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RENEWAL THEORY FOR FUNCTIONALS OF A MARKOV CHAIN WITH GENERAL STATE SPACE[1]

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We prove an analogue of Blackwell's renewal theorem or the "key renewal theorem" and the existence of the limit distribution of the residual waiting time in the following setup: $X_0, X_1, \cdots$ is a Markov chain with separable metric state space and $u_0, u_1, \cdots$ is a sequence of random variables, such that the conditional distribution of u_i, given all X_j and u_l, $l \neq i$, depends on X_i and X_{i+1} only. Here the $V_n \equiv \sum_0^{n-1} u_i$, $n \geqq 1$, take the role of the partial sums of independent identically distributed random variables in ordinary renewal theory. E.g. the key renewal theorem in this setup states that $\lim_{t\to\infty} E\{\sum_{n=0}^\infty g(X_n, t - V_n) \mid X_0 = x\}$ exists for suitable $g(\cdot, \cdot)$, and is independent of x.

1. Introduction and statement of results. Let $X_0, X_1, \cdots$ be a Markov chain with general state space S, on which a σ-field $\mathscr{S}$ is given. We assume that it has stationary transition probabilities and denote them by

$$P^m(x, A) = P\{X_{n+m} \in A \mid X_n = x\};$$

$P(x, A)$ will be written for $P^1(x, A)$. We consider a setup where at time n a (real valued) random quantity u_n is picked, whose distribution depends only on the state of the underlying Markov chain at times n and $n+1$. More precisely, for each fixed $x, y \in S$ and Borel set A

$$P\{u_n \in A \mid X_n = x, X_{n+1} = y, X_j, j \neq n, n+1, u_j, j \neq n\} = F(A \mid x, y)$$

for some probability distribution $F(\cdot \mid x, y)$, independent of n. Note that this permits $F(d\lambda \mid x, y)$ to be concentrated on one point. In other words u_i is allowed to be a deterministic function of X_i, X_{i+1}. As usual, we take as our basic probability space the space $\Omega = \prod_{i\geqq 0} (S \times \mathbb{R})$ with the σ-field $\mathscr{F} = \prod_{i\geqq 0} (\mathscr{S} \times \mathscr{B})$, where $\mathscr{B}$ is the collection of Borel sets of $\mathbb{R}$. The X_i and u_i are taken as coordinate functions, i.e., if $\omega = \{(\omega_i', \omega_i'')\}_{i\geqq 0}$, then $X_i(\omega) = \omega_i'$, $u_i(\omega) = \omega_i''$. We write P_x for the measure pertaining to "paths" with $X_0 = x$, so that our basic assumption states that for $A_i \in \mathscr{S}$, $B_i \in \mathscr{B}$

$$\begin{aligned} &P_x\{X_i \in A_i, 0 \leqq i \leqq n, u_i \in B_i, 0 \leqq i < n\} \\ (1.1) \qquad &= I_{A_0}(x) \int_{A_1} P(x, dy_1) \cdots \int_{A_n} P(y_{n-1}, dy_n) \int_{B_0} F(d\lambda_0 \mid x, y_1) \\ &\quad \times \int_{B_1} F(d\lambda_1 \mid y_1, y_2) \cdots \int_{B_{n-1}} F(d\lambda_{n-1} \mid y_{n-1}, y_n) . \end{aligned}$$

E_x will be the expectation operator w.r.t. P_x.

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Throughout this paper the following standard assumptions will be in force even though we shall not repeat them: For fixed x, $P(x, \cdot)$ is a probability measure on $\mathscr{S}$ and for fixed $A_i \in \mathscr{S}$, $B_i \in \mathscr{B}$, the probability in (1.1) is an $\mathscr{S}$-measurable function of x. We shall also make frequent use of the Markov property

$$P_x\{X_i \in A_i, u_i \in B_i, n \leqq i \leqq n+m \mid X_i, 0 \leqq i \leqq n, u_j, 0 \leqq j < n\} = P_{X_n}\{X_i \in A_{n+i}, u_i \in B_{n+i}, 0 \leqq i \leqq m\} \quad \text{a.e.} \quad [P_x],$$

without explicit reference. This property is immediate from (1.1).

Here we are concerned with renewal theorems for the sums

$$V_n = \sum_{i=0}^{n-1} u_i \qquad (V_0 = 0). \tag{1.2}$$

Our main results (1.20) and (1.16) are generalizations of Blackwell's renewal theorem or the equivalent key renewal theorem (see [9] pages 360–363 and [23] page 247) and the limit distribution of the residual waiting time in renewal theory (compare [9] page 369, 370 and [23] page 260). Renewal theorems for V_n have already been considered in [3], [4], [12], [15], [16], [21], [22], [27]; (see also some of their references; for related results see [5] and [25] Chapter X). If $\{X_n\}$ visits some point $s_0 \in S$ infinitely often with probability one (e.g., if S is countable and $\{X_n\}$ recurrent), then one can reduce the renewal theory for V_n to standard renewal theory by Doeblin's trick of looking at the excursions between successive visits by X_n to s_0 (see [4], [21], [22] and [27]). Noncountable S is harder to deal with, but has been considered by[2] Orey [15], [16] Runnenburg [21], Jacod [12] and Zaslavskii [28]. All these papers assume that $\{X_n\}$ is Harris recurrent and, with the exception of [28], $u_i \geqq 0$. [15] and [21] even make several further restrictions and we cannot understand the proof of [28] at all. In particular the definition of "non-arithmetic" in [28] does not seem to be the appropriate one and the use of selection principles in Lemma 2 and Theorem 5 seems unjustified because one may need different subsequences for different (α, Γ) (see [28]). (There are also some errors in [16] page 392 L 1–7 f.b and [12], Theorem 1 and Proposition 8. These errors have been corrected by the authors (private communication) and the correction to [12] will appear.) However, in the application discussed below there are interesting cases which have neither Harris recurrence nor $u_i \geqq 0$. We have therefore adapted our conditions to this application. We do not insist on $u_i \geqq 0$ but assume throughout that[3]

$$\alpha \equiv \lim_{n\to\infty} \frac{1}{n} V_n \quad \text{exists and is constant} \quad \text{a.s.}, \tag{1.3}$$

(i.e., a.e. w.r.t. P_x for every x) with $0 < \alpha < \infty$. Only in Section 4 do we take $\alpha < 0$. We indicate there how one can sometimes apply an exponential transformation familiar from the theory of random walk (see [8] or [9] Chapter XI.6) to change the process with $\alpha < 0$ into a new one with positive drift i.e., with

[2] We are grateful to Professor Çinlar for pointing out references [12] and [16] to us.

[3] a.s. stands for almost surely, i.e., of P_x measure one for all $x \in S$.

$\alpha > 0$. We also indicate how this can be used to find the asymptotic behavior ($t \to \infty$) of

$$P\{\max_{n \geqq 0} V_n > t\} \tag{1.4}$$

when $\alpha < 0$. u_i is also taken to be non-lattice, by means of the aperiodicity condition I.3 below. S could be countable, but our main interest is in the case where it is not. Harris recurrence is replaced by the unpleasant continuity condition I.4. Still our proofs are very similar to those in [12], [17], [21] as well as the proof of the renewal theorem in [9] Chapter XI.2.

We now describe the application which motivated this paper and is our only excuse for giving yet another generalization of renewal theory. Consider the difference equation

$$Y_n = M_n Y_{n-1} + Q_n\,, \qquad n \geqq 1\,, \tag{1.5}$$

where Y_n and Q_n are column vectors of size d and M_n are $d \times d$ matrices. Assume that the M_n and Q_n are random and such that $\{M_n, Q_n\}_{n \geqq 1}$ are independent and identically distributed. For any d-vector $y = (y(1), \cdots, y(d))$ (row or column) put

$$|y| = \{\textstyle\sum_{i=1}^{d} y^2(i)\}^{\frac{1}{2}}$$

and for any $d \times d$ matrix m

$$||m|| = \max_{|y|=1} |my|\,.$$

It is known ([10], Theorem 2) that if

$$E \log^+ ||M_1|| < \infty\,,$$

then

$$\alpha_1 = \lim \frac{1}{n} \log ||M_1 \cdots M_n|| \quad \text{exists and is a constant} \quad < \infty \quad \text{w.p. 1.}$$

It is not hard to show that if $\alpha_1 < 0$ and if $E|Q_1|^\beta < \infty$ for some $\beta > 0$, then the series

$$R \equiv \textstyle\sum_{n=1}^{\infty} M_1 \cdots M_{n-1} Q_n$$

converges w.p. 1 and the distribution of the solution Y_n of (1.5) converges to that of R, independently of Y_0. Questions about the tail of the distribution of R reduce to the study of

$$P\{\max_n |xM_1 \cdots M_n| > t\} \tag{1.6}$$

for large t and fixed unit vectors x. But (1.6) is the same as (1.4) if we choose for $\{X_n\}$ the Markov chain defined on the unit sphere in $\mathbb{R}^d$ by

$$X_n = |xM_1 \cdots M_n|^{-1}(xM_1 \cdots M_n)\,,$$

and

$$u_i = \log \frac{|xM_1 \cdots M_{i+1}|}{|xM_1 \cdots M_i|}\,.$$

Indeed, for this choice

$$V_n = \log |xM_1 \cdots M_n|\,,$$

and our theorems yield under certain conditions that if $\alpha_1 > 0$ and $h \geqq 1$, then[4]

$$\lim_{n\to\infty} E\#\{n \geqq 0: t \leqq |xM_1 \cdots M_n| \leqq th\} = \frac{\log h}{\alpha_1};$$

also, if $\alpha_1 < 0$, then

$$P\{\max_{n\geqq 0} |xM_1 \cdots M_n| > t\} \sim K(x)t^{-\kappa}$$

for suitable $\kappa > 0$ and $0 \leqq K(x) < \infty$, with $K(x) > 0$ for some x. The details will be worked out in [13]. Note also that the recent work of Arjas [1] yields an expression for the Laplace transform of (1.6). It is not clear to us, however, how to use Arjas' work to derive our asymptotic result.

We conclude this introduction with some notation and an explicit statement of our theorems.

$$(1.7) \qquad N(t) = \min\{n \geqq 0: V_n > t\} \quad (= \infty \text{ if no such } n \text{ exists}),$$

$$(1.8) \qquad W(t) = V_{N(t)} - t,$$

$$(1.9) \qquad Z(t) = X_{N(t)}.$$

$W(t)$ and $Z(t)$ are defined on $\{N(t) < \infty\}$ only and $P_x\{Z(t) \in A, W(t) \in B\}$ will always mean $P_x\{N(t) < \infty, Z(t) \in A, W(t) \in B\}$. Similarly $E_x f(Z(t), W(t))$ stands for the integral of $f(Z(t), W(t))$ w.r.t. P_x over $\{N(t) < \infty\}$ only.

$$(1.10) \qquad C_k = \{x \in S: P_x\{V_m \geqq mk^{-1} \text{ for all } m \geqq k\} \geqq \tfrac{1}{2}\}, \qquad k \geqq 1,$$

and $C_0 = \emptyset$.

Definition 1. A function $g: S \times \mathbb{R} \to \mathbb{R}$ is called *directly Riemann integrable* if it is $\mathscr{S} \times \mathscr{B}$ measurable and satisfies

$$(1.11) \qquad \sum_{k=0}^{\infty} \sum_{l=-\infty}^{+\infty} (k+1) \sup\{|g(x,t)|: x \in C_{k+1}\backslash C_k, l \leqq t \leqq l+1\} < \infty,$$

and if for every fixed $x \in S$ and $0 < L < \infty$ the function $t \to g(x, t)$ is Riemann integrable on $[-L, +L]$.

Definition 1 describes the class of functions for which we shall prove the key renewal theorem (see (1.20); see also Remark 2 for some comments to this definition).

Definition 2. If f is any function from $\prod_{i=0}^{\infty}(S \times \mathbb{R})$ into $\mathbb{R}$ and $\delta > 0$, then

$$(1.12) \qquad \begin{aligned} &f^\delta(x_0, v_0, x_1, v_1, \cdots) \\ &= \lim_{n\to\infty} \sup\{f(y_0, w_0, y_1, w_1, \cdots): d(x_i, y_i) \\ &\qquad + |v_i - w_i| < \delta \text{ for } i \leqq n\}. \end{aligned}$$

Note that f^δ is automatically measurable w.r.t. $\prod_{i=0}^{\infty}(\mathscr{S} \times \mathscr{B})$.

Our principal conditions follow.

[4] $\#\{C\}$ denotes the number of elements in C.

CONDITIONS I.

I.1. S is a separable metric space with distance function $d(\cdot, \cdot)$ and $\mathscr{S}$ is the σ-field generated by the open sets. There exists a probability measure φ on $\mathscr{S}$ such that

$$\varphi(A) = \varphi P(A) = \int \varphi(dx)P(x, A) , \qquad A \in \mathscr{S}, \tag{1.13}$$

and such that

$$P_x\{X_n \in A \text{ for some } n\} = 1 \quad \text{for all } x \in S \text{ and } \textit{open } A \text{ with } \varphi(A) > 0 .$$

I.2.

$$\int \varphi(dx)E_x|u_0| = \int \varphi(dx) \int P(x, dy) \int |\lambda|F(d\lambda \mid x, y) < \infty ,$$
$$\alpha \equiv \int \varphi(dx)E_x u_0 = \int \varphi(dx) \int P(x, dy) \int \lambda F(d\lambda \mid x, y) > 0 ,$$

and

$$\lim_{n\to\infty} \frac{1}{n} V_n = \alpha \quad \text{a.s.}$$

I.3. There exists a sequence $\{\zeta_\nu\} \subset \mathbb{R}$ such that the group generated by $\{\zeta_\nu\}$ is dense in $\mathbb{R}$ and such that for each ζ_ν and $\delta > 0$ there exists a $y = y(\nu, \delta) \in S$ with the following property: For each $\varepsilon > 0$ there exist an $A \in \mathscr{S}$ with $\varphi(A) > 0$ and integers m_1, m_2 and $\tau \in \mathbb{R}$ such that

$$P_x\{d(X_{m_1}, y) < \varepsilon, |V_{m_1} - \tau| \leqq \delta\} > 0 , \tag{1.14}$$

as well as

$$P_x\{d(X_{m_2}, y) < \varepsilon, |V_{m_2} - \tau - \zeta_\nu| \leqq \delta\} > 0 , \tag{1.15}$$

whenever $x \in A$.

I.4. For each $x \in S$, $\delta > 0$ there exists an $r_0 = r_0(x, \delta) > 0$ such that for all functions $f\colon \prod_{i\geq 0} (S \times \mathbb{R}) \to \mathbb{R}$ for which $f(X_0, V_0, X_1, V_1, \cdots)$ is an $\mathscr{F}$ measurable function, and for all y with $d(x, y) < r_0$ one has

$$E_x f(X_0, V_0, X_1, V_1, \cdots) \leqq E_y f^\delta(X_0, V_0, X_1, V_1, \cdots) + \delta \sup |f|$$

and

$$E_y f(X_0, V_0, X_1, V_1, \cdots) \leqq E_x f^\delta(X_0, V_0, X_1, V_1, \cdots) + \delta \sup |f| .$$

In Section 3 we prove that $Z(t)$ and $W(t)$ have a joint limit distribution.

THEOREM 1. *Assume Conditions* I.1–I.4 *are satisfied. Then there exists a finite measure* ψ *on* $\mathscr{S}$, *defined in* (3.10), *such that for every bounded and (jointly) continuous function* $f\colon S \times (0, \infty) \to \mathbb{R}$ *one has for each fixed* $x \in S$

$$\lim_{t\to\infty} E_x f(Z(t), W(t)) = \alpha^{-1} \int_S \psi(dy) \int_{S\times(0,\infty)} P_y\{X_{N(0)} \in dz, V_{N(0)} \in d\lambda\} \int_{0<s\leqq\lambda} f(z, s)\, ds . \tag{1.16}$$

In particular, for $w \geqq 0$

$$\lim_{t\to\infty} P_x\{W(t) \geqq w\} = \alpha^{-1} \int \psi(dy) \int_{\lambda\geqq w} (\lambda - w)P_y\{V_{N(0)} \in d\lambda\} . \tag{1.17}$$

Also, if $A \in \mathscr{S}$ satisfies[5]

(1.18) $$\int \psi(dy) \int_{\lambda>0} \lambda P_y\{X_{N(0)} \in \partial A,\ V_{N(0)} \in d\lambda\} = 0\,,$$

then

(1.19) $$\lim_{t\to\infty} P_x\{Z(t) \in A\} = \alpha^{-1} \int \psi(dy) \int_{\lambda>0} \lambda P_y\{X_{N(0)} \in A,\ V_{N(0)} \in d\lambda\}\,.$$

Section 3 also contains the following key renewal theorem.

THEOREM 2. *Assume Conditions* I.1–I.4 *are satisfied. Then for every jointly continuous function $g: S \times \mathbb{R} \to \mathbb{R}$ which is directly Riemann integrable and every $x \in S$*

(1.20) $$\lim_{t\to\infty} E_x\{\textstyle\sum_{n=0}^{\infty} g(X_n, t - V_n)\} = \alpha^{-1} \int_S \varphi(dy) \int_{-\infty}^{+\infty} g(y, s)\, ds\,.$$

REMARK 1. Once we have (1.16) for bounded continuous functions f it can of course be extended to functions f for which there exist bounded continuous functions $f_n^{\pm}$ such that $f_n^- \leqq f \leqq f_n^+$ and

$$\lim_{n\to\infty} \alpha^{-1} \int \psi(dy) \int_{S\times(0,\infty)} P_y\{X_{N(0)} \in dz,\ V_{N(0)} \in d\lambda\} \int_{0<s\leqq\lambda} f_n^{\pm}(z, s)\, ds$$
$$= \alpha^{-1} \int \psi(dy) \int_{S\times(0,\infty)} P_y\{X_{N(0)} \in dz,\ V_{N(0)} \in d\lambda\} \int_{0<s\leqq\lambda} f(z, s)\, ds\,.$$

A similar remark applies to (1.20). In this way one can show for instance that (1.20) holds when g is directly Riemann integrable, and such that the family of functions $\{x \to g(x, t)\}_{t\in\mathbb{R}}$ is equicontinuous.

REMARK 2. If (1.3) holds with $\alpha > 0$ then

$$S = \bigcup_{k=1}^{\infty} C_k\,,$$

and Definition 1 extends Feller's definition of direct Riemann integrability ([9] page 362) and Orey's of class $\mathscr{K}$ ([17] page 950) which correspond to the case where the underlying Markov chain $\{X_n\}$ is trivial, i.e., where S is a one point set. However, in the case where S is countable, (1.11) is more stringent than the convergence of the sums (3.2) and (3.3) in [4]. Çinlar's definition ((3.1) in [4] or Section 4 of [3]) of direct Riemann integrability would replace (1.11) by the weaker condition

$$\int \varphi(dx) \textstyle\sum_{l=-\infty}^{+\infty} \sup\{|g(x, t)| : l \leqq t \leqq l + 1\} < \infty$$

where φ is an invariant measure for $\{X_n\}$ as in (1.13). However, our proof of Theorem 2 only works with the more stringent condition (1.11) and we cannot follow the proof of (27) in [3] (specifically we do not understand the estimate following (32) in [3]). We point out though, that if there exist $\Delta > 0$ and $0 < p \leqq 1$ for which

$$F(0- \mid x, y) = 0\,, \qquad F(\Delta- \mid x, y) \leqq 1 - p$$

for all $x, y \in S$, then (by a simple comparison with binomial variables)

$$\begin{aligned} P_x\{V_m < \tfrac{1}{2}mp\Delta \ \text{ for some } \ m \geqq k_0\} &\leqq P_x\{\text{at most } \tfrac{1}{2}mp\, u_i \ \text{ with } \ i < m \ \text{ exceed } \ \Delta\} \\ &\leqq \textstyle\sum_{m=k_0}^{\infty} \sum_{l\leqq\frac{1}{2}mp} \binom{m}{l} p^l (1 - p)^{m-l}\,, \qquad x \in S\,. \end{aligned}$$

[5] We write $\bar{A}$ for the closure of A, $\mathring{A}$ for the interior of A, and $\partial A = \bar{A} \backslash \mathring{A}$, the boundary of A.

If we take $k_0 \geqq (\frac{1}{2}p\Delta)^{-1}$ so large that the last sum is less than $\frac{1}{2}$, then every x belongs to C_{k_0} and (1.11) is implied by

$$(1.21) \qquad \sum_{l=-\infty}^{+\infty} \sup\{(g(x, t)) : x \in S, l \leqq t \leqq l+1\} < \infty \, .$$

Thus, if the u_i are positive in a uniform way then (1.11) may be replaced by (1.21). In particular this is permissable when $u_i \geqq \Delta > 0$ a.s.

The continuity condition I.4 is not too hard to check in the matrix example discussed above when the matrices M_i have positive entries. However, it seems awkward in general and hard to check in practice. If the M_i can have negative entries, and conceivably in other applications as well, alternative conditions are more useful. We give such a set of conditions and the corresponding theorems below. Conditions II require even more than Harris recurrence of $\{X_n\}$ in that φ in II.1 is assumed to be a probability measure; Orey in [16] and Jacod in [12] allow infinite invariant measures φ. Theorems 3 and 4 can be obtained from [16] and [12]. They can also be proved by methods of this paper; only Section 2 needs changes whereas Section 3 simplifies considerably with Conditions II in the place of Conditions I. The reader should also notice that the continuity requirements on f and g for (1.16)—(1.20) are considerably weakened in Theorems 3 and 4.

CONDITIONS II.

II.1. There exists a probability measure φ on $\mathscr{S}$ such that $\varphi = \varphi P$ (i.e., (1.13)) and[6]

$$(1.22) \qquad \lim_{n\to\infty} \|P^n(x, \cdot) - \varphi(\cdot)\| = 0 \qquad \text{for each} \quad x \in S \, .$$

II.2. Same as I.2.

II.3. There exists a sequence $\{\zeta_\nu\} \subset \mathbb{R}$ such that the group generated by $\{\zeta_\nu\}$ is dense in $\mathbb{R}$ and such that for each ζ_ν and $\delta > 0$ there exist $A_\nu \in \mathscr{S}$ and integers $m_1, m_2 \geqq 0$ satisfying

$$\varphi(A_\nu) > 0 \qquad \text{and for each} \quad x \in A_\nu{}^7$$

$$(1.23) \qquad \int_S \min\,(P_x\{X_{m_1} \in dy, \tau - \delta < V_{m_1} < \tau + \delta\}, \\ P_x\{X_{m_2} \in dy, \tau + \zeta_\nu - \delta < V_{m_2} < \tau + \zeta_\nu + \delta\}) > 0 \, .$$

THEOREM 3. *Assume that Conditions* II.1–II.3 *are satisfied. Then* (1.16) *holds for every bounded,* $\mathscr{S} \times \mathscr{B}$ *measurable function* $f: S \times (0, \infty) \to \mathbb{R}$ *for which* $t \to f(x, t)$ *is continuous for each fixed* x. *Also* (1.17) *holds, and* (1.19) *holds for every* $A \in \mathscr{S}$.

THEOREM 4. *Assume Conditions* II.1–II.3 *are satisfied. Then* (1.20) *holds for every directly Riemann integrable function* $g: S \times \mathbb{R} \to \mathbb{R}$.

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[6] $\|\nu\|$ denotes the total variation of the signed measure ν. If $\nu(S) = 0$, as is the case for the measure in (1.22) then one easily sees that $\|\nu\| = 2\sup_{A \in \mathscr{S}} \nu(A)$.

[7] The integral in (1.23) equals $P_x\{X_{m_1} \in S^-, \tau - \delta < V_{m_1} < \tau + \delta\} + P_x\{X_{m_2} \in S^+, \tau + \zeta_\nu - \delta < V_{m_2} < \tau + \zeta_\nu + \delta\}$, where S^+ is the subset of S on which $P_x\{X_{m_1} \in \cdot, \tau - \delta : < V_{m_1} < \tau + \delta\} - P_x\{X_{m_2} \in \cdot, \tau + \zeta_\nu - \delta < V_{m_2} < \tau + \zeta_\nu + \delta\}$ is a positive measure and $S^- = S/S^+$ (see [19] Chapter 11.5, especially problem 31).

2. The Choquet–Deny lemma. In this section we prove the analogue for our situation of the Choquet–Deny lemma (see [14] Section VIII.6) which figures prominently in the proof of the renewal theorem in [9] (see Lemma XI.2.1 and Corollary to Lemma XI.9.1) as well as in [12] and [28]. As in [12] our proof closely follows the proof of Theorem 3.1 of [17].

LEMMA 1. *Assume that Conditions* I.1–I.3 *are satisfied. Let* $H: S \times \mathbb{R}$ *be a bounded function which satisfies*

$$\text{(2.1)} \qquad \lim_{t\to\infty} |H(x, t) - E_x H(X_1, t - u_0)| = \lim_{t\to\infty} |H(x, t) - \textstyle\int P(x, dy) \int F(d\lambda \mid x, y) H(y, t - \lambda)| = 0$$

plus the continuity condition

$$\text{(2.2)} \qquad \lim_{y\to x, \delta\downarrow 0} \sup_{|t'-t''|<\delta} |H(x, t') - H(y, t'')| = 0$$

for each fixed $x \in S$. *Then any sequence* $\{t_n\}_{n\geq 1} \subset \mathbb{R}$ *with* $t_n \to \infty$ *contains a subsequence* $\{t_{n_k}\}_{k\geq 1}$ *for which*

$$\text{(2.3)} \qquad \lim_{k\to\infty} H(x, t_{n_k} + s)$$

exists and is independent of x *and* s.

PROOF. Let $\{t_n\}$ with $t_n \to \infty$ be given. Since the family of functions $\{(x, s) \to H(x, t_n + s)\}_{n\geq 1}$ is equicontinuous (by (2.2)) and $S \times \mathbb{R}$ is separable, we can by standard diagonal selection methods (compare [20] Theorem 7.23) find a subsequence t_{n_k} such that

$$G(x, s) \equiv \lim_{k\to\infty} H(x, t_{n_k} + s)$$

exists for all $(x, s) \in S \times \mathbb{R}$. By (2.1) G satisfies

$$\text{(2.4)} \qquad G(x, s) = E_x G(X_1, s - u_0) = \textstyle\int P(x, dy) \int F(d\lambda \mid x, y) G(y, s - \lambda) .$$

In addition G is bounded and satisfies (2.2) with G replaced by H. Since $s \to G(x, s)$ is continuous it suffices to show that

$$G_h(x, s) = \textstyle\int_{-\infty}^{+\infty} G(x, s + r) h(r)\, dr$$

is constant for every continuous function $h: \mathbb{R} \to \mathbb{R}$ with compact support. Again (2.2) and (2.4) hold with H, respectively G, replaced by G_h, and one easily sees that G_h even satisfies

$$\text{(2.5)} \qquad \lim_{\delta\downarrow 0} \sup_x \sup_{|t'-t''|<\delta} |G_h(x, t') - G_h(x, t'')| = 0 .$$

For the remainder of the proof we fix h and drop the subscript h. Thus we want to prove that a solution G of (2.4) which satisfies (2.2) and (2.5) must be a constant. Let $\{\zeta_\nu\}$ be as in I.3 and consider the random variables

$$\lambda_n(s) \equiv G(X_n, s - V_n) , \qquad \lambda_n(s + \zeta_\nu) \equiv G(X_n, s + \zeta_\nu - V_n)$$

and

$$\Lambda_n(s, \nu) = \lambda_n(s + \zeta_\nu) - \lambda_n(s) .$$

Each of these is measurable w.r.t. $\mathscr{F}_n$, the σ-field generated by X_i, $0 \leqq i \leqq n$, and u_i, $0 \leqq i \leqq n-1$. By (2.4), each of the sequences $\{\lambda_n(s)\}_{n\geqq 0}$, $\{\lambda_n(s+\zeta_\nu)\}_{n\geqq 0}$, $\{\Lambda_n(s,\nu)\}_{\nu\geqq 0}$ is even bounded $\mathscr{F}_n$ martingale under each of the measures P_x. Thus for all $(x, s) \in S \times \mathbb{R}$ they converge a.e. $[P_x]$. Now let φ be as in I.1 and assume that for each $\nu \geqq 1$, $k \geqq 1$ there exists an $x = x(\nu, k)$ in the support of φ for which

$$(2.6) \qquad P_x\left\{|\lim_{n\to\infty} \Lambda_n(s,\nu)| \leqq \frac{1}{k}\right\} \geqq 1 - \frac{1}{k} \qquad \text{for all} \quad s \in \mathbb{R}\,.$$

Then for this x, we have for all $s \in \mathbb{R}$

$$(2.7) \qquad \begin{aligned} &|G(x, s+\zeta_\nu) - G(x,s)| \\ &\quad = |E_x \Lambda_0(s,\nu)| = |E_x\{\lim_{n\to\infty} \Lambda_n(s,\nu)\}| \\ &\quad \leqq \frac{1}{k} + 2\sup_{y,w} |G(y,w)| P_x\left\{|\lim_{n\to\infty} \Lambda_n(s,\nu)| > \frac{1}{k}\right\} \\ &\quad \leqq \frac{1}{k}(1 + 2\sup_{y,w}|G(y,w)|)\,. \end{aligned}$$

Moreover, since $x \in \operatorname{supp}(\varphi)$

$$(2.8) \qquad \varphi\left\{z \in S : d(z,x) < \frac{1}{m}\right\} > 0 \qquad \text{for all} \quad m \geqq 1\,.$$

(See [18] Theorem II.2.1). Thus, if we put

$$(2.9) \qquad W_m = W_m(x) = \left\{z \in S : d(z,x) < \frac{1}{m}\right\},$$

and

$$(2.10) \qquad T_m = T_m(x) = \min\{n \geqq 0 : X_n \in W_m(x)\}\,,$$

then

$$(2.11) \qquad P_z\{X_{T_m} \in W_m\} = P_z\{T_m < \infty\} = 1$$

for all $z \in S$ (by virtue of (2.8) and I.1). The optional sampling theorem (see [7] Theorem VII.2.2 or [14] Theorem V.28) now implies for all $(z, s) \in S \times \mathbb{R}$

$$G(z,s) = E_z \lambda_0(s) = E_z \lambda_{T_m}(s) = E_z G(X_{T_m}, s - V_{T_m})\,,$$

and, hence by (2.11)

$$|G(z,s) - E_z G(x, s - V_{T_m})| \leqq \sup_{d(x',x)<m^{-1}} \sup_t |G(x',t) - G(x,t)|\,.$$

As $m \to \infty$ the right-hand side of this inequality tends to zero so that

$$(2.12) \qquad G(z,s) = \lim_{m\to\infty} E_z G(x, s - V_{T_m})\,.$$

From (2.7) we then obtain

$$\begin{aligned} |G(z, s+\zeta_\nu) - G(z,s)| &\leqq \lim_{m\to\infty} E_z |G(x, s+\zeta_\nu - V_{T_m}) - G(x, s - V_{T_m})| \\ &\leqq \frac{1}{k}(1 + 2\sup_{y,w}|G(y,w)|)\,. \end{aligned}$$

Since we assumed (2.6) for all $k \geqq 1$ and $\nu \geqq 1$ this would imply

$$G(z, s + \zeta_\nu) = G(z, s), \qquad z \in S,\ s \in \mathbb{R},\ \nu \geqq 1,$$

and then also

$$G(z, s + \Sigma) = G(z, s)$$

whenever Σ is a finite sum of the form

$$\Sigma r_\nu \zeta_\nu, \qquad r_\nu \ \text{integral.}$$

But these sums are dense in $\mathbb{R}$, so that we would obtain with the help of (2.5) that $G(z, s)$ is independent of s, for each z. In particular, the right-hand side of (2.12) would be independent of s, so that $G(z, s)$ would be independent of z and s (by (2.12)). The proof has therefore been reduced to finding $x(\nu, k) \in \operatorname{supp}(\varphi)$ satisfying (2.6), for each $\nu, k \geqq 1$.

We now assume that for some $\nu_0, k_0 \geqq 1$ there does not exist an $x(\nu, k) \in \operatorname{supp}(\varphi)$ satisfying (2.6) and derive a contradiction from this. Fix $\delta > 0$ such that

$$\text{(2.13)} \qquad \sup_{z \in S} \sup_{|t'-t''|<\delta} |G(z, t') - G(z, t'')| \leqq (17k_0)^{-1}.$$

Such a δ exists by (2.5). Next, fix $y = y(\nu_0, \delta)$ as in I.3 and $\varepsilon > 0$ such that

$$\text{(2.14)} \qquad \sup_{d(z,y)<\varepsilon} \sup_{|t'-t''|<\delta} |G(z, t') - G(y, t'')| \leqq 2(17k_0)^{-1}.$$

Such an ε exists by (2.2) and (2.13). We also choose A with $\varphi(A) > 0$, m_1, m_2 and τ such that (1.14) and (1.15) hold for all $x \in A$. We can find $\eta_0 > 0$ such that

$$A_0 \equiv \{x \in A : \text{left-hand sides of (1.14) and (1.15) are} \geqq \eta_0\}$$

satisfies $\varphi(A_0) > 0$. Finally we define the measure ξ on $\mathscr{F}$ by

$$\xi(C) = \int \varphi(dx) P_x(C), \qquad C \in \mathscr{F}.$$

The measure ξ is invariant under the shift on Ω by virtue of (1.1) and $\varphi P = \varphi$ (see (I.1)). Thus, by the ergodic theorem ([11] page 18)

$$\text{(2.15)} \qquad L = \lim_{n\to\infty} \frac{1}{n+1} \Sigma_0^n I_{A_0}(X_j) \qquad \text{exists a.e. } [\xi]$$

and

$$\int \varphi(dx) \int L\, dP_x = \int L\, d\xi = \int I_{A_0}(X_0)\, d\xi = \varphi(A_0) > 0.$$

In particular

$$\xi\{L > 0\} > 0.$$

By the martingale convergence theorem ([7] Corollary 1 to Theorem VII.4.3 or [2] Corollary 5.22)

$$P_{X_n}\{L > 0\} = \xi\{L > 0 \mid \mathscr{F}_n\} \to 1 \qquad (n \to \infty)$$

a.e. $[\xi]$ on the set $\{L > 0\}$. Thus, for some large n

$$\text{(2.16)} \qquad \xi\left\{P_{X_n}\{L > 0\} \geqq 1 - \frac{1}{2k_0}\right\} > 0.$$

But the distribution of X_n under ξ is just φ, and any sample path with $L > 0$

visits A_0 infinitely often so that (2.16) equals[8]

$$\varphi\left\{x \in S: P_x\{L > 0\} \geqq 1 - \frac{1}{2k_0}\right\} \leqq \varphi\left\{x \in S: P_x\{X_n \in A \text{ i.o.}\} \geqq 1 - \frac{1}{2k_0}\right\}. \tag{2.17}$$

(2.16) and (2.17) show that there exists an $x_0 \in \operatorname{supp}(\varphi)$ for which

$$P_{x_0}\{X_n \in A_0 \text{ i.o}\} \geqq 1 - \frac{1}{2k_0}. \tag{2.18}$$

By assumption, when $(\nu, k) = (\nu_0, k_0)$ then (2.6) fails for all $x \in \operatorname{supp}(\varphi)$ and in particular for x_0, so that there must be an $s_0 \in \mathbb{R}$ for which

$$P_{x_0}\left\{|\lim_{n\to\infty} \Lambda_n(s_0, \nu_0)| > \frac{1}{k_0}\right\} > \frac{1}{k_0}. \tag{2.19}$$

(2.18) and (2.19) together show that

$$P_{x_0}\left\{X_n \in A_0 \text{ i.o. and } |\lim_{n\to\infty} \Lambda_n(s_0, \nu_0)| > \frac{1}{k_0}\right\} \geqq \frac{1}{2k_0}.$$

Since each of $\lambda_n(s_0 + \zeta_{\nu_0})$ and $\lambda_n(s_0)$ converge a.e. $[P_{x_0}]$ there must exist two closed intervals J_1 and J_2, at least a distance $(4k_0)^{-1}$ apart, and an m_0 for which

$$P_{x_0}\{X_n \in A_0 \text{ i.o., } G(X_r, s_0 + \zeta_{\nu_0} - V_r) \in J_1 \text{ and } G(X_r, s_0 - V_r) \in J_2 \text{ for all } r \geqq m_0\} > 0. \tag{2.20}$$

Again by the martingale convergence theorem we have for $n \to \infty$, $n \geqq m_0$

$$\begin{aligned} &P_x\{G(X_r, s_0 + \zeta_{\nu_0} - v - V_r) \in J_1 \text{ and } G(X_r, s_0 - v - V_r) \in J_2 \text{ for all } r \geqq 0\} \\ &\quad \text{evaluated at } (x, v) = (X_n, V_n) \\ &\geqq P_{x_0}\{G(X_r, s_0 + \zeta_{\nu_0} - V_r) \in J_1 \text{ and } G(X_r, s_0 - V_r) \in J_2 \text{ for all } r \geqq m_0 \mid \mathscr{F}_n\} \to 1 \end{aligned} \tag{2.21}$$

a.e. $[P_{x_0}]$ on the set

$$\{G(X_r, s_0 + \zeta_{\nu_0} - V_r) \in J_1 \text{ and } G(X_r, s_0 - V_r) \in J_2 \text{ for all } r \geqq m_0\}.$$

(2.20) and (2.21) finally show that there exist a $z_0 \in A_0$ and $s_1 = s_0 - v$, for some v in the support of the distribution of some V_n for which

$$\begin{aligned} &P_{z_0}\{G(X_r, s_1 + \zeta_{\nu_0} - V_r \in J_1 \text{ for all } r \geqq 0\} \geqq 1 - \tfrac{1}{2}\eta_0, \\ &P_{z_0}\{G(X_r, s_1 - V_r) \in J_2 \text{ for all } r \geqq 0\} \geqq 1 - \tfrac{1}{2}\eta_0. \end{aligned} \tag{2.22}$$

This, however, is impossible as the following argument shows. Since $z_0 \in A_0$

$$P_{z_0}\{d(X_{m_2}, y) < \varepsilon, |V_{m_2} - \tau - \zeta_{\nu_0}| \leqq \delta\} \geqq \eta_0,$$

[8] i.o. stands for "infinitely often".

which together with the first inequality of (2.22) implies

$$(2.23)\qquad P_{z_0}\{G(X_{m_2}, s_1 + \zeta_{\nu_0} - V_{m_2}) \in J_1,\ d(X_{m_2}, y) < \varepsilon, |V_{m_2} - \tau - \zeta_{\nu_0}| \leqq \delta\} \geqq \tfrac{1}{2}\eta_0 > 0 .$$

In turn (2.23) implies

$$G(x, s_1 - w) \in J_1 \quad \text{for some } x \text{ with } d(x, y) < \varepsilon \text{ and } |w - \tau| \leqq \delta ,$$

and then, by (2.14)

$$(2.24)\qquad \min\{|r - G(y, s_1 - \tau)| : r \in J_1\} \leqq \frac{2}{17k_0} .$$

In the same way the second inequality of (2.22) together with

$$P_{z_0}\{d(X_{m_1}, y) < \varepsilon, |V_{m_1} - \tau| \leqq \delta\} \geqq \eta_0$$

implies

$$(2.25)\qquad \min\{|r - G(y, s_1 - \tau)| : r \in J_2\} \leqq \frac{2}{17k_0} ,$$

and (2.24) together with (2.25) contradict the fact that J_1 and J_2 are at least a distance $(4k_0)^{-1}$ apart. □

COROLLARY. *Assume that Conditions* I.1–I.3 *are satisfied and that H satisfies the conditions of Lemma* 1. *If there exists a measure ρ on $\mathscr{S} \times \mathscr{B}$ of total mass*

$$(2.26)\qquad 0 < \rho_0 < \infty ,$$

and such that

$$(2.27)\qquad \lim_{t\to\infty} \int_{S\times\mathbb{R}} H(z, t - w)\rho(dz, dw) = \beta ,$$

then

$$(2.28)\qquad \lim_{t\to\infty} H(x, t) = \frac{\beta}{\rho_0} \qquad \textit{for all } x \in S .$$

PROOF. Let x_0 and $t_k \to \infty$ be such that

$$(2.29)\qquad \beta^* \equiv \lim_{k\to\infty} H(x_0, t_k) \quad \text{exists.}$$

By Lemma 1 we then have along some subsequence of the t_k

$$H(z, t_k - w) \to \beta^* , \qquad (z, w) \in S \times \mathbb{R}$$

and, hence by the dominated convergence theorem

$$\int_{S\times\mathbb{R}} H(z, t_k - w)\rho(dz, dw) \to \rho_0\beta^* .$$

Thus $\beta^* = \rho_0^{-1}\beta$ for any x_0 and t_k satisfying (2.29). □

3. The renewal theorem. In this Section we prove Theorem 1 and 2, by applying the Corollary to Lemma 1 to suitable functions H. V_n, N, W and Z are still as in (1.2), (1.7)—(1.9) and, as in (1.13), φ is an invariant measure for the underlying Markov chain $\{X_n\}$. $W(t)$ is clearly the excess of the first jump of $V_\bullet$ across t, and $Z(t)$ is the state of the underlying chain $\{X_n\}$ immediately after the jump which takes $V_{\bullet}$ across t. $W(t)$ is often called the residual waiting time (compare [9] page 188). We shall also need the ladder indices ν_i for the sequence $\{V_n\}_{n\geq 0}$. These are defined by

$$(3.1)\qquad \begin{aligned} &\nu_0 = 0\,, \\ &\nu_{i+1} = \min\{n > \nu_i : V_n > V_{\nu_i}\} \quad (= \infty \text{ if no such } n \text{ exists})\,. \end{aligned}$$

It is easily seen that $\nu_1 = N(0)$ and that $N(t)$ must be one of the ν_i. Moreover, $\{X_{\nu_i}\}_{i\geq 0}$ is a Markov chain under any of the measures P_x. For fixed k and B the conditional probability

$$(3.2)\qquad \begin{aligned} P_x\{\nu_{i+1} - \nu_i = k,\, V_{\nu_{i+1}} - V_{\nu_i} \in B \mid \nu_0, \nu_1, \cdots, \nu_i, \\ u_j, j < \nu_i, u_{\nu_i+k+l}, l \geq 0, X_{\nu_k}, k \geq 0\} \end{aligned}$$

can be taken as a function of the values of X_{ν_i} and $X_{\nu_{i+1}}$ only.

Merely as a tool for proving the first lemma we also have to introduce a two sided process $\{X_n^{\#}, u_n^{\#}\}_{-\infty<n<\infty} = \{X_n^{\#}(\omega^{\#}), u_n^{\#}(\omega^{\#})\}_{-\infty<n<\infty}$. We define this as the process of coordinate functions on the probability space $\{\Omega^{\#}, \mathscr{F}^{\#}, P^{\#}\}$, where $\Omega^{\#} = \prod_{n=-\infty}^{+\infty} (S \times \mathbb{R})$, $\mathscr{F}^{\#} = \prod_{n=-\infty}^{+\infty} (\mathscr{S} \times \mathscr{B})$. If $\omega^{\#} = \{\omega_n^{\#}(1), \omega_n^{\#}(2)\}_{-\infty<n<\infty}$, then $X_n^{\#}(\omega^{\#}) = \omega_n^{\#}(1)$, $u_n^{\#}(\omega^{\#}) = \omega_n^{\#}(2)$ and $P^{\#}$ is determined by

$$(3.3)\qquad \begin{aligned} &P^{\#}\{X_{k+i}^{\#} \in A, 0 \leq i \leq n, u_{k+i}^{\#} \in B_i, 0 \leq i < n\} \\ &\quad = \int_{A_0} \varphi(dx_0) \int_{A_1} P(x_0, dx_1) \cdots \int_{A_n} P(x_{n-1}, dx_n) \\ &\qquad \times \int_{B_0} F(d\lambda_0 \mid x_0, x_1) \int_{B_1} F(d\lambda_1 \mid x_1, x_2) \cdots \int_{B_{n-1}} F(d\lambda_{n-1} \mid x_{n-1}, x_n)\,, \end{aligned}$$

for $A_i \in \mathscr{S}$, $B_i \in \mathscr{B}$ and any integer k. The above construction of a two sided process is of course standard (see [2] Section 6.1 or [7] page 456) as are the following facts: $\{X_n^{\#}, u_n^{\#}\}_{-\infty<n<\infty}$ is a stationary Markov chain,

$$(3.4)\qquad \begin{aligned} &P^{\#}\{X_{k+i}^{\#} \in A_i, 0 \leq i \leq n, u_{k+i}^{\#} \in B_i, 0 \leq i < n\} \\ &\quad = \int \varphi(dx) P_x\{X_i \in A_i, 0 \leq i \leq n, u_i \in B_i, 0 \leq i < n\}\,, \end{aligned}$$

and in particular

$$(3.5)\qquad P^{\#}\{X_k \in A\} = \varphi(A)\,.$$

Finally for any set $C \in \prod_{n=0}^{\infty} (\mathscr{S} \times \mathscr{B})$

$$(3.6)\qquad \begin{aligned} &P^{\#}\{\{X_{n+k}^{\#}, u_{n+k}^{\#}\}_{n\geq 0} \in C \mid X_i^{\#}, i \leq k, u_j, j < k\} \\ &\quad = P_x\{\{X_n, u_n\}_{n\geq 0} \in C\} \qquad \text{a.e. on } \{X_k^{\#} = x\}\,. \end{aligned}$$

In analogy with (1.2) and (3.1) we now define

$$(3.7)\qquad \begin{aligned} V_n^{\#} &= \textstyle\sum_{i=0}^{n-1} u_i^{\#} && \text{if } n > 0\,, \\ &= 0 && \text{if } n = 0\,, \\ &= -\textstyle\sum_{i=n}^{-1} u_i^{\#} && \text{if } n < 0\,, \end{aligned}$$

and ladder indices for the sequence $V_n^{\#}$:

$$(3.8)\qquad \begin{aligned} &\nu_0^{\#} = \max\{n \leq 0 : V_n^{\#} > \sup_{j<n} V_j^{\#}\} \quad (= -\infty \text{ if no such } n \text{ exists})\,, \\ &\nu_{i+1}^{\#} = \min\{n > \nu_i^{\#} : V_n^{\#} > V_{\nu_i}^{\#}\} \quad (= \infty \text{ if no such } n \text{ exists})\,. \end{aligned}$$

Note that for all $n_1 < n_2$

$$\sum_{n_1 \leqq i < n_2} u_i^{\#} = V_{n_2}^{\#} - V_{n_1}^{\#} . \tag{3.9}$$

Also $\nu_0^{\#}$ is the index of the last strict maximum of $V_n^{\#}$, $n \leqq 0$, and $\nu_i^{\#}$ for $i > 0$ is defined w.r.t. $\nu_0^{\#}$ in the same way as ν_i w.r.t. ν_0. Lastly we define the measure ψ on $\mathscr{S}$ by

$$\psi(A) = P^{\#}\{\nu_0^{\#} = 0, X_0^{\#} \in A\} = P^{\#}\{\sup_{n<0} V_n^{\#} < 0, X_0^{\#} \in A\} , \tag{3.10}$$

for $A \in \mathscr{S}$.

LEMMA 2. *If Conditions* I *hold, then*

$$P_x\{\nu_i < \infty\} = 1 , \qquad x \in S, i \geqq 0 , \tag{3.11}$$

$$P^{\#}\{-\infty < \nu_0^{\#} < \nu_1^{\#} < \cdots < \nu_i^{\#} < \infty\} = 1 , \qquad i \geqq 0 , \tag{3.12}$$

$$q \equiv P^{\#}\{\nu_0^{\#} = 0\} > 0 . \tag{3.13}$$

ψ is an invariant measure for the Markov chain $X_{\nu_0} = X_0, X_{\nu_1}, X_{\nu_2}, \cdots$, i.e.,

$$\int \psi(dy) P_y\{X_{\nu_1} \in A\} = \psi(A) , \qquad A \in \mathscr{S}. \tag{3.14}$$

Lastly

$$\int \psi(dy) E_y\{\nu_1\} = 1 \tag{3.15}$$

and

$$\int \psi(dy) E_y V_{\nu_1} = \alpha . \tag{3.16}$$

REMARK 3. Let T be the shift on $\Omega^{\#}$, i.e., $(T\omega^{\#})_n(j) = \omega_{n+1}^{\#}(j)$, and let

$$\Omega_0^{\#} = \{\omega^{\#} : \nu_0^{\#} = 0\} .$$

Then (3.14)—(3.16) are equivalent to statements about the expected length and distribution of the excursions between successive visits to $\Omega_0^{\#}$ by the stationary process $\{T^n\omega^{\#}\}_{-\infty<n<\infty}$. In this context these results are known and usually dealt with under the "Poincaré recurrence theorem." Much of the proof below is a transcription of [26] and [2], Section 6.9.

PROOF. By I.2

$$P_x\left\{\frac{V_n}{n} \to \alpha > 0\right\} = 1 , \qquad x \in S , \tag{3.17}$$

and hence (see (3.4)) also

$$P^{\#}\left\{\frac{V_n^{\#}}{n} \to \alpha > 0\right\} = 1 . \tag{3.18}$$

Clearly $\nu_i < \infty$ on $\{V_n \to \infty\}$ so that (3.11) follows from (3.17). Similarly (3.12) will follow from (3.18) once we prove

$$P^{\#}\{\nu_0^{\#} > -\infty\} = 1 . \tag{3.19}$$

However, from the stationarity of $\{X_n^{\#}, u_n^{\#}\}$ and Birkhoff's ergodic theorem it follows that

$$\lim_{n\to\infty} \frac{V^{\#}_{-n}}{n} = \lim_{n\to\infty} -\frac{1}{n} \sum_{i=1}^{n} u^{\#}_{-i}$$

exists a.e. $[P^{\#}]$. Moreover, the distribution of $V^{\#}_{-n}$ is the same as that of $-V^{\#}_{n-1}$, so that, by (3.18)

$$P^{\#}\left\{\lim_{n\to\infty} \frac{V^{\#}_{-n}}{n} = -\alpha\right\} = 1 .$$

In particular $V^{\#}_{-n} \to -\infty$ and there exists a maximal $0 \leqq n < \infty$ such that $V^{\#}_{-n} = \sup_{j\leqq 0} V_j^{\#}$. $-\nu_0^{\#}$ equals this index n, so that (3.19) indeed holds. (3.13) is also immediate from (3.19) because

$$\begin{aligned} 1 &= P^{\#}\{\nu_0^{\#} > -\infty\} = \textstyle\sum_{n=0}^{\infty} P^{\#}\{\nu_0^{\#} = -n\} \leqq \sum_{n=0}^{\infty} P^{\#}\{V^{\#}_{-n} > \sup_{j<-n} V_j^{\#}\} \\ &= \textstyle\sum_{n=0}^{\infty} P^{\#}\{V_0^{\#} > \sup_{j<0} V_j^{\#}\} = \sum_{n=0}^{\infty} q . \end{aligned}$$

Because the process $\{X_n^{\#}, u_n^{\#}\}_{-\infty<n<\infty}$ is invariant under the shift the left-hand side of (3.14) equals

$$\begin{aligned} &\textstyle\int P^{\#}\{\sup_{n<0} V_n^{\#} < 0,\ X_0^{\#} \in dy\} P_y\{X_{\nu_1} \in A\} \\ &\quad = P^{\#}\{\sup_{n<0} V_n^{\#} < 0,\ X^{\#}_{\nu_1^{\#}} \in A\} \qquad \text{(see (3.6))} \\ (3.20) &\quad = \textstyle\sum_{k=1}^{\infty} P^{\#}\{\sup_{n<0} V_n^{\#} < 0,\ X_k^{\#} \in A,\ V_k^{\#} > 0, \text{ but } \\ &\qquad\qquad V_l^{\#} \leqq 0 \text{ for } 0 < l < k\} \\ &\quad = \textstyle\sum_{k=1}^{\infty} P^{\#}\{\sup_{j<-k} (V_j^{\#} - V^{\#}_{-k}) < 0,\ X_0^{\#} \in A,\ -V^{\#}_{-k} > 0, \text{ but } \\ &\qquad\qquad V^{\#}_{l-k} - V^{\#}_{-k} \leqq 0 \text{ for } 0 < l < k\} . \end{aligned}$$

The kth event in the last member of (3.20) is just the event where $X_0^{\#} \in A$ and the smallest index at which $\sup_{j<0} V_j^{\#}$ is taken on is $-k$ and

$$\sup_{j<0} V_j^{\#} \leqq \max\,(\sup_{j<-k} V_j^{\#},\ V^{\#}_{-k},\ \max_{0<l<k} V^{\#}_{l-k}) = V^{\#}_{-k} < 0 .$$

Thus these events are disjoint and their union is

$$\{X_0^{\#} \in A,\ \sup_{j<0} V_j^{\#} < 0,\ \exists\, k \geqq 1 \text{ with } \sup_{j<-k} V_j^{\#} < V^{\#}_{-k}\} .$$

But

$$P^{\#}\{\exists\, k \geqq 1 \text{ with } \sup_{j<-k} V_j^{\#} < V^{\#}_{-k}\} = 1$$

for the same reason as (3.19), so that the last member of (3.20) is precisely

$$P^{\#}\{X_0^{\#} \in A,\ \sup_{j<0} V_j^{\#} < 0\} = \psi(A) .$$

This proves (3.14) and the proof of (3.15) is quite similar (compare also [26] and [2] Proposition 6.38):

$$\begin{aligned} \textstyle\int \psi(dy) E_y\{\nu_1\} &= \textstyle\sum_{k=1}^{\infty} k \int P^{\#}\{\sup_{j<0} V_j^{\#} < 0,\ X_0^{\#} \in dy\} P_y\{\nu_1 = k\} \\ &= \textstyle\sum_{k=1}^{\infty} k P^{\#}\{\sup_{j<0} V_j^{\#} < 0,\ V_l^{\#} \leqq 0,\ 0 < l < k,\ V_k^{\#} > 0\} \\ &= \textstyle\sum_{k=1}^{\infty} \sum_{r=0}^{k-1} P^{\#}\{\sup_{j<-r} \{V_j^{\#} - V^{\#}_{-r}\} < 0,\ V^{\#}_{k-r} - V^{\#}_{-r} > 0, \text{ but } \\ &\qquad V^{\#}_{l-r} - V^{\#}_{-r} \leqq 0 \text{ for } 0 < l < k\} \\ &= \textstyle\sum_{r=0}^{\infty} \sum_{m=1}^{\infty} P^{\#}\{\nu_0^{\#} = -r,\ \nu_1^{\#} = m\} = 1 . \end{aligned}$$

To prove (3.16) we appeal once more to the ergodic theorem. Let $\mu(\cdot)$ be the probability measure on $(\Omega, \mathscr{F})$ defined by

$$\mu(C) = \frac{1}{q} \int \psi(dx) P_x(C) , \qquad C \in \mathscr{F},$$

and consider the following functions on $(\Omega, \mathscr{F})$:

$$L_j = (\nu_{j+1} - \nu_j, V_{\nu_{j+1}} - V_{\nu_j}, X_{\nu_{j+1}}) , \qquad j \geqq 0 . \tag{3.21}$$

From the fact that (3.2) can be taken as a function of X_{ν_i} and $X_{\nu_{i+1}}$ only, and from (3.14) it follows that $\{L_j\}_{j\geqq 0}$ is a stationary Markov chain on $\{\Omega, \mathscr{F}, \mu\}$, and thus by the ergodic theorem ([11] page 18)

$$\lim_{n\to\infty} \frac{\nu_n}{n} = \beta \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n} V_{\nu_n} = \gamma \tag{3.22}$$

exist a.e. $[\mu]$ and

$$\int \beta \, d\mu = \int (\nu_1 - \nu_0) \, d\mu = \frac{1}{q} \int \psi(dy) E_y \nu_1 = \frac{1}{q} , \tag{3.23}$$

$$\int \gamma \, d\mu = \frac{1}{q} \int \psi(dy) E_y V_{\nu_1} .$$

Note that we may draw these conclusions for V_{ν_n} even before we proved (3.16) because $V_{\nu_{i+1}} - V_{\nu_i} > 0$ by definition (see [2] page 116 for the proper truncation argument). By virtue of (3.17)

$$\lim_{n\to\infty} \frac{V_{\nu_n}}{\nu_n} = \lim_{n\to\infty} \frac{V_n}{n} = \alpha \qquad \text{a.e.} \quad [\mu] .$$

Together with (3.22) this implies $\gamma = \alpha\beta$. But then (3.16) follows from (3.15) and (3.23). □

In Lemma 2 we showed that the L_j, $j \geqq 0$, of (3.21) form a stationary Markov chain on $\{\Omega, \mathscr{F}, \mu\}$. We now use this to construct a stationary process $\{Z^*(t), W^*(t)\}_{t\geqq 0}$ in continuous time. The interpretation of $Z^*(t)$, $W^*(t)$ is the same as that of $Z(t)$, $W(t)$ (see beginning of this section) except that now we do not necessarily start the process off as if we just had a ladder index. We permit arbitrary initial values $(z, w) \in S \times (0, \infty)$ for $(Z^*(0), W^*(0))$. The initial state (z, w) corresponds to a situation where the process ran already for some time before our time zero, and at time zero the remaining excess of the $V_\bullet$ process is w, and this excess will decrease linearly till the time of the next ladder index, i.e.,

$$W^*(t) = w - t , \qquad 0 \leqq t < w .$$

The position of the underlying X process at the next ladder index is z, and this will remain so till the next ladder index is passed, i.e.,

$$Z^*(t) = z , \qquad 0 \leqq t < w .$$

From the time w on the starred process will develop as the original Z, W process with initial position of the underlying X chain equal to z and the time axis shifted by w. Formally, we define $\omega^* \to (Z^*(t, \omega^*), W^*(t, \omega^*))$ as the coordinate functions on the space Ω^* of right continuous functions from $[0, \infty)$ into $S \times (0, \infty)$, $\mathscr{F}^*$ is the σ-field in Ω^* generated by all the coordinate functions $\omega^* \to (Z^*(t, \omega^*), W^*(t, \omega^*))$, $t \geqq 0$. The measure on $\mathscr{F}^*$ corresponding to paths starting at (z, w) is defined by

$$\begin{aligned} Q_{z,w}\{Z^*(t) = z \text{ and } W^*(t) = w - t \text{ for } 0 \leqq t < w \text{ and} \\ (Z^*(w + t_i), W^*(w + t_i)) \in E_i, 0 \leqq i \leqq n\} \\ = P_z\{(Z(t_i), W(t_i)) \in E_i, 0 \leqq i \leqq n\} \end{aligned} \tag{3.24}$$

for $t_i \geqq 0$, $E_i \in \mathscr{B} \times \mathscr{S}$, $0 \leqq i \leqq n$. In particular

$$Q_{z,w}\{Z^*(t) = z \text{ and } W^*(t) = w - t \text{ for } 0 \leqq t < w\} = 1\,. \tag{3.25}$$

The next two lemmas give an invariant measure for the (Z^*, W^*) process. This measure in question is obtained heuristically by assigning to the set $\{Z^*(0) \in A, W^*(0) \leqq s\}$ a value proportional to[9]

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \int \phi(dx) E_x\{\min(V_{\nu_{i+1}} - V_{\nu_i}, s); X_{\nu_{i+1}} \in A\}\,.$$

This should correspond to the expectation of

$$\lim_{T\to\infty} \frac{1}{T} \text{ Lebesgue measure of } \{t \in [0, T] : Z^*(t) \in A, W^*(t) \leqq s\}\,.$$

For completeness sake we verify in Lemmas 3 and 4 that this guess yields the correct answer, but the reader should notice that this distribution has been derived in several places when S is countable. For uncountable S it is given in [16] without proof and in [12] with a proof using Harris' recurrence of $\{X_n\}$ (see Theorem 2). Our proofs of Lemmas 3 and 4 are quite similar to [21], Theorem 3.1.1 and make use of the function

$$\begin{aligned} R(z, A, t) &\equiv E_z \#\{i \geqq 0 : X_{\nu_i} \in A, V_{\nu_i} \leqq t\} \\ &= \sum_{i=0}^{\infty} P_z\{X_{\nu_i} \in A, V_{\nu_i} \leqq t\}\,, \qquad z \in S\,, \quad A \in \mathscr{S}\,. \end{aligned} \tag{3.26}$$

Note that $R(z, A, t) = 0$ for $t < 0$ since $V_{\nu_i} \geqq 0$. Also, without restricting A we do not know a priori that $R(z, A, t) < \infty$. However, if we put

$$D_n = \left\{x \in S : P_x\left\{V_{\nu_1} \geqq \frac{1}{n}\right\} \geqq \frac{1}{n}\right\}\,,$$

then, every time X_{ν_i} takes a value in D_n, there is a conditional probability of at least n^{-1} of $V_{\nu_{i+1}} \geqq V_{\nu_i} + n^{-1}$. A simple adaptation of the argument in [24] therefore shows for any $A \subset D_n$

$$R(z, A, t) \leqq R(z, D_n, t) \leqq 1 + n(nt + 1)\,. \tag{3.27}$$

[9] For any set of conditions $\mathscr{C}$, $E_x\{f, \mathscr{C}\}$ denotes the integral of f w.r.t. P_x over the set where $\mathscr{C}$ is satisfied, i.e. $E_x\{f; \mathscr{C}\} = E_x\{fI_{\mathscr{C}}\}$.

LEMMA 3. *If Conditions* I *hold, then for any positive* $\mathscr{S} \times \mathscr{B}$ *measurable function* $f\colon S \times \mathbb{R} \to \mathbb{R}$ *and* $t \geqq 0$

$$\begin{aligned}(3.28)\qquad &\int_S \phi(dy) \int_{S\times(0,\infty)} P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \int_{0\leqq w\leqq\lambda} dw \\ &\quad\times \int_{x\in S,\tau\leqq t-w} R(z, dx, d\tau) f(x, t-w-\tau) \\ &= \int_S \phi(dy) \int_{0\leqq s\leqq t} f(y, s)\, ds\,.\end{aligned}$$

PROOF. We may assume that $0 \leqq f(x, s) \leqq n$ and $f(x, s) = 0$ for $x \notin D_n$ for some fixed n, so that all integrals below are finite. The general case can be reduced to this by first truncating $f(x, s)$ at n and multiplying it with $I_{D_n}(x)$ and then letting $n \to \infty$. We introduce

$$\begin{aligned} f^+(x, s) &= f(x, s) \qquad \text{if}\quad s \geqq 0\,. \\ &= 0 \qquad\qquad \text{if}\quad s < 0\,.\end{aligned}$$

With this notation we have by a familiar renewal argument

$$\begin{aligned}&\int_{x\in S,\tau\leqq t-w} R(z, dx, d\tau) f(x, t-w-\tau) \\ &\quad= E_z \textstyle\sum_{i=0}^{\infty} f^+(X_{\nu_i}, t-w-V_{\nu_i}) \\ &\quad= f^+(z, t-w) + \int P_z\{X_{\nu_1} \in dz',\ V_{\nu_1} \in d\tau\} \int_{S\times\mathbf{R}} R(z', dx, d\tau') f^+(x, t-w-\tau-\tau')\,,\end{aligned}$$

because

$$E_z\{f^+(X_{\nu_i}, t-w-V_{\nu_i}) \mid X_{\nu_1} = z',\ V_{\nu_1} = \tau\} = E_{z'} f^+(X_{\nu_{i-1}}, t-w-\tau-V_{\nu_{i-1}})\,, \qquad i \geqq 1\,.$$

We can therefore rewrite the left-hand side of (3.28) as

$$\begin{aligned}(3.29)\qquad &\int \phi(dy) \int P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \int_{0\leqq w\leqq t} dw\, [f^+(z, t-w) \\ &\quad+ \int P_z\{X_{\nu_1} \in dz',\ V_{\nu_1} \in d\tau\} \int R(z', dx, d\tau') f^+(x, t-w-\tau-\tau')] \\ &\quad- \int \phi(dy) \int P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \int_{\min(t,\lambda)<w\leqq t} dw \\ &\quad\times \int R(z, dx, d\tau) f^+(x, t-w-\tau)\,.\end{aligned}$$

In the first multiple integral of (3.29) we can integrate out λ and then use (3.14). This changes the first integral to

$$\begin{aligned}(3.30)\qquad &\int_S \phi(dz) \int_{0\leqq w\leqq t} dw\, f^+(z, t-w) + \int_S \phi(dz) \int P_z\{X_{\nu_1} \in dz',\ V_{\nu_1} \in d\tau\} \\ &\quad\times \int_{0\leqq w\leqq t} dw \int R(z', dx, d\tau') f^+(x, t-w-\tau-\tau')\,.\end{aligned}$$

The first term in (3.30) is just equal to the right-hand side of (3.28) and the second term can be seen to be the opposite of the second multiple integral in (3.29) (replace $w + \tau$ by w'; w' then runs from τ to $t + \tau$ but $f^+(x, t-w-\tau-\tau') = f^+(x, t-w'-\tau') = 0$ for $w' > t$ and $\tau' \geqq 0$, so that we may restrict w' to lie between τ and t or equivalently, between $\min(\tau, t)$ and t). □

LEMMA 4. *If Conditions* I *hold, then for any* $t \geqq 0$, $A \in \mathscr{S}$ *and Borel set* $B \subset (0, \infty)$ *we have*

$$\begin{aligned}(3.31)\qquad &\alpha^{-1} \int_S \phi(dy) \int_{S\times(0,\infty)} P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \\ &\quad\times \int_{0<w\leqq\lambda} dw\, Q_{z,w}\{Z^*(t) \in A,\ W^*(t) \in B\} \\ &= \alpha^{-1} \int_S \phi(dy) \int_{\lambda>0} P_y\{X_{\nu_1} \in A,\ V_{\nu_1} \in d\lambda\} \int_{w\in B,0\leqq w\leqq\lambda} dw\,.\end{aligned}$$

Consequently

$$\alpha^{-1} \int_S \psi(dy)\, dw \int_{\lambda \geq w} P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \tag{3.32}$$

defines an invariant probability distribution for the $\{Z^*, W^*\}$ process.

PROOF. To start with, we note that (3.32) defines an honest probability measure, i.e., its integral over $z \in S$, $w \in (0, \infty)$ equals one (by virtue of (3.16)). Now the left-hand side of (3.31) is just the probability of

$$\{Z^*(t) \in A,\ W^*(t) \in B\} \tag{3.33}$$

when the initial state $(Z^*(0), W^*(0))$ is chosen according to the distribution (3.32). Also, if $Z^*(0) = z$, $W^*(0) = w$ with $w > t$, then

$$Z^*(t) = z\,, \qquad W^*(t) = w - t\,,$$

so that in this case (3.33) occurs only if $z \in A$, $w - t \in B$. If, however, $w \leqq t$, then the $Q_{z,w}$ measure of (3.33) equals

$$P_z\{Z(t-w) \in A,\ W(t-w) \in B\} \tag{3.34}$$

(see (3.24)). Now $W(t)$ is defined in terms of $N(t)$ and we already pointed out (just below (3.1)) that for $t \geqq 0$

$$N(t) = \nu_{i(t)}$$

for some $i(t)$. Necessarily $i(t) > 0$ because $V_{\nu_0} = V_0 = 0$. We decompose the event between braces in (3.34) into subevents according to the value of $i(t-w) - 1$ and $V_{\nu_{i(t-w)-1}}$. This leads to the following evaluation of (3.34):[10]

$$\begin{aligned}\sum_{j=0}^{\infty} \int_{x \in S, 0 \leqq \tau \leqq t-w} P_z\{X_{\nu_j} \in dx,\ V_{\nu_j} \in d\tau\} P_x\{X_{\nu_1} \in A,\ V_{\nu_1} + \tau - (t-w) \in B\} \\ = \int_{x \in S, 0 \leqq \tau \leqq t-w} R(z, dx, d\tau) P_x\{X_{\nu_1} \in A,\ V_{\nu_1} + \tau - (t-w) \in B\}\,.\end{aligned}$$

Consequently, the left-hand side of (3.31) equals

$$\begin{aligned}\alpha^{-1} \int_S \psi(dy) \int_{z \in A, \lambda > 0} P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \int_{t < w \leqq \lambda, w - t \in B} dw \\ + \alpha^{-1} \int_S \psi(dy) \int_{S \times (0,\infty)} P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \int_{0 < w \leqq \min(t,\lambda)} dw \\ \times \int_{x \in S, 0 \leqq \tau \leqq t-w} R(z, dx, d\tau) P_x\{X_{\nu_1} \in A,\ V_{\nu_1} + \tau - (t-w) \in B\}\,.\end{aligned} \tag{3.35}$$

In the second term of (3.35) we may extend the integral over w from 0 to λ, because there are no τ with $0 \leqq \tau \leqq t - w$ for $w > t$. Thus, by Lemma 3 this term equals

$$\begin{aligned}\alpha^{-1} \int \psi(dy) \int_{0 \leqq s \leqq t} P_y\{X_{\nu_1} \in A,\ V_{\nu_1} - s \in B\}\, ds \\ = \alpha^{-1} \int \psi(dy) \int_{\lambda > 0} P_y\{X_{\nu_1} \in A,\ V_{\nu_1} \in d\lambda\} \int_{0 \leqq s \leqq t, \lambda - s \in B} ds\,.\end{aligned} \tag{3.36}$$

Clearly

$$\int_{t < w \leqq \lambda, w - t \in B} dw + \int_{0 \leqq s \leqq t, \lambda - s \in B} ds = \int_{w' \in B, 0 \leqq w' \leqq \lambda} dw'\,,$$

so that (3.36) and the first term in (3.35) add up to right hand side of (3.31). □

[10] Note that $B \subset (0, \infty)$ so that $V_{\nu_1} + \tau - (t - w) \in B$ implies $V_{\nu_1} + \tau > t - w$.

LEMMA 5. *Assume Conditions* I *hold and let* $f: S \times \mathbb{R} \to \mathbb{R}$ *be* uniformly *continuous and bounded. Let* H_0 *and* H *be defined by*

$$(3.37)\qquad H_0(x, t) = E_x f(Z(t), W(t)) \quad \text{if } t \geqq 0$$
$$= 0 \quad \text{if } t < 0\,,$$

respectively

$$(3.38)\qquad H(x, t) = \int H_0(x, t + s) g(s)\, ds\,,$$

where $g(\cdot)$ *is some continuous function with compact support. Then* $H(\cdot, \cdot)$ *satisfies* (2.2).

PROOF. As a first step we show that for fixed x and $\varepsilon > 0$ there exists an $N = N(x, \varepsilon)$ such that

$$(3.39)\qquad P_x\{\#\{n: V_n \in [t-2, t+2]\} \geqq N\} \leqq \varepsilon \qquad \text{for all } \ t \geqq 0\,.$$

Let $x_0 \in \operatorname{supp}(\varphi)$ so that (2.8) holds with x_0 for x, and let $\varepsilon > 0$ be given. Take $r_0 = r_0(x_0, \varepsilon)$ as in I.4 and define

$$f(N, k, t, L, y_0, v_0, y_1, v_1, \cdots) = 1 \qquad \text{if } \ \#\{n \leqq k: t - L \leqq v_n \leqq t + L\} \geqq N\,,$$
$$= 0 \qquad \text{otherwise.}$$

Then for $\varepsilon > 0$ $f^\varepsilon(N, k, t, L, y_0, v_0, y_1, v_1, \cdots) = 0$ whenever

$$\#\{n \leqq k: t - L - 1 < v_n < t + L + 1\} < N\,. \qquad \text{(See Def. 2.)}$$

Consequently, by I.4 one has for $d(y, x_0) < r_0$

$$\begin{aligned}(3.40)\qquad & P_y\{\#\{n: t - L - 1 < V_n < t + L + 1\} \geqq N\} \\ &\geqq \lim_{k\to\infty} E_y f^\varepsilon(N, k, t, L, X_0, V_0, X_1, V_1, \cdots) \\ &\geqq -\varepsilon + \lim_{k\to\infty} E_{x_0} f(N, k, t, L, X_0, V_0, X_1, V_1, \cdots) \\ &= -\varepsilon + P_{x_0}\{\#\{n: t - L \leqq V_n \leqq t + L\} \geqq N\}\,.\end{aligned}$$

The inequality (3.40) holds for all t, N and L simultaneously. Now put

$$(3.41)\qquad T = \min\{n: d(X_n, x_0) < r_0\} \quad (= \infty \ \text{ if no such } \ n \ \text{ exists})\,.$$

We shall prove the following string of relations for $t - w - L - 1 \geqq 0$:

$$\begin{aligned}& \int_{S\times(0,\infty)} Q_{z,w}\{Z^*(t-L-1) \in dz', W^*(t-L-1) \in d\tau\} \\ &\quad \times P_{z'}\{\#\{n: V_n \in (-\tau, 2L+2-\tau)\} \geqq N\} \\ &\quad = \int_{S\times(0,\infty)} P_z\{Z(t-w-L-1) \in dz', W(t-w-L-1) \in d\tau\} \\ &\qquad \times P_{z'}\{\#\{n: V_n \in (-\tau, 2L+2-\tau)\} \geqq N\} \\ (3.42)\qquad &\quad = P_z\{\#\{n: V_n \in (t-w-L-1, t-w+L+1)\} \geqq N\} \\ &\quad \geqq \int_{S\times\mathbb{R}} P_z\{T < \infty, X_T \in dy, V_T \in ds\} \\ &\qquad \times P_y\{\#\{n: V_n \in (t-w-s-L-1, t-w-s+L+1)\} \geqq N\} \\ &\quad \geqq -\varepsilon + \int_{\mathbb{R}} P_z\{T < \infty, V_T \in ds\} \\ &\qquad \times P_{x_0}\{\#\{n: V_n \in [t-w-s-L, t-w-s+L]\} \geqq N\}\,.\end{aligned}$$

The first equality in (3.42) is immediate from (3.24), whereas the second equality is just a decomposition w.r.t. the first entry by V_n into $(t-w-L-1,\infty)$; no V_n with $n < N(t-w-L-1)$ can lie in $(t-w-L-1, t-w+L+1)$. The first inequality in (3.42) simply comes from ignoring all V_n with $n < T$ and the last inequality is immediate from (3.40) and $d(X_T, x_0) < r_0$ on $\{T < \infty\}$.

We now integrate the first and last member of (3.42) w.r.t. the distribution (3.32), with (z, w) ranging over $S \times (0,\infty)$. Since the distribution in (3.32) is a stationary one for the (Z^*, W^*) process, the integral of the first member equals for $t - L - 1 \geqq 0$

$$(3.43) \qquad \alpha^{-1} \int_S \psi(dy) \int_{S\times(0,\infty)} P_y\{X_{\nu_1} \in dz, V_{\nu_1} \in d\lambda\} \int_{0<w\leqq\lambda} dw \\ \times P_z\{\#\{n: V_n \in (-w, 2L+2-w)\} \geqq N\}.$$

On the other hand the integral of the last member of (3.42) w.r.t. the distribution (3.32) is of the form

$$-\varepsilon + \int M(d\beta) P_{x_0}\{\#\{n: V_n \in [t-\beta-L, t-\beta+L]\} \geqq N\}$$

for some probability measure M on $\mathbb{R}$[11]. It is important to notice that this measure depends only on the distribution of V_T. M therefore may depend on ε, but not on t, L and N. Thus we can choose $L = L(\varepsilon)$ such that

$$\int_{|\beta|\leqq L-3} M(d\beta) \geqq \tfrac{1}{2}.$$

From the inequality (3.42) we then finally obtain

$$(3.44) \qquad \begin{aligned} &-\varepsilon + \tfrac{1}{2} P_{x_0}\{\#\{n: V_n \in [t-3, t+3]\} \geqq N\} \\ &\leqq -\varepsilon + \int_{|\beta|\leqq L-3} M(d\beta) P_{x_0}\{\#\{n: V_n \in [t-\beta-L, t-\beta+L]\} \geqq N\} \\ &\leqq \text{expression in (3.43)} + \alpha^{-1} \int_S \psi(dy) \int_{w>t-L-1} dw\, P_y\{V_{\nu_1} \geqq w\}. \end{aligned}$$

The last term in the right-hand side of (3.44) has to be added because (3.42) is only proved for $w \leqq t - L - 1$. Since $V_n \to \infty$ a.s. as $n \to \infty$ (see I.2) we can, with $L = L(\varepsilon)$ fixed, find $t_0 = t_0(\varepsilon)$ and $N_0 = N_0(\varepsilon)$ such that the last member of (3.44) is at most ε for $t \geqq t_0$ and $N \geqq N_0$. This at last gives

$$(3.45) \qquad P_{x_0}\{\#\{n: V_n \in [t-3, t+3]\} \geqq N\} \leqq 4\varepsilon, \qquad t \geqq t_0, \quad N \geqq N_0.$$

This essentially is (3.39) with x replaced by x_0. To obtain (3.39) for any x, observe first, that just as in (3.40) for $d(y, x_0) < r_0$

$$(3.46) \qquad P_{x_0}\{\#\{n: t-w-L-1 < V_n < t-w+L+1\} \geqq N\} \\ \geqq -\varepsilon + P_y\{\#\{n: t-w-L \leqq V_n \leqq t-w+L\} \geqq N\}.$$

Thus, with T as in (3.41) and $x \in S$ arbitrary

$$\begin{aligned} &P_x\{\#\{n: V_n \in [t-2, t+2] \geqq N\} \\ &\quad\leqq P_x\{\max_{n\leqq T} V_n \geqq t-2\} + \int_{S\times\mathbb{R}} P_x\{Z_T \in dy, V_T \in dw\} \\ &\qquad\quad \times P_y\{\#\{n: t-w-2 \leqq V_n \leqq t-w+2\} \geqq N\} \\ &\quad\leqq P_x\{\max_{n\leqq T} V_n \geqq t-2\} + \varepsilon \\ &\qquad\quad + \int_{\mathbb{R}} P_x\{V_T \in dw\} P_{x_0}\{\#\{n: t-w-3 < V_n < t-w+3\} \geqq N\} \end{aligned}$$

[11] $M(\mathbb{R}) = 1$ because $T < \infty$ a.s. (by virtue of (2.8) and I.1).

Since $P_x\{T < \infty\} = 1$ by (2.8) and I.1, we now have for $N \geqq N_0(\varepsilon)$

$$\begin{aligned}\limsup_{t\to\infty} P_x\{\#\{n: V_n \in [t-2, t+2] \geqq N\} \\ \leqq \varepsilon + \limsup_{t\to\infty} P_{x_0}\{\#\{n: t-3 < V_n < t+3\} \geqq N\} \\ \leqq 5\varepsilon\,.\end{aligned}$$

This implies (3.39) because for fixed x and t

$$P_x\{\#\{n: V_n \in [t-2, t+2]\} \geqq N\} \to 0$$

as $N \to \infty$ (recall $P_x\{V_n \to \infty\} = 1$ by I.2). Actually we get a bit more; if we combine (3.39) with (3.46) with $w = 0$, $L = 1$ and x_0 replaced by an arbitrary x, we find for fixed x and ε and $N = N(x, \varepsilon)$ that

$$P_y\{\#\{n: V_n \in [t-1, t+1]\} \geqq N\} \leqq 2\varepsilon \quad \text{for all} \quad t \geqq 0 \quad \text{and all} \quad y \quad \text{with} \quad d(y, x) < r_0(x, \varepsilon)\,. \tag{3.47}$$

With (3.39) established the proof of (2.2) is easily completed. Firstly we can restrict ourselves to positive f. Secondly, it is almost immediate from (3.38) that

$$\sup_x |H(x, t') - H(x, t'')| \to 0 \qquad \text{as} \quad |t' - t''| \to 0$$

so that we only have to estimate $H(x, t) - H(y, t)$. Now if $\{x_n, v_n\}_{n\geqq 0}$ is any sequence in $S \times \mathbb{R}$ we define $n(t)$, $z(t)$, $w(t)$ in terms of x_n, v_n in the same way as $N(t)$, $Z(t)$, $W(t)$ in terms of $\{X_n, V_n\}_{n\geqq 0}$ (see (1.7)—(1.9)). Also we put

$$\begin{aligned} f(t, \delta, x_0, v_0, x_1, v_1, \cdots) &= f(z(t), w(t)) \qquad \text{if no} \quad v_n \in [t-3\delta, t+3\delta]\,, \\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Then

$$\begin{aligned} H(y, t) &= E_y \int_{t+s\geqq 0} f(Z(t+s), W(t+s))g(s)\,ds \\ &= E_y \int_{t+s\geqq 0} f(t+s, \delta, X_0, V_0, X_1, V_1, \cdots)g(s)\,ds \\ &\quad + \theta \sup f \cdot E_y \int_{\Lambda(t,\delta)} |g(s)|\,ds\,,\end{aligned}$$

where $|\theta| \leqq 1$ and Λ is the random set

$$\Lambda(t, \delta) = \{s: \text{some} \quad V_n \in [t+s-3\delta, t+s+3\delta]\}\,.$$

Now let $\eta < \infty$ be such that $g(s) = 0$ for $|s| > \eta$. Then

$$\int_{\Lambda(t,\delta)} |g(s)|\,ds \leqq \sup_s |g(s)| 6\delta \cdot \#\{n: V_n \in [t-\eta-3\delta, t+\eta+3\delta]\}$$

and

$$\begin{aligned} E_y \int_{\Lambda(t,\delta)} |g(s)|\,ds \\ \leqq \sup_s |g(s)|[6\delta N \\ + 2\eta P_y\{\#\{n: V_n \in [t-\eta-3\delta, t+\eta+3\delta]\} \geqq N\}]\,.\end{aligned} \tag{3.48}$$

Now by (3.47) we can pick first N and $r_1(x, \varepsilon) > 0$ and then $\delta_0 = \delta(x, \varepsilon)$ such that for the right hand side of (3.48) is at most ε for all $\delta \leqq \delta_0$, $d(y, x) < r_1(x, \varepsilon)$. Thus for such δ and y with $d(y, x) < \min(r_0(x, \delta), r_1(x, \varepsilon))$,

$$\begin{aligned} H(y, t) &\leqq \varepsilon \sup f + E_y \int_{t+s\geqq 0} f(t+s, \delta, X_0, V_0, X_1, V_1, \cdots)g(s)\,ds \\ &\leqq (\varepsilon + \delta)\sup f + E_x \int_{t+s\geqq 0} f^\delta(t+s, \delta, X_0, V_0, X_1, V_1, \cdots)g(s)\,ds\,.\end{aligned} \tag{3.49}$$

To estimate $f^\delta(t+s, \delta, x_0, v_0, x_1, v_1, \cdots)$ we distinguish two cases. First assume $v_n \in [t+s-2\delta, t+s+2\delta]$ for some n. Then for any $\{x_i', v_i'\}_{i\geq 0}$ with $|v_n - v_n'| \leqq \delta$ we have $v_n' \in [t+s-3\delta, t+s+3\delta]$ and hence $f(t+s, \delta, x_0', v_0', \cdots) = 0$. Consequently

$$f^\delta(t+s, \delta, x_0, v_0, \cdots) = 0 \qquad \text{when some} \quad v_n \in [t+s-2\delta, t+s+2\delta] .$$

Next assume no $v_n \in [t+s-2\delta, t+s+2\delta]$. Then for any $\{x_i', v_i'\}_{i\geq 0}$ with $|v_i' - v_i| \leqq \delta$ for $i \leqq n(t+s) \equiv \min\{n: v_n > t+s\}$ we have

$$v_i' \leqq t+s-\delta, \qquad i < n(t+s), \qquad v'_{n(t+s)} \geqq t+s+\delta .$$

Thus, in this case

$$\begin{aligned} n'(t+s) &\equiv \min\{n: v_n' > t+s\} = n(t+s), \\ z'(t+s) &\equiv x'_{n'(t+s)} = x'_{n(t+s)}, \\ w'(t+s) &\equiv v'_{n'(t+s)} - (t+s) = v'_{n(t+s)} - (t+s) . \end{aligned}$$

These observations show that if there is no v_n in $[t+s-2\delta, t+s+2\delta]$ and $n(t+s) < \infty$, then

$$\begin{aligned} &f^\delta(t+s, \delta, x_0, v_0, \cdots) \\ &\quad \leqq \sup\{f(z', w'): d(z', z(t+s)) + |w' - w(t+s)| < \delta\} \\ &\quad \leqq f(z(t+s), w(t+s)) + \sup\{|f(z', w') - f(z, w)|: d(z', z) + |w' - w| < \delta\} . \end{aligned}$$

When this is substituted into (3.49) we obtain

$$\begin{aligned} H(y, t) \leqq (\varepsilon + \delta) \sup f &+ E_x \textstyle\int_{t+s\geq 0} f(Z(t+s), W(t+s)) g(s)\, ds \\ &+ \sup\{|f(z', w') - f(z, w)|: d(z', z) + |w' - w| \leqq \delta\} . \end{aligned}$$

Since the only restrictions are $\varepsilon > 0$, $\delta \leqq \delta_0(x, \varepsilon)$ and $d(y, x) < \min(r_0(x, \delta), r_1(x, \varepsilon))$, and since f is uniformly continuous, this implies

$$\limsup_{y\to x} \sup_t (H(y, t) - H(x, t)) \leqq 0 .$$

In (3.49) and the following estimates we may interchange x and y so that also

$$\liminf_{y\to x} \inf_t (H(y, t) - H(x, t)) \geqq 0 . \qquad \square$$

PROOF OF THEOREM 1. Let $f: S \times (0, \infty) \to \mathbb{R}$ be bounded and continuous and define H_0 and H as in (3.37), (3.38) with a continuous g satisfying for some $0 < \eta < \infty$

$$g(s) \geqq 0, \qquad \int g(s)\, ds = 1, \qquad g(s) = 0 \quad \text{for} \quad |s| > \eta . \tag{3.50}$$

Then $H(\cdot, \cdot)$ is bounded and it is easily seen to satisfy (2.1) (check this for H_0 first, using the fact that $N(t) = \min\{n: \sum_{i=1}^{n-1} u_i > t - u_0\}$ on $u_0 < t$.) Moreover, because (3.32) is an invariant probability distribution for the (Z^*, W^*) process we have for $t > \eta$

$$\begin{aligned} &\alpha^{-1} \int \varphi(dy) \int P_y\{X_{\nu_1} \in dz, V_{\nu_1} \in d\lambda\} \textstyle\int_{0<w\leqq\lambda} dw \\ &\qquad \times \int g(s)\, ds\, E_{z,w} f(Z^*(t+s), W^*(t+s)) \\ &\qquad\qquad = \alpha^{-1} \int \varphi(dy) \int P_y\{X_{\nu_1} \in dz, V_{\nu_1} \in d\lambda\} \textstyle\int_{0<w\leqq\lambda} f(z, w)\, dw . \end{aligned} \tag{3.51}$$

($E_{z,w}f$ in (3.51) of course stands for the expectation of f w.r.t. $Q_{z,w}$.) On the other hand, by (3.24) and (3.25), the left-hand side of (3.51) can also be written as

$$\begin{aligned}\alpha^{-1} \int g(s)\, ds \int \psi(dy) \int P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \\ \times [\textstyle\int_{t+s<w\leq\lambda} f(z, w - t - s)\, dw + \int_{0\leq w\leq \min(t+s,\lambda)} H_0(z, t + s - w)\, dw] \\ = o(1) + \alpha^{-1} \int \psi(y)\, dy \int P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \textstyle\int_{0\leq w\leq\lambda} H(z, t - w)\, dw\,,\end{aligned}$$

where $o(1) \to 0$ as $t \to \infty$. Consequently, if we take

$$\rho(dz, dw) = \alpha^{-1} \int \psi(dy)\, dw \int_{\lambda\geq w} P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\}$$

on $S \times (0, \infty)$, then

$$\begin{aligned}\lim_{t\to\infty} \textstyle\int_{S\times \mathbf{R}} H(z, t - w)\rho(dz, dw) \\ = \alpha^{-1} \int \psi(dy) \int P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \textstyle\int_{0<w\leq\lambda} f(z, w)\, dw\,.\end{aligned}$$

Thus, if f is bounded and uniformly continuous, then the corollary to Lemma 1 applies (by virtue of Lemma 5) and

$$\lim_{t\to\infty} H(x, t) = \alpha^{-1} \int \psi(dy) \int P_y\{X_{\nu_1} \in dz,\ V_{\nu_1} \in d\lambda\} \textstyle\int_{0<w\leq\lambda} f(z, w)\, dw \tag{3.52}$$

for all $x \in S$. We apply (3.52) first with $f(z, w) = k(w)$ for some uniformly continuous function $k(\cdot)$ which satisfies

$$0 \leqq k(w) \leqq 1\,, \qquad k(w) = 1 \quad \text{if} \quad 0 < w \leqq 4\eta \qquad \text{and}$$
$$k(w) = 0 \quad \text{if} \quad w \geqq 5\eta\,.$$

Then, for $t > \eta$ the corresponding H becomes

$$\begin{aligned}H(x, t) &= \int g(s)\, ds\, E_x k(W(t + s)) \geqq \int g(s)\, ds\, P_x\{W(t + s) \leqq 4\eta\} \\ &= \int g(s)\, ds\, P_x\{\text{some } V_{\nu_i} \text{ lies in } (t + s, t + s + 4\eta]\} \\ &\geqq P_x\{\text{some } V_{\nu_i} \text{ lies in } (t + \eta, t + 3\eta]\} \\ &= P_x\{W(t + \eta) \leqq 2\eta\}\,.\end{aligned}$$

Thus, by (3.52)

$$\begin{aligned}\limsup_{t\to\infty} P_x\{W(t - \eta) \leqq 2\eta\} \\ \leqq \lim_{t\to\infty} H(x, t - 2\eta) \\ \leqq \alpha^{-1} \textstyle\int_S \psi(dy) \int_{\lambda>0} P_y\{V_{\nu_1} \in d\lambda\} \min(\lambda, 5\eta)\,.\end{aligned} \tag{3.53}$$

Denoting the last member of (3.53) by $\varepsilon_1(\eta)$ we have

$$\limsup_{t\to\infty} P_x\{W(t - \eta) \leqq 2\eta\} \leqq \varepsilon_1(\eta) \downarrow 0 \qquad \text{as} \quad \eta \downarrow 0\,. \tag{3.54}$$

Observe now that

$$\{W(t - \eta) > 2\eta\} = \{\text{no } V_{\nu_i} \text{ lies in } (t - \eta, t + \eta]\}\,,$$

and that whenever this event occurs, one has for $|s| < \eta$

$$Z(t + s) = Z(t - \eta) = Z(t)\,, \qquad W(t + s) = W(t) - s\,.$$

Thus the $H(x, t)$ of (3.38) equals

$$H(x, t) = E_x \int f(Z(t+s), W(t+s))g(s)\, ds$$
$$= E_x \int f(Z(t), W(t) - s)g(s)\, ds + 2\theta \sup_{z,w} |f(z, w)| P_x\{W(t - \eta) \leqq 2\eta\}$$

for $t > \eta$ and some $\theta = \theta(x, t) \in [-1, +1]$. Thus for bounded uniformly continuous f we obtain from (3.52), (3.53) and (3.50)

$$\begin{aligned} &\limsup_{t\to\infty} |E_x f(Z(t), W(t)) \\ &\quad - \alpha^{-1} \int \psi(dy) \int P_y\{X_{\nu_1} \in dz, V_{\nu_1} \in dx\} \int_{0<w\leqq\lambda} f(z, w)\, dw| \\ &\qquad \leqq 2 \sup_{z,w} |f(z, w)| \varepsilon_1(\eta) \\ &\qquad\quad + \limsup_{t\to\infty} E_x\{\sup_{|s|\leqq\eta} |f(Z(t), W(t) - s) \\ &\qquad\quad - f(Z(t), W(t))|\}\,. \end{aligned} \tag{3.55}$$

By letting $\eta \downarrow 0$ in (3.55) we obtain (1.16) for bounded, *uniformly* continuous f (recall $N(0) = \nu_1$). But since $S \times (0, \infty)$ is a metric space (with the obvious distance $d(x_1, x_2) + |t_1 - t_2|$ between the points (x_1, t_1) and (x_2, t_2)) this actually implies the theorem in full (see for instance Theorem II.6.1 in [18]). □

LEMMA 6. *Let* $h: S \times \mathbb{R} \to \mathbb{R}$ *be a bounded continuous function such that* $h(x, t) = 0$ *for* $x \notin C^*_{k_0}$ *or* $|t| \geqq L$ *for some* k_0 *and* $L < \infty$, *where*

$$C_k^* = \{x \in S: P_x\{V_m \geqq mk^{-1} \text{ for all } m \geqq k\} \geqq \tfrac{1}{4}\}\,. \tag{3.56}$$

If Conditions I *hold, then*

$$G(x, t) \equiv E_x \textstyle\sum_{n=0}^\infty h(X_n, t - V_n) \tag{3.57}$$

is bounded and (jointly) continuous on $S \times \mathbb{R}$.

PROOF. As a first step we prove the uniform bound (in z and t)

$$E_z \#\{n \geqq 0: X_n \in C_k^*, t \leqq V_n \leqq t + b\} \leqq 4(k + 1 + kb)\,, \qquad b \geqq 0\,. \tag{3.58}$$

To establish (3.58) we note that if $V_n \in [t, t+b]$ and $V_{n+m} - V_n \geqq mk^{-1}$ and $m > kb$, then $V_{n+m} \notin [t, t+b]$. Also, by the very definition (3.56) and the Markovian properties of the measures (1.1) we have

$$P_x\{V_{n+m} - V_n \geqq mk^{-1} \text{ for all } m \geqq k \mid X_0, \cdots, X_n, V_0, \cdots, V_n\}$$
$$= P_{X_n}\{V_m \geqq mk^{-1} \text{ for all } m \geqq k\} \geqq \tfrac{1}{4}$$

a.e. $[P_x]$ on $\{X_n \in C_k^*\}$. Thus, if we define the stopping times

$$\tau_0 = \min\{n \geqq 0: X_n \in C_k^*, t \leqq V_n \leqq t + b\}$$
$$\tau_{i+1} = \min\{n > \tau_i + k: X_n \in C_k^*, t \leqq V_n \leqq t + b, V_n - V_{\tau_i} < (n - \tau_i)k^{-1}\}$$

($\tau_i = \infty$ if no n with the required properties exists), then

$$P_x\{\tau_{l+1} < \infty \mid \tau_0, \cdots, \tau_l\} \leqq \tfrac{3}{4} \qquad \text{a.e.} \quad [P_x]\,, \tag{3.59}$$

and

$$\begin{aligned} &\#\{n \geqq 0: X_n \in C_k^*, t \leqq V_n \leqq t + b\} \\ &\quad \leqq \textstyle\sum_{j=0}^\infty I_{[\tau_j<\infty]} \#\{n: \tau_j \leqq n < \tau_{j+1}, V_n \in [t, t+b]\} \\ &\quad \leqq (k + 1 + kb) \textstyle\sum_{j=0}^\infty I_{[\tau_j<\infty]}\,. \end{aligned}$$

This implies (3.58), since its left-hand side is at most

$$(k+1+kb)\sum_{j=0}^{\infty} P_z\{\tau_j < \infty\} \leqq (k+1+kb)\sum_{j=0}^{\infty} (\tfrac{3}{4})^j \qquad \text{(see (3.59))}.$$

Now let the support of h be contained in $C_{k_0}^* \times [-L, +L]$ and let

$$M = M(t, N) = \min\{n \geqq N: -L \leqq t - V_n \leqq +L\} \qquad (= \infty \text{ if no such } n \text{ exists}).$$

An immediate consequence of (3.58) and a first entry decomposition is, for any $y \in S$, $N \geqq 0$

$$\begin{aligned} &|E_y\{\textstyle\sum_{n=N}^{\infty} h(X_n, t - V_n)| \\ (3.60)\qquad &= |\textstyle\int P_y\{M < \infty, X_M \in dz, V_M \in dw\} E_z\{\sum_{n=0}^{\infty} h(X_n, t - w - V_n)\}| \\ &\leqq \sup_{x,s} |h(x, s)| 4(k_0 + 1 + 2k_0 L) P_y\{M < \infty\} \\ &\leqq \Gamma P_y\{t - L \leqq V_n \leqq t + L \text{ for some } n \geqq N\}, \end{aligned}$$

where

$$\Gamma = \sup_{x,s} |h(x, s)| 4(k_0 + 1 + 2k_0 L).$$

In particular $G(\cdot, \cdot)$ is bounded.

Now fix $(x, t) \in S \times \mathbb{R}$. Then, by (3.60), for $|s| \leqq 1$ and any $y \in S$, $N \geqq 0$ and suitable $|\theta| \leqq 1$

$$\begin{aligned} &G(y, t + s) - G(x, t) \\ (3.61)\qquad &= E_y \textstyle\sum_{n=0}^{N-1} h(X_n, t + s - V_n) - E_x \sum_{n=0}^{N-1} h(X_n, t - V_n) \\ &\quad + \theta \Gamma P_y\{V_n \leqq t + s + L \text{ for some } n \geqq N\} \\ &\quad + \theta \Gamma P_x\{V_n \leqq t + L \text{ for some } n \geqq N\}. \end{aligned}$$

Fix $\gamma > 0$ and N for the moment and take

$$\begin{aligned} f_k(x_0, v_0, x_1, v_1, \cdots) &= 1 \qquad \text{if } v_m < m(2\gamma)^{-1} \text{ for some } m \in [N, k], \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then by I.4 for fixed x, γ, $0 < \varepsilon < 1$, $N \geqq 2\gamma$ and $d(y, x) < r_0(x, \varepsilon)$

$$\begin{aligned} &P_y\{V_m < m(2\gamma)^{-1} \text{ for some } m \geqq N\} \\ (3.62)\qquad &\leqq \varepsilon + \lim_{k\to\infty} E_x f_k^{\varepsilon}(X_0, V_0, X_1, V_1, \cdots) \\ &\leqq \varepsilon + P_x\{V_m - \varepsilon < m(2\gamma)^{-1} \text{ for some } m \geqq N\} \\ &\leqq \varepsilon + P_x\{V_m < m\gamma^{-1} \text{ for some } m \geqq N\}. \end{aligned}$$

Thus, if $0 < \varepsilon < 1$ is given, we can first pick $N_0 = N_0(x, \varepsilon, t)$ (by I.2) such that

$$P_x\left\{V_m \leqq m\left(\frac{N_0}{2|t + L| + 2}\right)^{-1} \text{ for some } m \geqq N_0\right\} \leqq \varepsilon,$$

and then $r_0(x, \varepsilon)$ such that for $d(y, x) < r_0(x, \varepsilon)$

$$\begin{aligned} &P_y\{V_m \leqq t + 1 + L \text{ for some } m \geqq N_0\} \\ &\qquad \leqq P_y\left\{V_m \leqq m\left(\frac{N_0}{|t + L| + 1}\right)^{-1} \text{ for some } m \geqq N_0\right\} \leqq 2\varepsilon. \end{aligned}$$

By virtue of (3.61) and I.4 we then have for all $|s| \leqq 1$ and $\delta < \varepsilon$ and $d(y, x) < r_0(x, \delta)$

$$
\begin{aligned}
&G(y, t+s) - G(x, t) \\
(3.63)\qquad &\leqq 4\Gamma\varepsilon + E_y \sum_{n=0}^{N_0-1} h(X_n, t+s-V_n) - E_x \sum_{n=0}^{N_0-1} h(X_n, t-V_n) \\
&\leqq 4\Gamma\varepsilon + N_0\delta \sup_{z,w} |h(z, w)| \\
&\quad + E_x \sum_{n=0}^{N_0-1} [h^\delta(X_n, t+s-V_n) - h(X_n, t-V_n)] .
\end{aligned}
$$

Since h is continuous it is clear that we can make the last sum in (3.63) small by choosing δ and s small (note N_0 does not depend on y, s or δ). Thus

$$
(3.64)\qquad \limsup_{y\to x, s\to 0} G(y, t+s) - G(x, t) \leqq 0 .
$$

Similarly, two successive applications of I.4 yield

$$
\begin{aligned}
G(x, t) - G(y, t+s) &\leqq 4\Gamma\varepsilon + N_0\delta \sup_{z,w} |h(z, w)| \\
&\quad + E_y \sum_{n=0}^{N_0-1} [h^\delta(X_n, t-V_n) - h(X_n, t+s-V_n)] \\
&\leqq 4\Gamma\varepsilon + 3N_0\delta \sup_{z,w} |h(z, w)| \\
&\quad + E_x \sum_{n=0}^{N_0-1} [h^\delta(X_n, t-V_n) - h(X_n, t+s-V_n)]^\delta
\end{aligned}
$$

(in the last sum the notation $[r(X_0, V_0, \cdots)]^\delta$ is used for $r^\delta(X_0, V_0, \cdots)$ where $r(\cdot)$ is the function between square brackets). Thus also

$$
\liminf_{y\to x, s\to 0} G(y, t+s) - G(x, t) \geqq 0
$$

which together with (3.64) proves the lemma. □

PROOF OF THEOREM 2. First we prove (1.20) with g replaced by a function h satisfying the conditions of Lemma 6. Since $h(X_n, t - V_n) = 0$ as long as $V_n \leqq t - L$, a simple decomposition w.r.t. the first entry of V_n into $(t - L, \infty)$ gives

$$
\begin{aligned}
G(x, t) &= \int P_x\{Z(t-L) \in dz, W(t-L) \in dw\} E_z \sum_{n=0}^{\infty} h(X_n, L-w-V_n) \\
(3.65)\qquad &= \int P_x\{Z(t-L) \in dz, W(t-L) \in dw\} G(z, L-w) \\
&= E_x G(Z(t-L), L - W(t-L)) .
\end{aligned}
$$

By Lemma 6 the function $f(z, w) \equiv G(z, L - w)$ on $S \times (0, \infty)$ is bounded and continuous, so that by (1.16) and (3.65)

$$
(3.66)\qquad \lim_{t\to\infty} G(x, t) = \lim_{t\to\infty} E_x f(Z(t-L), W(t-L))
$$

exists, is independent of x, and is given by the right-hand side of (1.16). However, there appears to be no simple way of evaluating the right-hand side of (1.16) and we therefore use an Abelian argument to evaluate (3.66) (for another method of attack see [3], page 389). For the time being take h positive. Then, by the boundedness of $G(\cdot, \cdot)$ (Lemma 6) and the dominated convergence theorem (3.66) equals

$$
\begin{aligned}
&\lim_{T\to\infty} \int_S \varphi(dx) \frac{1}{T} \int_0^T G(x, t)\, dt \\
&\qquad = \lim_{T\to\infty} \frac{1}{T} \int_S \varphi(dx) E_x \sum_{n=0}^{\infty} \int_0^T h(X_n, t - V_n)\, dt
\end{aligned}
$$

$$(3.67)\qquad = \lim_{T\to\infty}\Big[\frac{1}{T}\int_S \varphi(dx)E_x \sum_{M_1(T)}^{M_2(T)} \int_{-\infty}^{+\infty} h(X_n, s)\, ds$$

$$+ \frac{1}{T}\int_S \varphi(dx)E_x\{\sum_{n<M_1(T)} + \sum_{n>M_2(T)}\}\int_0^T h(X_n, t - V_n)\, dt$$

$$- \frac{1}{T}\int_S \varphi(dx)E_x \sum_{M_1(T)}^{M_2(T)} \int_{s<-V_n \text{ or } s>T-V_n} h(X_n, s)\, ds\Big].$$

(3.67) holds for any $M_2(T) \geqq M_1(T) \geqq 0$, but we take $M_1(T) = [\alpha^{-1}\varepsilon T]$, $M_2(T) = [\alpha^{-1}(1 - \varepsilon)T]$, for some small $\varepsilon > 0$. Because φ is an invariant measure for the $\{X_n\}$, the first term in the last member of (3.67) equals

$$(3.68)\qquad \int_S \varphi(dx) \int_{-\infty}^{+\infty} h(x, s)\, ds \lim_{T\to\infty} \frac{1}{T}([\alpha^{-1}(1 - \varepsilon)T] - [\alpha^{-1}\varepsilon T])$$

$$= (1 - 2\varepsilon)\alpha^{-1} \int_S \varphi(dx) \int_{-\infty}^{+\infty} h(x, s)\, ds .$$

Since $h(x, s) = 0$ for $|s| \geqq L$ the last term in (3.67) is bounded by

$$\sup_z \int_{-\infty}^{+\infty} |h(z, s)|\, ds \cdot \frac{1}{T} \sum_{M_1}^{M_2} \int \varphi(dx)P_x\{-V_n \geqq -L \text{ or } T - V_n \leqq L\} .$$

Now

$$\sup_z \int_{-\infty}^{+\infty} |h(z, s)|\, ds \leqq 2L \sup_{z,s} |h(z, s)| ,$$

and since

$$(3.69)\qquad \int \varphi(dx)P_x\Big\{\frac{V_n}{n} \to \alpha\Big\} = 1 ,$$

we have for each fixed $\varepsilon > 0$

$$(3.70)\qquad \sup_{M_1(T)\leqq n\leqq M_2(T)} \int \varphi(dx)P_x\{-V_n \geqq -L \text{ or } T - V_n \leqq L\} \to 0$$

as $T \to \infty$ and consequently the last term in the last member of (3.67) tends to zero as $T \to \infty$. Clearly

$$\frac{1}{T}\int_S \varphi(dx)E_x \sum_{n<M_1(T)} \int_0^T h(X_n, t - V_n)\, dt$$

$$\leqq \frac{M_1(T)}{T} \sup_{z,s} |h(z, s)|2L \leqq \varepsilon\alpha^{-1} \cdot 2L \sup_{z,s} |h(z, s)| .$$

Finally, by (3.60)

$$E_x \sum_{n>M_2(T)} \int_0^T h(X_n, t - V_n)\, dt$$

$$\leqq \sup_{z,s} |h(z, s)|2L \cdot 2\alpha^{-1}\varepsilon T + \int_0^T dt\, E_x \sum_{n>\alpha^{-1}(1+\varepsilon)T} h(X_n, t - V_n)$$

$$\leqq \sup_{z,s} |h(z, s)|[4L\alpha^{-1}\varepsilon T$$

$$+ 4(k_0 + 1 + 2k_0 L)TP_x\{V_n \leqq T + L \text{ for some } n \geqq \alpha^{-1}(1 + \varepsilon)T\}] .$$

Thus

$$\frac{1}{T}\int \varphi(x)\, dx\, E_x \sum_{n>M_2(T)} \int_0^T h(X_n, t - V_n)\, dt$$

$$(3.71)\qquad \leqq \sup_{z,s} |h(z, s)|4L\alpha^{-1}\varepsilon$$

$$+ 0(\int \varphi(x)\, dx\, P_x\{V_n \leqq T + L \text{ for some } n \geqq \alpha^{-1}(1 + \varepsilon)T\}) .$$

The last term in the right-hand side of (3.71) tends to zero as $T \to \infty$ by (3.69), and combining the estimates (3.67)—(3.71) we find upon letting $\varepsilon \downarrow 0$

$$\lim_{t\to\infty} G(x, t) = \alpha^{-1} \int \varphi(dx) \int h(x, s)\, ds . \tag{3.72}$$

Thus, (1.20) holds with g replaced by a positive function h satisfying the conditions of Lemma 6. The condition $h \geqq 0$ may be dropped since the above can be applied to the positive and negative part of h separately.

Now let g be an arbitrary continuous and directly Riemann integrable function on $S \times \mathbb{R}$ and let $\lambda\colon \mathbb{R} \to [0, 1]$ be a continuous function such that $\lambda(t) = 1$ for $|t| \leqq L - 1$, $\lambda(t) = 0$ for $|t| \geqq L$. Then

$$\begin{aligned} &|E_x \textstyle\sum_{n=0}^\infty g(X_n, t - V_n) - E_x \sum_{n=0}^\infty g(X_n, t - V_n)\lambda(t - V_n)| \\ &\qquad \leqq \textstyle\sum_{|j| \geqq L-2} E_x \sum_{n=0}^\infty |g(X_n, t - V_n)| I_{[j,j+1]}(t - V_n) . \end{aligned} \tag{3.73}$$

Moreover, by virtue of I.2 and the definition (1.10)

$$S = \bigcup_{k=1}^\infty C_k = \bigcup_{k=0}^\infty C_{k+1} \backslash C_k ,$$

so that the right-hand side of (3.73) is further bounded by

$$\begin{aligned} &\textstyle\sum_{|j| \geqq L-2} \sum_{k=0}^\infty \sup\{|g(z, w)|: z \in C_{k+1}\backslash C_k, j \leqq w \leqq j + 1\} \\ &\quad \times E_x \#\{n \geqq 0: X_n \in C_{k+1}, t - j - 1 \leqq V_n \leqq t - j\} \\ &\qquad \leqq \textstyle\sum_{|j| \geqq L-2} \sum_{k=0}^\infty 4(2k + 3) \sup\{|g(z, w)|: z \in C_{k+1}\backslash C_k, j \leqq w \leqq j + 1\} \end{aligned}$$

(see (3.58)). This last expression tends to zero as $L \to \infty$ because g is directly Riemann integrable. Thus, it suffices to prove (1.20) with $g(x, t)$ replaced by $g(x, t)\lambda(t)$ and we may assume that $g(x, t) = 0$ for $|t| \geqq L$ for some L. In the same way it follows that we may assume $g(x, t) = 0$ for $x \notin C_{k_0}^*$ for some k_0 once we prove that for each N there exist a continuous function $\theta\colon S \to [0, 1]$ and a k_0 such that $\theta(x) = 1$ for $x \in C_N$, $\theta(x) = 0$ for $x \notin C_{k_0}^*$. This will bring us to the case already dealt with in (3.72).

We complete the proof by showing how such functions θ can be found. When we interchange the role of x and y in (3.62) we find

$$P_x\{V_m < m(2\gamma)^{-1} \text{ for some } m \geqq N\} \leqq \varepsilon + P_y\{V_m < m\gamma^{-1} \text{ for some } m \geqq N\} ,$$

whenever $N \geqq 2\gamma$ and $d(y, x) < r_0(x, \varepsilon)$ and thus, for $x \in \bar{C}_N =$ closure of C_N

$$\begin{aligned} &P_x\{V_m < m(2N)^{-1} \text{ for some } m \geqq 2N\} \\ &\qquad \leqq \limsup_{y\to x, y \in C_N} P_y\{V_m < mN^{-1} \text{ for some } m \geqq 2N\} \leqq \tfrac{1}{2} . \end{aligned}$$

Thus $\bar{C}_N \subset C_{2N}$. Similarly for $x \in C_{2N}$, by (3.62)

$$\begin{aligned} &\limsup_{y\to x} P_y\{V_m < m(4N)^{-1} \text{ for some } m \geqq 4N\} \\ &\qquad \leqq P_x\{V_m < m(2N)^{-1} \text{ for some } m \geqq 4N\} \leqq \tfrac{1}{2} , \end{aligned}$$

so that some neighborhood of x is contained in C_{4N}^* (see (3.56)). Consequently

$$\bar{C}_N \subset C_{2N} \subset \mathring{C}_{4N}^* = \text{interior of } C_{4N}^* ,$$

Therefore, if $N \geqq 1$ is given, we can take $k_0 = 4N$ and

$$\theta(x) = \gamma(\inf_{z_1 \in C_N} d(x, z_1), \inf_{z_2 \notin C^*_{4N}} d(x, z_2)) ,$$

where γ is some continuous function from

$$\{(s_1, s_2): s_1 \geqq 0, s_2 \geqq 0 \text{ but not } s_1 = s_2 = 0\}$$

into $[0, 1]$ such that $\gamma(s_1, 0) = 0$, $s_1 > 0$, and $\gamma(0, s_2) = 1$, $s_2 > 0$. Such a θ is continuous, equals 1 on C_N and vanishes on the complement of C^*_{4N}. □

4. The case of negative drift. In this section we briefly indicate how Theorem 1 can be used to find the behavior for large t of

$$P_x\{\max_n V_n > t\} ,$$

or more generally

$$E_x\{f(Z(t), W(t)); N(t) < \infty\} ,$$

when $n^{-1}V_n \to \alpha$ a.s., but now with $\alpha < 0$ instead of $\alpha > 0$ as in I.2 ($N(t)$, $W(t)$ and $Z(t)$ are as in (1.7)—(1.9)). We take our cue from the method of Chapter XI.6 in [9]. Suppose that we can find a function $r: S \to (0, \infty)$ and $\kappa > 0$ such that

$$r(x) > 0 , \qquad x \in S ,$$

and

$$r(x) = E_x e^{\kappa u_0} r(X_1) = \int P(s, dy) r(y) \int F(d\lambda \mid s, y) e^{\kappa\lambda} .$$

One can then define new measures $\tilde{P}_x$ on $\mathscr{F}$ by

$$\tilde{P}_x\{X_i \in A_i, 0 \leqq i \leqq n, u_i \in B_i, 0 \leqq i < n\}$$

$$= \frac{1}{r(x)} I_{A_0}(x) \int_{A_1} P(x, dy_1) \cdots \int_{A_n} P(y_{n-1}, dy_n) r(y_n) \tag{4.1}$$

$$\times \int_{B_0} F(d\lambda_0 \mid x_1, y_1) e^{\kappa\lambda_0} \cdots \int_{B_{n-1}} F(d\lambda_{n-1} \mid y_{n-1}, y_n) e^{\kappa\lambda_{n-1}}$$

for $A_i \in \mathscr{S}$, $B_i \in \mathscr{B}$ (compare (1.1)). When governed by these measures, the coordinate functions X_n on Ω still form a Markov chain, now with the new transition function

$$\tilde{P}(x, A) = \frac{1}{r(x)} \int_A P(x, dy) r(y) \int_{-\infty}^{+\infty} F(d\lambda \mid x, y) e^{\kappa\lambda}$$

and the conditional distribution function of u_0 given $X_0 = x$, $X_1 = y$ is

$$\tilde{P}(d\lambda \mid x, y) = [\int_{-\infty}^{+\infty} e^{\kappa\xi} F(d\xi \mid x, y)]^{-1} e^{\kappa\lambda} F(d\lambda \mid x, y) .$$

(The use of the measures $\tilde{P}$ is a familiar trick in the boundary theory of Markov chains; see for instance [8].) For practical purposes it is more convenient to define $\tilde{P}_x$ by the relation

$$\tilde{E}_x f(X_0, \cdots, X_n, u_0, \cdots, u_{n-1})$$

$$= \frac{1}{r(x)} E_x\{f(X_0, \cdots, X_n, u_0, \cdots, u_{n-1}) r(X_n) e^{\kappa V_n}\} , \tag{4.2}$$

for f positive and $\mathscr{S}^{n+1} \times \mathscr{B}^n$ measurable. (4.2) is easily seen to be equivalent to (4.1). Assume now that the new measures $\tilde{P}_x$ satisfy Conditions I.1–I.4. We can then apply (1.16) to the function

$$\tilde{f}(z, w) = \frac{1}{r(z)} e^{-\kappa w} f(z, w)$$

to obtain

$$\lim_{t\to\infty} \tilde{E}_x \tilde{f}(Z(t), W(t)) = K(f) \tag{4.3}$$

for some finite $K(f)$, independent of x. However, by (4.2)

$$\begin{aligned}\tilde{E}_x \tilde{f}(Z(t), W(t)) &= \sum_{k=0}^{\infty} \tilde{E}_x\{\tilde{f}(X_k, V_k - t); N(t) = k\} \\ &= \frac{1}{r(x)} \sum_{k=0}^{\infty} E_x\{r(X_k)e^{\kappa V_k}\tilde{f}(X_k, V_k - t); N(t) = k\} \\ &= \frac{e^{\kappa t}}{r(x)} \sum_{k=0}^{\infty} E_x\{f(X_k, V_k - t); N(t) = k\} \\ &= \frac{e^{\kappa t}}{r(x)} E_x\{f(Z(t), W(t); N(t) < \infty\} .\end{aligned}$$

Consequently, by (4.3)

$$\lim_{t\to\infty} e^{\kappa t} E_x\{f(Z(t), W(t); N(t) < \infty\} = K(f) r(x) .$$

For the matrix example discussed in the introduction, (especially (1.6)) the details of this procedure will be worked out in [13].

REFERENCES

[1] ARJAS, E. (1972). On a fundamental identity in the theory of semi-Markov processes. *Adv. Appl. Probability* **4** 258–270.

[2] BREIMAN, L. (1968). *Probability.* Addison-Wesley, Reading.

[3] ÇINLAR, E. (1969). On semi-Markov processes on arbitrary spaces. *Proc. Cambridge Philos. Soc.* **66** 381–392.

[4] ÇINLAR, E. Periodicity in Markov renewal theory. *J. Appl. Prob.* To appear.

[5] CRUMP, K. S. (1970). On systems of renewal equations. *J. Math. Appl.* **30** 425–434.

[6] DOOB, J. L. (1948). Renewal theory from the point of view of the theory of probability. *Trans. Amer. Math. Soc.* **63** 422–438.

[7] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[8] DOOB, J. L. SNELL, J. L. and WILLIAMSON, R. E. (1960). Application of boundary theory to sums of independent random variables, in *Contributions to Probability and Statistics*, ed. I. Olkin *et al.* Stanford Univ. Press, 182–197.

[9] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications* **2** 2nd ed. Wiley, New York.

[10] FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Statist.* **31** 457–469.

[11] HALMOS, P. R. (1956). *Lectures on Ergodic Theory.* Chelsea, New York.

[12] JACOD, J. (1971). Théorème de renouvellement et classification pour les chaînes semi-markoviennes. *Ann. Inst. Henri Poincaré, Sect. B* **7** 83–129. Correction Note to appear.

[13] KESTEN, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131** 207–248.

[14] MEYER, P. A. (1966). *Probability and Potentials.* Blaisdell, New York.

[15] OREY, S. (1959). Recurrent Markov chains. *Pacific J. Math.* **9** 805–827.

[16] OREY, S. (1961). Change of time scale for Markov processes. *Trans. Amer. Math. Soc.* **99** 384–397.

[17] OREY, S. (1966). Tail events for sums of independent random variables. *J. Math. Mech.* **15** 937–952.

[18] PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces*. Academic Press, New York.

[19] ROYDEN, H. L. (1968). *Real Analysis*, 2nd ed. McMillan, New York.

[20] RUDIN, W. (1964). *Principles of Mathematical Analysis*, 2nd ed. McGraw-Hill, New York.

[21] RUNNENBURG, J. Th. (1960). On the use of Markov processes in one server waiting-time problems and renewal theory. Ph. D. dissertation, Univ. of Amsterdam.

[22] Smith, W. L. (1955). Regenerative stochastic processes. *Proc. Roy. Soc. London, Ser. A.* **232** 6–31.

[23] SMITH, W. L. (1958). Renewal theory and its ramifications. *J. Roy. Statist. Soc. Ser. B* **20** 243–302.

[24] STEIN, C. (1947). A note on cumulative sums. *Ann. Math. Statist.* **17** 498–499.

[25] VANDERGENUGTEN, B. B. (1969). The random walk with stochastic sojourn times. Ph. D. dissertation, Univ. of Groningen.

[26] WOLFOWITZ, J. (1949). Remarks on the notion of recurrence. *Bull. Amer. Math. Soc.* **55** 394–395.

[27] YACKEL, J. (1966). Limit theorems for semi Markov processes. *Trans. Amer. Math. Soc.* **123** 402–424.

[28] ZASLAVSKIĬ, A. E. (1971). A generalization of the renewal theorem. *Sibirsk. Mat. Z.* **12** 513–535; trans. *Siberian Math. J.* **12** 362–377.

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