assume that* $E\{-\log(\mathbf{1}, \mathbf{1} - \varphi_{\zeta_0}(\mathbf{0}))\} < \infty$. *Then,*

$$\pi < 0 \Rightarrow P\{\bar{\zeta} : q(\bar{\zeta}) < \mathbf{1}\} = 0$$

and

$$\pi > 0 \Rightarrow P\{\bar{\zeta} : q(\bar{\zeta}) < \mathbf{1}\} = 1 .$$

Theorem C is not that surprising for in both situations we know exactly how the mean matrix behaves, i.e., if $\pi < 0$, then $||Y_n(\bar{\zeta})||$ converges to 0 and if $\pi > 0$, then $||Y_n(\bar{\zeta})||$ converges to ∞. The case of interest is what happens if $\pi = 0$, That is the content of the next theorem.

THEOREM 2. *Let* $\{Z_n\}_{n\geq 0}$ *be an* MBPRE *with the associated environmental process satisfying the conditions of Theorem* 1. *Then,*

$$\pi = 0 \Rightarrow P\{\bar{\zeta} : q(\bar{\zeta}) = \mathbf{1}\} = 1 .$$

So we see that, modulo certain regularity conditions, the value of π is the critical value for the extinction problem. That this should be so, is not that unexpected, in view of what π turns out to be in the classical model. In this case M_{ζ_0} equals some constant matrix, say B, w.p. 1. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \log ||B^n|| = \pi .$$

However, we recognize π to be the log of the spectral radius of B, which if B is a positive matrix, is equal to the log of the largest eigenvalue of B. From the classical theory, we know that extinction occurs iff the largest eigenvalue is less than or equal to 1, or equivalently whether $\pi \leq 0$.

The proof of Theorem 2 is given in Section 4.

3. Proof of the transience of nonzero states. Our goal is to prove:

$$(3.1)\qquad P\{\lim_{n\to\infty} (Z_n, \mathbf{1}) = 0 \,|\, |Z_0| = 1; F(\bar{\zeta})\} + P\{\lim_{n\to\infty} (Z_n, \mathbf{1}) = \infty \,|\, |Z_0| = 1; F(\bar{\zeta})\} = 1 \qquad \text{w.p. } 1.$$

By the Borel Cantelli lemma, it is enough to show that for each $1 \leq i \leq p$ and for each positive integer K,

$$(3.2)\qquad A_i(K) = \sum_{j=1}^{\infty} P\{1 \leq (Z_n, \mathbf{1}) \leq K \,|\, Z_0 = e_i; F(\bar{\zeta})\} < \infty \qquad \text{w.p. } 1.$$

The remaining part of this section will be devoted to proving (3.2). Set

$$\pi_n(s, \bar{\zeta}) = \phi_{\zeta_0}(\phi_{\zeta_1}(\cdots \phi_{\zeta_{n-1}}(Rs)))$$
$$\pi_0(s, \bar{\zeta}) = Rs \qquad 0 \leq s \leq 1,\ n \geq 1 .$$

Then for $0 \leq s \leq 1$ and $1 \leq i \leq p$,

$$P\{1 \leq (Z_n, \mathbf{1}) \leq K \,|\, Z_0 = e_i; F(\bar{\zeta})\}$$
$$\leq s^{-K} \sum_{j=1}^{K} s^j P\{(Z_n, \mathbf{1}) = j \,|\, Z_0 = e_i; F(\bar{\zeta})\}$$
$$\leq s^{-K}(\pi_n(s, \bar{\zeta}) - \pi_n(0, \bar{\zeta}), \mathbf{1}) .$$

So to prove (3.2), it is sufficient to show

$$(3.3)\qquad \sum_{j=1}^{\infty} (\pi_j(s, \bar{\zeta}) - \pi_j(0, \bar{\zeta}), \mathbf{1}) < \infty \qquad \text{w.p. } 1.$$

Since the terms in the sum in (3.3) are all positive, the convergence in (3.3) is equivalent to the convergence of $\lim_{k\to\infty} \sum_{j=1}^{kn} (\pi_j(s, \bar{\zeta}) - \pi_j(0, \bar{\zeta}), \mathbf{1})$ for any n. For the present fix n. Then,

$$\sum_{j=n+1}^{kn} (\pi_j(s, \bar{\zeta}) - \pi_j(0, \bar{\zeta}), \mathbf{1}) = \sum_{j=1}^{k-1} W_j(\bar{\zeta})$$

where

$$W_j(\bar{\zeta}) = \sum_{l=jn+1}^{(j+1)n} (\pi_l(s, \bar{\zeta}) - \pi_l(0, \bar{\zeta}), \mathbf{1}) .$$

By the Mean Value Theorem for p-dimensions and the monotonicity of the pgf vector, we conclude that

$$(3.4) \qquad (\pi_l(s, \bar{\zeta}) - \pi_l(0, \bar{\zeta}), \mathbf{1}) \leqq p\|\prod_{i=1}^{l} M_{\zeta_{i-1}}(\pi_{l-i}(s, T^i\bar{\zeta}))\| .$$

The assumptions of Theorem 1 imply that $\|M_{\zeta_0}\| \leqq pD$ w.p. 1 and without loss of generality we assume $pD > 1$. It follows that for $jn + 1 \leqq l < (j + 1)n$,

$$(3.5) \qquad \|\prod_{i=1}^{l} M_{\zeta_{i-1}}(\pi_{l-i}(s, T^i\bar{\zeta}))\| \leqq [pD]^{l-jn}\{\prod_{r=0}^{j-1} \|\prod_{i=rn+1}^{(r+1)n} M_{\zeta_{i-1}}((\pi_{l-i}(s, T^i\bar{\zeta})))\|\} .$$

Define:

$$f_r^{(n)}(\bar{\zeta}) = \max_{rn \leqq j \leqq (r+1)n-1} \|\prod_{i=1}^{n} M_{\zeta_{i-1}}(\pi_{j-i}(s, T^i\bar{\zeta}))\| , \qquad r \geqq 1 .$$

It follows from (3.4) and (3.5) that

$$W_j(\bar{\zeta}) \leqq p[\sum_{i=1}^{n} (pD)^i]\{\prod_{r=1}^{j} f_r^{(n)}(T^{(j-r)n}\bar{\zeta})\}$$

and

$$\sum_{j=1}^{k} W_j(\bar{\zeta}) \leqq p[\sum_{i=1}^{n} (pD)^i]\{\sum_{j=1}^{k} \prod_{r=1}^{j} f_r^{(n)}(T^{n(j-r)}\bar{\zeta})\} .$$

(3.3) will therefore be valid for those sample paths, $\bar{\zeta}$, such that

$$G_n(\bar{\zeta}) = \limsup_{j\to\infty} \left\{\frac{1}{j} \sum_{k=1}^{j} \log [f_k^{(n)}(T^{n(j-k)}\bar{\zeta})]^{1/n}\right\} < 0 .$$

To complete the proof of the Theorem, it is enough to show that the set

$$\Omega_1 = \{\bar{\zeta}: \inf_{n\geqq 1} G_n(\bar{\zeta}) < 0\}$$

has probability 1. The next set of lemmas will accomplish this. For the remainder of this section, we assume that the conditions of Theorem 1 are in effect.

It has already been noted in Section 2 that for $s \in [0, 1)$,

$$\lim_{n\to\infty} \pi_n(s, \bar{\zeta}) = g(s, \bar{\zeta}) \quad \text{exists} \qquad \text{w.p. 1.}$$

The next two lemmas will show that either (3.1) is valid or w.p. 1, $g(s, \bar{\zeta}) = g(0, \bar{\zeta})$, $s \in [0, 1)$.

LEMMA 3.1. *Either* (3.1) *holds, or for some* n_0,

$$(3.6) \qquad P\{\bar{\zeta}: \prod_{i=1}^{p} \pi_{n_0}^i(0, \bar{\zeta}) > 0\} > 0 .$$

PROOF. Suppose (3.6) is false. Then for every n, $P\{\bar{\zeta}: \prod_{i=1}^{p} \pi_n^i(0, \bar{\zeta}) = 0\} = 1$. This implies that $P\{\bar{\zeta}: \prod_{i=1}^{p} q^i(\bar{\zeta}) = 0\} = 1$. It follows now by stationarity that the set

$$\Omega' = \{\bar{\zeta}: \prod_{i=1}^{p} q^i(T^n\bar{\zeta}) = 0; n \geqq 0\}$$

has probability 1. We will show that (3.1) is valid for a.e. $\bar{\zeta} \in \Omega'$. Choose a $\bar{\zeta} \in \Omega'$ and assume $Z_0 = e_i$, $1 \leqq i \leqq p$. Define the following subsets of the integers $\{1, 2, \cdots, p\}$.

$$A_n(\bar{\zeta}) = \{i: q^i(T^n\bar{\zeta}) = 0\} \qquad n \geqq 0 .$$

Since $\bar{\zeta} \in \Omega'$, we know that $A_n(\bar{\zeta}) \neq \emptyset$ for all n. On the $\{Z_n\}$ process, define the function:

$$\eta(Z_n) = \sum_{i\in A_n(\bar{\zeta})} Z_n^i .$$

Our first step is to show

$$(3.7) \qquad P\{\eta(Z_n) \geqq \eta(Z_{n-1}) \mid Z_0 = e_i; F(\bar{\zeta})\} = 1 \qquad n \geqq 1 .$$

Let $j \in A_{n-1}(\bar{\zeta})$. It follows that

$$(3.8) \qquad P\{\eta(Z_n) \geqq 1 \mid Z_{n-1} = e_j; F(\bar{\zeta})\} = 1 .$$

If (3.8) was not true, then necessarily by the Markovian property of the process $q^j(T^{n-1}\bar{\zeta}) > 0$ and this violates the fact that $j \in A_{n-1}(\bar{\zeta})$. (3.8) together with the definition of the process implies (3.7). Thus, $P\{\eta(Z_n)$ is increasing $\mid F(\bar{\zeta})\} = 1$. We next show

$$(3.9) \qquad P\{\lim_{n\to\infty} \eta(Z_n) = \infty \mid Z_0 = e_i; F(\bar{\zeta})\} + P\{\lim_{n\to\infty} \eta(Z_n) = 0 \mid Z_0 = e_i; F(\bar{\zeta})\} = 1 .$$

Since $\eta(Z_n)$ is increasing, we know that $\lim_{n\to\infty} \eta(Z_n)$ exists. All we need do to verify (3.9) is prove

$$P\{\lim_{n\to\infty} \eta(Z_n) = K \mid Z_0 = e_i; F(\bar{\zeta})\} = 0 \qquad \text{for any } K > 0 .$$

Since $\eta(Z_n)$ is integer valued, $\lim_{n\to\infty} \eta(Z_n) = K$ implies that $\eta(Z_n) = K$ for all n sufficiently large. However,

$$P\{\eta(Z_n) = K \text{ for all } n \geqq M \mid Z_0 = e_i; F(\bar{\zeta})\} \leqq \prod_{n=M+1}^{\infty} R_n$$

where

$$R_n = P\{\eta(Z_n) = K \mid \eta(Z_j) = K, M \leqq j \leqq n-1, Z_0 = e_i; F(\bar{\zeta})\} .$$

Using (3.8) it is not difficult to see that

$$\begin{aligned} R_n &\leqq P\{\text{each of the } K \text{ particles of type belonging to } A_{n-1}(\bar{\zeta}) \text{ must} \\ &\qquad \text{produce } \textit{exactly } 1 \text{ particle}\} \\ &\leqq [\max_{1\leqq j\leqq p} [\textstyle\sum_{i=1}^{p} b^j_{e_i}(\zeta_{n-1})]]^K = \alpha(\zeta_{n-1}) . \end{aligned}$$

Our assumptions on the environmental process imply that $E\{\log \alpha(\zeta_0)\} < 0$. It follows then by the ergodic theorem that $\prod_{n=M}^{\infty} \alpha(\zeta_n) = 0$ w.p. 1. This proves (3.9).

We next observe that

$$(3.10) \qquad P\{\liminf_{n\to\infty} |Z_n| > 0; \eta(Z_n) = 0 \text{ all } n \mid Z_0 = e_i; F(\bar{\zeta})\} = 0 .$$

To see this we argue as follows.

$$\begin{aligned} P\{\liminf_{n\to\infty} |Z_n| > 0, \eta(Z_n) = 0 \text{ all } n \mid Z_0 = e_i; F(\bar{\zeta})\} \\ \leqq P\{\eta(Z_n) = 0; |Z_n| > 0 \text{ all } n \mid Z_0 = e_i; F(\bar{\zeta})\} \\ \leqq \prod_{n=1}^{\infty} S_n \end{aligned}$$

where

$$S_n = P\{\eta(Z_n) = 0 \mid \eta(Z_j) = 0, |Z_j| > 0, 1 \leqq j \leqq n-1, Z_0 = e_i; F(\bar{\zeta})\} .$$

Let

$$\Theta = \{\theta = (\theta_1, \cdots, \theta_p) : \theta_i = 0 \text{ or } 1 \text{ and at least one } \theta_i = 0\} .$$

It follows now from the definition of the process that

$$S_n \leqq \max_{1\leqq i\leqq p}\,[\max_{\theta\in\Theta}\phi^i_{\zeta_{n-1}}(\theta)] = \beta(\zeta_{n-1})\,.$$

Using the assumptions of Theorem 1, it is easy to show that $E\{\log\beta(\zeta_0)\} < 0$. It follows again by the ergodic theorem that

$$\prod_{n=1}^{\infty}\beta(\zeta_n) = 0 \qquad \text{w.p. } 1.$$

This proves (3.10). (3.9) and (3.10) imply (3.1). □

Without loss of generality we will therefore, assume for the remainder of this section that for some n, (3.6) is valid.

LEMMA 3.2. *$P\{\bar\zeta: g(s, \bar\zeta) = g(0, \bar\zeta)$ for $s \in [0, 1)\} = 1$.*

PROOF. It was shown in Section 2 that w.p. 1, $g(s, \bar\zeta)$ is either a constant function of s, or is strictly increasing in s. It suffices therefore, to show that there exists an $\varepsilon > 0$ such that

$$P\{\bar\zeta: g(0, \bar\zeta) \geqq g(\varepsilon, \bar\zeta)\} = 1\,. \tag{3.11}$$

It follows from (3.6) that for some integer n, there exists an $\varepsilon > 0$ such that

$$P\{\bar\zeta: \pi_n{}^i(0, \bar\zeta) > \varepsilon,\ 1 \leqq i \leqq p\} > 0\,.$$

Using the ergodic theorem, we deduce that w.p. 1 there exists an increasing sequence of integers, $\{n_k\}$, depending on the sample path, such that for all n_k,

$$\pi_n{}^i(0, T^{n_k}\bar\zeta) > \varepsilon \qquad 1 \leqq i \leqq p\,.$$

It follows that

$$\pi_{n_k+n}(0, \bar\zeta) \geqq \pi_{n_k-1}(\varepsilon, \bar\zeta)\,. \tag{3.12}$$

(3.12) implies (3.11). □

COROLLARY 3.1.

$$\lim_{k\to\infty} f_k{}^{(n)}(\bar\zeta) = \|\prod_{j=0}^{n-1} M_{\zeta_j}(q(T^{j+1}\bar\zeta))\| \qquad \text{w.p. } 1.$$

PROOF. Immediate.

It is convenient to introduce some additional notation. Let

$$P_n(\bar\zeta) = \prod_{j=0}^{n-1} M_{\zeta_j}(q(T^{j+1}\bar\zeta))$$
$$g_n(\bar\zeta) = \log\|P_n(\bar\zeta)\|^{1/n} \qquad n \geqq 1\,.$$

It follows from Corollary 3.1 that

$$\lim_{k\to\infty}\frac{1}{n}\log f_k{}^{(n)}(\bar\zeta) = g_n(\bar\zeta) \qquad \text{w.p. } 1. \tag{3.13}$$

It is also easy to see that the assumptions of Theorem 1 imply

$$E\{\sup_{k\geqq 1}(\log^+[f_k{}^{(n)}(\bar\zeta)]^{1/n})\} < \infty\,. \tag{3.14}$$

We now state two simple lemmas from probability theory. The proofs of these results are very similar in spirit to the proof of Lemma 6′ in [1]. Therefore, the proofs will be omitted.

LEMMA 3.3. *Let T be a measure preserving transformation. Assume that* $f_n \to f$ w.p. 1 *and* $E\{\sup_{n\geqq 1} f_n^+\}$. *Then* w.p. 1,

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} f_j(T^{n-j}) \leqq E\{f^+ \mid I\} - E\{f^- \wedge 1 \mid I\}$$

where I is the σ-field of invariant sets under T.

LEMMA 3.4. *Assume* $f_n \to f$ w.p. 1 *and* $E\{\sup_{n\geqq 1} |f_n|\} < \infty$. *Let* $\{U_n\}_{n\geqq 1}$ *be a monotone increasing sequence of σ-fields and let U be the smallest σ-field generated by the* $\{U_n\}_{n\geqq 1}$. *Then* w.p. 1,

$$\lim_{n\to\infty} E\{f_n \mid U_n\} = E\{f \mid U\} .$$

It follows from (3.13), (3.14) and Lemma 3.3 that

$$\limsup_{j\to\infty} \frac{1}{j} \sum_{k=1}^{j} \log f_k^{(n)}(T^{n(j-k)}\bar{\zeta}) \leqq E\{g_n^+(\bar{\zeta}) \mid I_n(\bar{\zeta})\} - E\{g_n^-(\bar{\zeta}) \wedge 1 \mid I_n(\bar{\zeta})\}$$

where $I_n(\bar{\zeta})$ is the σ-field of invariant sets under T^n. So to prove $P\{\Omega_1\} = 1$, it is sufficient to show that

$$\inf_{n\geqq 1} \{E\{g_n^+(\bar{\zeta}) \mid I_n(\bar{\zeta})\} - E\{g_n^-(\bar{\zeta}) \wedge 1 \mid I_n(\bar{\zeta})\}\} < 0 \qquad \text{w.p. } 1. \tag{3.15}$$

The collection of random matrices $\{M_{\zeta_n}(q(T^{n+1}\bar{\zeta}))\}$ constitutes a stationary ergodic sequence. By the result of Furstenburg and Kesten [4],

$$\lim_{n\to\infty} g_n(\bar{\zeta}) = \lim_{n\to\infty} \frac{1}{n} \log ||P_n(\bar{\zeta})|| = \pi_0$$

exists w.p. 1 and

$$\pi_0 = \inf_{n\geqq 0} \frac{1}{n} E\{\log ||P_n(\bar{\zeta})||\} .$$

We assume for the present that $\pi_0 < 0$.

Choose $m_k = 2^k$, $k = 1, 2, \cdots$. By the assumptions of Theorem 1, $E\{\sup_{m_k} g_{m_k}^+(\bar{\zeta})\} < \infty$. Also the σ-fields $\{I_{m_k}(\bar{\zeta})\}$ are increasing. By Lemma 3.4

$$\lim_{m_k\to\infty} E\{g_{m_k}^+(\bar{\zeta}) \mid I_{m_k}(\bar{\zeta})\} = 0$$

and

$$\lim_{m_k\to\infty} E\{g_{m_k}^-(\bar{\zeta}) \wedge 1 \mid I_{m_k}(\bar{\zeta})\} = (-\pi_0) \wedge 1 > 0 \qquad \text{w.p. } 1.$$

This proves (3.15) and therefore Theorem 1, providing we can show $\pi_0 < 0$.

The next series of lemmas will do this.

We first state a simple matrix inequality. Let $\{U_n\}$ be any sequence of $p \times p$ matrices and set $V_n = \prod_{j=0}^{n-1} U_j$. Then,

LEMMA 3.5.

$$\min [V_n] \leqq ||V_n|| \leqq p \max [V_n] \leqq pB[U_0^t]B[U_{n-1}] \min [V_n] . \tag{3.16}$$

We have already noticed that

$$\lim_{n\to\infty} \frac{1}{n} \log ||P_n(\bar{\zeta})|| = \pi_0 \qquad \text{w.p. } 1.$$

Applying Lemma 3.5 to the sequence of matrices $\{M_{\zeta_n}(q(T^{n+1}\bar{\zeta}))\}$ $n \geqq 0$ we obtain,

$$\frac{1}{n} \log ||P_n(\bar{\zeta})|| \leqq \frac{1}{n} \{\log p + \log B[M_{\zeta_0}(q(T\bar{\zeta}))] + \log B[(M_{\zeta_{n-1}}(q(T^n\bar{\zeta}))] + \log \min [P_n(\bar{\zeta})]\} . \tag{3.17}$$

From the assumptions of Theorem 1, there exists a constant $L > 0$ such that

$$P\{\bar{\zeta}\colon B[M_{\zeta_0}(q(T\bar{\zeta}))] < L\} > 0 .$$

By the ergodic theorem then,

$$\liminf_{n\to\infty} \frac{1}{n} \log B[M_{\zeta_{n-1}}(q(T^n\bar{\zeta}))] = 0 \qquad \text{w.p. } 1. \tag{3.18}$$

From (3.17) and (3.18)

$$\liminf_{n\to\infty} \frac{1}{n} \log ||P_n(\bar{\zeta})|| \leqq \liminf_{n\to\infty} \frac{1}{n} \log \min [P_n(\bar{\zeta})] \qquad \text{w.p. } 1.$$

The next two lemmas prove that

$$\liminf_{n\to\infty} \frac{1}{n} \log \min [P_n(\bar{\zeta})] < 0 \qquad \text{w.p. } 1.$$

It was shown in the previous section that either $P\{\bar{\zeta}\colon q(\bar{\zeta}) = \mathbf{1}\} = 1$ or $P\{\bar{\zeta}\colon q(\bar{\zeta}) < \mathbf{1}\} = 1$. Without loss of generality, we may assume that $P\{\bar{\zeta}\colon q(\bar{\zeta}) < \mathbf{1}\} = 1$. Otherwise, Theorem 1 would trivially be true.

LEMMA 3.6. *Assume the conditions of Theorem* 1. *Also assume that* $P\{\bar{\zeta}\colon q(\bar{\zeta}) < \mathbf{1}\} = 1$. *Then,*

$$E\left\{\left|\log \frac{(\mathbf{1} - q(\bar{\zeta}), \mathbf{1})}{(\mathbf{1} - q(T\bar{\zeta}), \mathbf{1})}\right|\right\} < \infty \qquad \textit{and} \qquad E\left\{\log \frac{(\mathbf{1} - q(\bar{\zeta}), \mathbf{1})}{(\mathbf{1} - q(T\bar{\zeta}), \mathbf{1})}\right\} = 0 .$$

The proof of this lemma is similar to that of Theorem 1 in [1] and will be omitted.

LEMMA 3.7. *Let the assumptions of Lemma* 3.6 *hold. Then,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \min [P_n(\bar{\zeta})] < 0 \qquad \text{w.p. } 1.$$

PROOF. Define,

$$\begin{aligned} \Gamma_n(\bar{\zeta}) &= (P_n(\bar{\zeta})\cdot(\mathbf{1} - q(T^n\bar{\zeta}))^t, \mathbf{1}) \qquad n \geqq 1 \\ \Gamma_0(\bar{\zeta}) &= (\mathbf{1} - q(\bar{\zeta}), \mathbf{1}) . \end{aligned} \tag{3.19}$$

Using (2.4) and Taylor's theorem, we obtain

$$\begin{aligned} \mathbf{1} - q(\bar{\zeta}) &= \mathbf{1} - \phi_{\zeta_0}(\phi_{\zeta_1}(\cdots \phi_{\zeta_{n-1}}(q(T^n\bar{\zeta})))) \\ &= P_n(\bar{\zeta})\cdot(\mathbf{1} - q(T^n\bar{\zeta}))^t + R_n(\bar{\zeta}) \qquad n \geqq 1 \end{aligned} \tag{3.20}$$

where $R_n(\bar{\zeta})$ is the remainder term for Taylor's theorem. It follows that

$$\Gamma_0(\bar{\zeta}) = \Gamma_n(\bar{\zeta}) + (R_n(\bar{\zeta}), \mathbf{1}) = \Gamma_n(\bar{\zeta}) \left\{1 + \frac{(R_n(\bar{\zeta}), \mathbf{1})}{\Gamma_n(\bar{\zeta})}\right\} \tag{3.21}$$

and

$$\Gamma_0(\bar{\zeta}) \geqq p[\min [P_n(\bar{\zeta})]](\mathbf{1} - q(T^n\bar{\zeta}), \mathbf{1}) \left\{1 + \frac{(R_n(\bar{\zeta}), \mathbf{1})}{\Gamma_n(\bar{\zeta})}\right\}.$$

Taking logs we obtain

$$\log \frac{(\mathbf{1} - q(\bar{\zeta}), \mathbf{1})}{(\mathbf{1} - q(T^n\bar{\zeta}), \mathbf{1})} \geqq \log p + \log \min [P_n(\bar{\zeta})] + \log\left[1 + \frac{(R_n(\bar{\zeta}), \mathbf{1})}{\Gamma_n(\bar{\zeta})}\right]. \tag{3.22}$$

It follows from Lemma 3.6 and the ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{(\mathbf{1} - q(\bar{\zeta}), \mathbf{1})}{(\mathbf{1} - q(T^n\bar{\zeta}), \mathbf{1})} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{(\mathbf{1} - q(T^j\bar{\zeta}), \mathbf{1})}{(\mathbf{1} - q(T^{j+1}\bar{\zeta}), \mathbf{1})} = 0 \qquad \text{w.p. } 1.$$

So to prove the lemma it is enough to show that

$$\liminf_{n\to\infty} \frac{1}{n} \log \left\{1 + \frac{(R_n(\bar{\zeta}), \mathbf{1})}{\Gamma_n(\bar{\zeta})}\right\} > 0 \qquad \text{w.p. } 1.$$

Observe that

$$\frac{1}{n} \log \left\{1 + \frac{(R_n(\bar{\zeta}), \mathbf{1})}{\Gamma_n(\bar{\zeta})}\right\} = \frac{1}{n} \log \Gamma_0(\bar{\zeta})/\Gamma_n(\bar{\zeta}) = \frac{1}{n} \sum_{j=0}^{n-1} \log \Gamma_j(\bar{\zeta})/\Gamma_{j+1}(\bar{\zeta}).$$

Again, using Taylor's Theorem, it is easy to see that

$$1 - q(T^n\bar{\zeta}) \geqq M_{\zeta_n}(q(T^{n+1}\bar{\zeta}))\cdot(1 - q(T^{n+1}\bar{\zeta}))^t \qquad n \geqq 1. \tag{3.23}$$

This implies that $\Gamma_n(\bar{\zeta})$ is decreasing in n w.p. 1. Therefore,

$$\Gamma_n(\bar{\zeta})/\Gamma_{n+1}(\bar{\zeta}) \geqq 1 \qquad \text{w.p. } 1., \ n \geqq 0. \tag{3.24}$$

From our assumptions on the environmental process, the inequality in (3.23) is strict in each of the components. In particular, there exists an $\varepsilon > 0$ and a $\delta > 0$ such that the set

$$E = \{\bar{\zeta}: (1 - \varepsilon)(\mathbf{1} - q(\bar{\zeta})) \geqq M_{\zeta_0}(q(T\bar{\zeta}))(\mathbf{1} - q(T\bar{\zeta}))^t\}$$

has probability greater than δ. Let $I_E(\bar{\zeta})$ be the indicator function of set E. By the ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} I_E(T^j\bar{\zeta}) > \delta \qquad \text{w.p. } 1.$$

Also, if $T^n(\bar{\zeta}) \in E$ then $\log [\Gamma_n(\bar{\zeta})/\Gamma_{n+1}(\bar{\zeta})] \geqq \log (1-\varepsilon)^{-1}$. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \Gamma_j(\bar{\zeta})/\Gamma_{j+1}(\bar{\zeta}) \geqq \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} I_E(T^j\bar{\zeta}) \log \frac{\Gamma_j(\bar{\zeta})}{\Gamma_{j+1}(\bar{\zeta})}$$

$$\geqq \left[\log \frac{1}{1-\varepsilon}\right]\delta > 0 \qquad \text{w.p. 1.}$$

This proves the result. □

The proof of Theorem 1 is now complete.

REMARK 1. If one checks the proof of Theorem 1, carefully, it becomes clear that, except for one place, all the arguments can be carried out with the condition of the theorem weakened to

$$E\{\log^+ ||M_{\zeta_0}||\} < \infty . \tag{3.25}$$

The strength of the condition of the theorem is needed to assert that

$$\lim_{n\to\infty} \phi_{\zeta_0}(\phi_{\zeta_1}(\cdots \phi_{\zeta_n}(Rs))) \quad \text{exists} \qquad \text{w.p. 1. } s \in [0, 1) . \tag{3.26}$$

Therefore, any set of conditions which imply (3.25) and (3.26) will imply the result of Theorem 1. One such set of conditions is:

There exists constants E and $\varepsilon > 0$ such that

$$\max [M_{\zeta_0}] < E \qquad \text{and} \qquad \mathbf{1} - \phi_{\zeta_0}(\mathbf{0}) > R\varepsilon \qquad \text{w.p. 1.}$$

REMARK 2. The proof of $\pi_0 < 0$ can be carried out with only assuming (3.25). An interesting consequence of this result is the following proposition.

PROPOSITION 3.1. *Let* $\{\phi_{\zeta_n}(s)\}_{n\geqq 0}$ *be an environmental process satisfying* (3.25). *Then the only solution* $q_0(\bar{\zeta})$ (*if one exists*) *of the functional equation*

$$\phi_{\zeta_0}(q_0(T\bar{\zeta})) = q_0(\bar{\zeta}) \qquad \text{w.p. 1.} \tag{3.27}$$

satisfying $P\{\bar{\zeta} : q_0(\bar{\zeta}) < \mathbf{1}\} = 1$ *is* $q_0(\bar{\zeta}) = q(\bar{\zeta})$ a.e.

PROOF. Iterating (3.27) we obtain

$$q_0(\bar{\zeta}) = \phi_{\zeta_0}(\phi_{\zeta_1}(\cdots \phi_{\zeta_{n-1}}(q_0(T^n\bar{\zeta})) \geqq \phi_{\zeta_0}(\phi_{\zeta_1}(\cdots \phi_{\zeta_{n-1}}(\mathbf{0}))) \qquad \text{w.p. 1., } n \geqq 1 .$$

Let $n \to \infty$. Then $q_0(\bar{\zeta}) \geqq q(\bar{\zeta})$ w.p. 1. Using the mean value theorem we now obtain

$$|q_0(\bar{\zeta}) - q(\bar{\zeta})| \leqq p||\textstyle\prod_{j=0}^{n} M_{\zeta_0}(q_0(T^{j+1}\bar{\zeta}))|| \qquad \text{w.p. 1.} \tag{3.28}$$

The collection of matrices $\{M_{\zeta_j}(q_0(T^{j+1}\bar{\zeta}))\}_{j\geqq 0}$ constitute a stationary ergodic sequence and we may therefore, apply the result of Furstenburg and Kesten [4]. Thus

$$\lim_{n\to\infty} \frac{1}{n} \log ||\textstyle\prod_{j=0}^{n-1} M_{\zeta_j}(q_0(T^{j+1}\bar{\zeta}))|| = \hat{\pi} \qquad \text{w.p. 1.}$$

It can be shown using the same techniques to show $\pi_0 < 0$, that $\hat{\pi} < 0$. It follows that

$$\lim_{n\to\infty} ||\textstyle\prod_{j=0}^{n-1} M_{\zeta_j}(q_0(T^{j+1}\bar{\zeta}))|| = 0 \qquad \text{w.p. 1.} \tag{3.29}$$

(3.28) and (3.29) prove the result. □

4. Extinction criteria for a critical MBPRE. We now prove Theorem 2. The proof is by contradiction. Thus, we suppose that the result is false. It then follows by the remarks of Section 2 that,

$$P\{\bar{\zeta}: q(\bar{\zeta}) < \mathbf{1}\} = 1 . \tag{4.1}$$

Using Theorem 1, together with Fatou's Lemma, we conclude that (4.1) implies

$$\lim_{n\to\infty} ||Y_n(\bar{\zeta})|| = \infty \qquad \text{w.p. } 1. \tag{4.2}$$

In the next series of lemmas we show that (4.2) implies

$$\lim_{n\to\infty} E\{\log ||Y_n(\bar{\zeta})||\} = \infty . \tag{4.3}$$

On the other hand, we prove from other considerations that (4.3) is impossible. This will give the needed contradiction.

We first prove that (4.2) implies (4.3). To establish (4.3), it is sufficient to show that

$$\lim_{n\to\infty} E\{\log ||Y_n(\bar{\zeta})^t||\} = \infty .$$

The random matrices $\{Y_n(\bar{\zeta})^t\}_{k\geqq 0}$ are positive w.p. 1. Therefore, we can apply theorems of Frobenius [8] and conclude that for $k \geqq 1$, $Y_k(\bar{\zeta})^t$ has w.p. 1 a positive eigenvalue $\lambda_k(\bar{\zeta})$ and a corresponding right eigenvector $U_k(\bar{\zeta})$ which is strictly positive. We normalize $U_k(\bar{\zeta})$ so that $(U_k(\bar{\zeta}), \mathbf{1}) = 1$. Using the inequality in Lemma 3.5 applied to the matrices $\{M_{\zeta_n}\}$ and the assumptions of Theorem 2, it is easy to show that:

$$E\{\log \lambda_n(\bar{\zeta})\} \leqq \log p + 2\log(D/C) + E\{\log ||Y_n(\bar{\zeta})^t||\} .$$

So to prove (4.3) it is sufficient to show that $\lim_{n\to\infty} E\{\log \lambda_n(\bar{\zeta})\} = \infty$. The next two lemmas do this.

LEMMA 4.1.

$$\lim_{n\to\infty} E\left\{\log \frac{(Y_n(\bar{\zeta})\cdot(\mathbf{1} - q(T^n\bar{\zeta}))^t, U_n(\bar{\zeta}))}{(\mathbf{1} - q(\bar{\zeta}), U_n(\bar{\zeta}))}\right\} = \infty .$$

PROOF. Choose $\varepsilon, L > 0$. From (4.2) $\lim_{n\to\infty} ||Y_n(\bar{\zeta})|| = \infty$ w.p. 1. By Egoroff's Theorem there exists a set H_ε and an integer N_0 such that $P\{H_\varepsilon\} > 1 - \varepsilon/2$ and if $n \geqq N_0$ then $||Y_n(\bar{\zeta})|| \geqq L$ a.e. on H_ε. Using the inequality in Lemma 3.5 applied to the matrices $\{M_{\zeta_n}\}_{n\geqq 0}$, together with the above remarks, it is easy to show that for $n \geqq N_0$

$$\min[Y_n(\bar{\zeta})] \geqq L/p(D/C)^2 \qquad \text{a.e. on } \quad H_\varepsilon . \tag{4.4}$$

Since we are assuming that (4.1) is valid, there exists an $\alpha > 0$ such that

$$P\{\bar{\zeta}: \mathbf{1} - q(\bar{\zeta}) \geqq R\alpha\} \geqq 1 - \varepsilon/2 . \tag{4.5}$$

Define $F_n = \{\bar{\zeta}: \mathbf{1} - q(T^n\bar{\zeta}) \geqq R\alpha\}$, $n \geqq 1$. By stationarity $P\{F_n\} \geqq 1 - \varepsilon/2$. Also observe that by (2.4) and Taylor's theorem:

$$\frac{(Y_k(\bar{\zeta})\cdot(\mathbf{1} - q(T^k\bar{\zeta}))^t, U_k(\bar{\zeta}))}{(\mathbf{1} - q(\bar{\zeta}), U_k(\bar{\zeta}))} \geqq 1 \qquad \text{w.p. } 1. \tag{4.6}$$

Set $Q_n = H_\varepsilon \cap F_n$. It follows from (4.4), (4.5) and (4.6) that

$$E\left\{\log \frac{(Y_k(\bar{\zeta}) \cdot (1 - q(T^k\bar{\zeta}))^t, U_k(\bar{\zeta}))}{(1 - q(\bar{\zeta}), U_k(\bar{\zeta}))}\right\} \geqq E[\log \{\alpha p \min [Y_n(\bar{\zeta})]\}; Q_n\}$$
$$\geqq \log \{\alpha L/(D/C)^2\}(1 - \varepsilon) .$$

Since L was arbitrary, the result follows. □

LEMMA 4.2.

$$\sup_{n\geqq 1}\left|E\left\{\log \frac{(1 - q(T^n\bar{\zeta}), U_n(\bar{\zeta}))}{(1 - q(\bar{\zeta}), U_n(\bar{\zeta}))}\right\}\right| < \infty .$$

PROOF. It is not difficult to show that for $n \geqq 1$

$$\left|E\left\{\log \frac{(1 - q(T^n\bar{\zeta}), U_n(\bar{\zeta}))}{(1 - q(\bar{\zeta}), U_n(\bar{\zeta}))}\right\}\right| \leqq E\{\log B[Y_n(\bar{\zeta})^t]\} + \left|E\left\{\log \frac{(1 - q(T^n\bar{\zeta}), 1)}{(1 - q(\bar{\zeta}), 1)}\right\}\right| .$$

It is a consequence of Lemma 3.6 that

$$E\left\{\log \frac{(1 - q(T^n\bar{\zeta}), 1)}{(1 - q(\bar{\zeta}), 1)}\right\} = 0 \qquad n \geqq 1 .$$

It also follows from the inequality in Lemma 3.5 that

$$\sup_{n\geqq 1} E\{\log B[Y_n(\bar{\zeta})]\} < \infty .$$

The result now follows. □

Finally observe that

$$(4.7) \qquad E\left\{\log \frac{(Y_n(\bar{\zeta}) \cdot (1 - q(T^n\bar{\zeta}))^t, U_n(\bar{\zeta}))}{(1 - q(\bar{\zeta}), U_n(\bar{\zeta}))}\right\}$$
$$= E\{\log \lambda_n(\bar{\zeta})\} + E\left\{\log \frac{(1 - q(T^n\bar{\zeta}), U_n(\bar{\zeta}))}{(1 - q(\bar{\zeta}), U_n(\bar{\zeta}))}\right\} .$$

Using Lemmas 4.1 and 4.2 we conclude that

$$\lim_{n\to\infty} E\{\log \lambda_n(\bar{\zeta})\} = \infty .$$

To complete the proof of Theorem 2, we show that (4.3) is impossible. In view of the inequality of Lemma 3.5, it is sufficient to prove that

$$(4.8) \qquad \limsup_{n\to\infty} E\{\log (Y_n(\bar{\zeta})^t)_{11}\} < \infty .$$

$\{M^t_{\zeta_n}\}_{n\geqq 0}$ is a stationary ergodic process. In their paper, Furstenburg and Kesten [4] prove that if there exists a constant G such that $B[M^t_{\zeta_0}] < G$ w.p. 1, then there exists a constant b and an integer n_0 such that

$$(4.9) \qquad E\{\log [(Y_{k+1}(\bar{\zeta})^t)_{11}/(Y_k(\bar{\zeta})^t)_{11}]\} = b + O(1 - G^{-3})^k \qquad k \geqq n_0 .$$

For our problem $G = D/C$. We show that (4.9) implies (4.8). Let

$$\xi_k = E\{\log [(Y_{k+1}(\bar{\zeta})^t)_{11}/(Y_k(\bar{\zeta})^t)_{11}]\} \qquad k \geqq 1 .$$

The following identity is valid for $n \geqq n_0$

$$\frac{1}{n+1} \sum_{k=n_0}^{n} \xi_k = \frac{1}{n+1} \{E\{\log [(Y_{n+1}(\bar{\zeta})^t)_{11}]\} - E\{\log [(Y_{n_0}(\bar{\zeta})^t)_{11}]\}\} . \tag{4.10}$$

From (4.9) and (4.10) we infer that

$$\lim_{n\to\infty} \frac{1}{n} E\{\log (Y_n(\bar{\zeta})^t)_{11}\} = b .$$

However,

$$\lim_{n\to\infty} \frac{1}{n} E\{\log (Y_n(\bar{\zeta})^t)_{11}\} = \lim_{n\to\infty} \frac{1}{n} E\{\log ||Y_n(\bar{\zeta})||\} = 0 .$$

Therefore $b = 0$. (4.9) now becomes

$$E\{\log [(Y_{k+1}(\bar{\zeta})^t)_{11}/(Y_k(\bar{\zeta})^t)_{11}]\} = O(1 - (D/C)^{-3})^k \qquad k \geqq n_0 . \tag{4.11}$$

Sum both sides of (4.11) for $k = n_0, n_0 + 1, \cdots, n$. The left side telescopes and we obtain

$$E\{\log (Y_n(\bar{\zeta})^t)_{11}\} - E\{\log (Y_{n_0}(\bar{\zeta})^t)_{11}\} = \sum_{k=n_0}^{n} O(1 - (D/C)^{-3})^k .$$

From this we conclude that

$$\limsup_{n\to\infty} E\{\log (Y_n(\bar{\zeta})^t)_{11}\} < \infty .$$

This concludes the proof of Theorem 2.

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