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"NORMAL" DISTRIBUTION FUNCTIONS ON SPHERES AND THE MODIFIED BESSEL FUNCTIONS[1]

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In R^n, Brownian diffusion leads to the normal or Gaussian distribution. On the sphere S^n, diffusion does not lead to the Fisher distribution which often plays the role of the normal distribution on S^n. On the circle (S^1) and sphere (S^2), they are known to be numerically close. It is shown that there exists a random stopping time for the diffusion which leads to the Fisher distribution. This follows from the fact, proved here, that the modified Bessel function $I_\nu(x)$ is a completely monotone function of ν^2 (for fixed $x > 0$). More generally, we study the class of distributions on S^n which can be represented as mixtures of diffusions. The stopping time distribution is characterized, but not given in computable form. Also, three new distribution functions involving Bessel functions are presented.

1. Introduction and summary. The density of a normal distribution with zero mean and variance $2v$ when reduced modulo 2π (or "rolled-up") is

$$f_1(\theta, v) = (2\pi)^{-1}[1 + 2 \textstyle\sum_{m=1}^{\infty} \exp(-m^2 v) \cos m\theta] , \tag{1.1.1}$$

where $-\pi \leqq \theta \leqq \pi$. $f_1(\theta, v)$ is also the density of a Brownian particle, released at the "north pole" $\theta = 0$ of a circle, after time v. Thus $f_1(\theta, v)$ has some claim to be called the "normal" distribution on the circle S^1. The corresponding density for the Brownian diffusion on the n-sphere $S^n = \{x \in E^{n+1}: |x| = 1\}$ is given by

$$f_n(\theta, v) = \omega_n^{-1} \textstyle\sum_{m=0}^{\infty} N_{nm} \exp[-m(m+n-1)v] P_{nm}(\cos\theta) , \tag{1.1}$$

where $\omega_n =$ area of $S^n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$, $m(m+n-1)$ for $m = 0, 1, \cdots$ are the eigenvalues of the Laplacian on S^n, P_{nm} is the Legendre polynomial of order m for E^{n+1}, and N_{nm} is the number of linearly independent homogeneous spherical harmonics of degree m in E^{n+1} (cf. Section 2 for notation). For $n = 2$, this follows from Yosida [16]. It is stated explicitly for $n = 2$, for example, by Roberts and Ursell ([10] page 321) and can be obtained for arbitrary n by specializing their formulas on pages 336–339 for general n-manifolds or from standard formulas for the fundamental solutions of the heat equation in S^n (see Section 2).

While $f_n(\theta, v)$ is, on probabilistic grounds, the analogue on S^n of the normal distribution, the analogue for the purposes of statistical inference is the density proportional to $\exp(x \cos\theta)$ with $x \geqq 0$, where θ is the angle between the

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observation and the mean or polar direction. On S^1, von Mises showed that only for this density is the direction of the vector resultant of a sample the maximum likelihood estimate of the polar direction. This vector is seen to be a sufficient statistic. The same holds on S^2 (cf. Fisher [3] and Breitenberger [1]) and, in fact, on S^n. On S^1, the density is, for $-\pi \leqq \theta \leqq \pi$,

$$(1.2.1) \qquad g_1(\theta, x) = \exp(x \cos\theta)/2\pi I_0(x) = (2\pi)^{-1}\{1 + 2 \textstyle\sum_{m=1}^{\infty} [I_m(x)/I_0(x)] \cos m\theta\} ,$$

and, on $S^n (n \geqq 2)$, for $0 \leqq \theta \leqq \pi$,

$$(1.2) \qquad g_n(\theta, x) = c_n(x) \exp(x \cos\theta) = \omega_n^{-1} \textstyle\sum_{m=0}^{\infty} N_{nm}[I_{m+(n-1)/2}(x)/I_{(n-1)/2}(x)]P_{nm}(\cos\theta) .$$

The normalizing factor $c_n(x)$ is determined by

$$1 = \int_{S^n} g_n(\theta, x)\, d\omega_n = \omega_{n-1} \int_0^\pi g_n(\theta, x) \sin^{n-1}\theta\, d\theta ,$$

since $d\omega_n = \sin^{n-1}\theta\, d\theta\, d\omega_{n-1}$. Hence

$$(1.3) \qquad c_n(x) = (x/2)^{(n-1)/2}/2\pi^{(n+1)/2} I_{(n-1)/2}(x) ;$$

cf. (2.5) below. The distribution with density $g_n(\cdot, x)$ on S^n is called the von Mises–Fisher distribution. In these formulae, $I_\nu(x) = \exp(-\nu\pi i/2)J_\nu(ix)$ is the modified Bessel function

$$(1.4) \qquad I_\nu(x) = \textstyle\sum_{k=0}^{\infty} (x/2)^{2k+\nu}/k!\, \Gamma(\nu + k + 1) .$$

$I_\nu(x)$ and $K_\nu(x)$ are standard linearly independent solutions of the modified Bessel equation; cf. [14] pages 77–80.

Roberts and Ursell [10] showed for $n = 2$ and a fixed x that there is a choice of $v = v_n(x)$ such that the maximum of $|\int_0^\theta [f_n(\theta', v) - g_n(\theta', x)]\, d\theta'|$ is "small". Stephens [12] found the same was true for $n = 1$. The "best" choice of v hardly differs from that obtained by equating the coefficients of $P_{n1}(\cos\theta)$ in (1.1) and (1.2), i.e., $I_{(n+1)/2}(x)/I_{(n-1)/2}(x) = \exp(-nv)$. The result on $|\int_0^\theta [f_n(\theta', v) - g_n(\theta', x)]\, d\theta'|$ led us to the conjecture that there is a random stopping time distribution depending only on n and x for the Brownian motion which leads exactly to $g_n(\theta, x)$. In Section 4, we prove this to be the case:

THEOREM 1.1. *For fixed $n = 1, 2, \cdots$ and $x > 0$, there exists a distribution function $W_n(v) = W_n(v, x)$ on $0 \leqq v < \infty$ such that*

$$(1.5) \qquad g_n(\theta, x) = \int_0^\infty f_n(\theta, v) W_n(dv, x) \qquad \textit{for all} \quad \theta ,$$

$W_n(+0, x) = W_n(0, x) = 0$, *Furthermore, $W_1, W_2, \cdots$ can be chosen so that*

$$(1.6) \qquad W_n(v, x) = [I_0(x)/I_{(n-1)/2}(x)] \int_0^v \exp[-(n-1)^2 t/4] W_1(dt, x) .$$

A possible explanation of the phenomenon first noticed by Roberts and Ursell may be that $W_n(v, x)$ is "nearly" the Dirac distribution $\varepsilon(v - v_n(x))$ with a jump of 1 at $v = v_n(x)$. In Theorem 4.1 we deduce a number of properties of W_1 but do not obtain a good estimate of $W_n(v, x) - \varepsilon(v - v_n(x))$.

At the end of Section 4 we use arguments similar to [5] pages 766–768, to prove

PROPOSITION 1.2. *Let $n = 1, 2, \cdots$ and $v > 0$ be fixed. Then there does not exist a distribution function U on $x \geqq 0$ such that*

$$(1.7) \qquad f_n(\theta, v) = \int_0^\infty g_n(\theta, x)U(dx) \qquad \text{for almost all} \quad \theta .$$

Thus we see another sense in which the diffusion density is more fundamental. The question of the uniqueness of W_n in (1.5) (or more generally of Q_n in (3.1) below) will remain open.

In Section 3, we introduce the class Ω_n of probability densities on S^n which can be obtained as mixtures of f_n and we obtain basic results about Ω_n. These results permit us to deduce Theorem 1.1 with $W_1 = W$ in Section 4 as a corollary of the following:

THEOREM 1.3. *For fixed $x > 0$, $I_\nu(x)$ is a completely monotone function of ν^2; hence there exists a distribution function $W(t) = W(t, x)$, $0 \leqq t < \infty$, such that*

$$(1.8) \qquad I_\nu(x)/I_0(x) = \int_0^\infty \exp(-\nu^2 t)W(dt, x) \qquad \text{for} \quad 0 \leqq \nu < \infty .$$

The last part of this theorem follows from the first and the theorem of Hausdorff–Bernstein; cf. [2] pages 415–418 or [15] page 60. For properties of a function of ν having an integral representation of the type (1.8), see [11]. It turns out that the exponent 2 in (1.8) is critical. In fact if $\mu = \nu^\alpha$ and $\alpha < 2$, then $\partial^2 I_\nu(x)/\partial\mu^2 \to -\infty$ as $x \to \infty$ for fixed μ.

The first part (Sections 2–4) of this paper concerns directly the spherical distributions f_n and g_n. In Section 2, we give our notation and a brief resumé of the required properties of the Legendre polynomials P_{nm}. Section 3 deals with the class Ω_n of probability densities on S^n. In Section 4, we assume Theorem 1.3 and give the proofs of Theorem 1.1 and Proposition 1.2.

The second part (Sections 5 and 6) of this paper deals with the modified Bessel function I_ν. We prove Theorem 1.3 in Section 5 and obtain properties of the function W in Section 6.

In the last part (Section 7), we observe the existence of three new families of distributions on $x \geqq 0$ depending on I_ν and K_ν.

2. Notation and the Legendre polynomials. For $n = 1$, the variable θ on $|\theta| \leqq \pi$ is the standard parameter on S^1 with $d\omega_1 = d\theta$. For $n > 1$, the variable θ in $0 \leqq \theta \leqq \pi$ is taken as the polar angle on S^n so that $d\omega_n = \sin^{n-1}\theta \, d\theta \, d\omega_{n-1}$. Thus functions of θ alone are axially symmetric functions on S^n.

Müller [9] has given an account of spherical harmonics in q-dimensions and our notation is suggested by his. He denotes the Legendre polynomials of degree m in E^q, normalized to be 1 at $t = 1$, by $P_m(q; t)$, whereas we use the notation $P_{nm}(t)$ when $q = n + 1$. Thus by Rodrigue's formula ([9] page 17),

$$(2.1) \qquad P_{nm}(t) = (-2)^{-m}[\Gamma(n/2)/\Gamma(m + n/2)](1 - t^2)^{(2-n)/2}(d/dt)^m(1 - t^2)^{m+(n-1)/2} .$$

$P_{n0}(t) \equiv 1$ for $n = 1, 2, \cdots$, $P_{1m}(\cos\theta) = \cos m\theta$ for $m = 0, 1, \cdots$ and $P_{2m}(t)$ is the standard Legendre mth order polynomial, $|P_{nm}(\cos\theta)| \leqq 1$. Further $P_{nm}(t) = [\Gamma(m + 1)\Gamma(n - 1)/\Gamma(m + n - 1)]C_m^{(n-1)/2}(t)$ where $C_m^{\,\nu}(t)$ is the usual Gegenbauer function; [7] page 218. The number N_{nm} of linearly independent homogeneous spherical harmonics of degree m in E^{n+1}, is given by

$$N_{nm} = \frac{(2m+n-1)\Gamma(m+n-1)}{\Gamma(m+1)\Gamma(n)}. \tag{2.2}$$

Since (see [7] page 221) $(d/dt)C_m{}^\nu(t) = 2\nu C_{m-1}^{\nu+1}(t)$, we have

$$P'_{nm}(t) = \frac{m(m+n-1)}{n} P_{n+2,m-1}(t). \tag{2.3}$$

We shall use this formula in the form

$$P'_{nm}(t) = 2\pi[N_{n+2,m-1}/\omega_{n+2}][\omega_n/N_{nm}]P_{n+2,m-1}(t). \tag{2.3'}$$

As is well known (cf. [9]),

$$\omega_{n-1} \int_0^\pi P_{nk}(\cos\theta)P_{nm}(\cos\theta)\sin^{n-1}\theta\, d\theta = \delta_{km}\omega_n/N_{nm}, \tag{2.4}$$

so that $[N_{nm}/\omega_n]^{\frac{1}{2}}P_{nm}(\cos\theta)$ for $m = 0, 1, \cdots$ are orthonormal on S^n. They span the space of axially symmetric functions in $L^2(S^n)$. Also from [7], page 221, it follows that

$$\begin{aligned}\omega_{n-1} \int_0^\pi \exp(x\cos\theta)P_{nm}(\cos\theta)\sin^{n-1}\theta\, d\theta \\ = 2\pi^{(n+1)/2}(2/x)^{(n-1)/2}I_{m+(n-1)/2}(x),\end{aligned} \tag{2.5}$$

from which (1.2) and (1.3) may be deduced.

Finally, (1.1) may be obtained by solving the diffusion equation in $n+1$ dimensions $\nabla^2 f = \partial f/\partial v$ by separation of variables and using the appropriate representation of the delta function on S_n; cf. Yosida [16]. Since the particle starts at $\theta = 0$, the motion is axially symmetric, so that only $P_{nm}(\cos\theta)$ appears.

For use below, we recall the following:

DEFINITION. A continuous function $f(\nu)$ is said to be *completely monotone* on a ν-interval if, on the interior of the interval, $f(\nu)$ is of class C^∞ and $(-1)^n d^n f/d\nu^n \geqq 0$ for $n = 0, 1, \cdots$.

3. The class Ω_n. This section deals with the class Ω_n of probability densities on S^n which can be represented as mixtures of the Brownian distribution (1.1). It is an analogue of the class Ω of distribution functions (rather than densities) on E^1 which are mixtures of centered normal distributions studied by Hartman and Wintner [5].

DEFINITION. Let $n = 1, 2, \cdots$ be fixed and Ω_n be the set of functions $q(t)$, $|t| \leqq 1$, with the properties that $q(t)$ is a Baire function, $q(-\cos\theta) \in L^1(S^n)$, $0 \leqq q(t) \not\equiv 0$ and if $C_n[q]$ is the normalizing constant defined by

$$C_n[q] \int_{S^n} q(-\cos\theta)\, d\omega_n = 1,$$

then there exists a distribution function $Q_n(v) = Q_n[q](v)$ on $v \geqq 0$ such that

$Q_n(+0) = Q_n(0) = 0$ and

$$C_n[q]q(-\cos\theta) = \int_0^\infty f_n(\theta, v)Q_n(dv) \tag{3.1}$$

for almost all θ.

First we shall verify the following elementary fact.

PROPOSITION 3.1. *Let $n = 1, 2, \cdots$ be fixed. Then $f_n(\theta, v)$ is of class C^∞ for $v > 0$, $f_n(\theta, v) > 0$ and its partial derivative*

$$\partial f_n(\theta, v)/\partial\theta = -2\pi e^{-nv} f_{n+2}(\theta, v)\sin\theta \tag{3.2}$$

is negative for $0 < \theta < \pi$.

PROOF. The series (1.1) is uniformly absolutely convergent for $v \geqq \varepsilon > 0$, since $|P_{nm}(\cos\theta)| \leqq 1$ (cf. [9] page 15) and $N_{nm} \sim 2m^{n-1}$ as $m \to \infty$ (cf. (2.2)). Easily justified formal differentiation of (1.1) gives (3.2) after using (2.3). All the assertions of Proposition 3.1 now follow from these facts and the relation of f_n to the fundamental solution of the heat equation on S^n.

THEOREM 3.2. *Let $q(t) \in \Omega_n$. Then, after alteration of $q(t)$ on a null set, $q(t) \in C^\infty(-1, 1]$; $q(t)$ is completely monotone on $(-1, 1]$, in fact, $(-)^j q^{(j)}(t) = (-)^j(d/dt)^j q(t) > 0$ for $-1 < t \leqq 1$; the integral $\int_0^\infty = \lim \int_\varepsilon^\infty$ as $\varepsilon \to 0$, in* (3.1) *is uniformly convergent on every interval $0 < \theta_0 \leqq \theta \leqq \pi$; and $(-)^j q^{(j)}(t) \in \Omega_{n+2j}$ for $j = 1, 2, \cdots$ with $Q_{n+2j}(v) = Q_{n+2j}[(-)^j q^{(j)}](v)$ given by*

$$Q_{n+2j}(v) = \int_0^v e^{-j(n+j-1)u}Q_n(du) \Big/ \int_0^\infty e^{-j(n+j-1)u}Q_n(du)\,. \tag{3.3}$$

PROOF. Since $f_n(\theta, v)$ is a decreasing function of θ for $0 \leqq \theta \leqq \pi$, $q \in \Omega_n$ implies that the integral (3.1) is uniformly convergent for $0 < \theta_0 \leqq \theta \leqq \pi$, $\theta_0 > 0$ arbitrary. Hence $q(t)$ can be taken as continuous on $-1 < t \leqq 1$ and is decreasing. It is easy to justify formal differentiation of (3.1) to obtain

$$-C_n[q]q'(-\cos\theta) = 2\pi \int_0^\infty f_{n+2}(\theta, v)e^{-nv}Q_n(dv)$$

by (3.2). It follows that $q'(t)$ is continuous for $-1 < t \leqq 1$ and the last integral is uniformly convergent on every interval $0 < \theta_0 \leqq \theta \leqq \pi$. Also by Fubini's theorem,

$$1/C_{n+2}[-q'] = -\int_{S^{n+2}} q'(-\cos\theta)\, d\omega_{n+2} = 2\pi \int_0^\infty e^{-nv}Q_n(dv)/C_n[q]\,.$$

The last two formulas imply $-q'(t) \in \Omega_{n+2}$ and the case $j = 1$ of (3.3). An induction on j completes the proof of Theorem 3.2.

PROPOSITION 3.3. *Let $q(t) \in \Omega_n \cap C^0[-1, 1]$. Then the integral in* (3.1) *is uniformly convergent and* (3.1) *holds for $0 \leqq \theta \leqq \pi$.*

PROOF. Since (3.1) is valid for $0 < \theta < \pi$ and $q(-\cos\theta)$ is decreasing,

$$C_n[q]q(-1) \geqq \int_0^\infty f_n(\theta, v)Q_n(dv) \qquad \text{for } 0 < \theta \leqq \pi\,;$$

Lebesgue's theorem on monotone convergence implies that the integral in (3.1) is convergent for $\theta = 0$ and

$$\int_0^\infty f_n(0, v)Q_n(dv) = \lim_{\theta\to+0} \int_0^\infty f_n(\theta, v)Q_n(dv) .$$

Our main criterion for $q \in \Omega_n$ is given by the following

THEOREM 3.4. *Let* $0 \leqq q(t) \in C^0(-1, 1]$ *and let* $0 \not\equiv q(-\cos\theta) \in L^1(S^n)$ *have the Fourier expansion*

$$C_n[q]q(-\cos\theta) \sim \omega_n^{-1} \sum_{m=0}^\infty N_{nm} q_{nm} P_{nm}(\cos\theta) , \tag{3.4}$$

where q_{nm} *is the* mth *Fourier coefficient,*

$$q_{nm} = C_n[q]\omega_{n-1} \int_0^\pi q(-\cos\theta)P_{nm}(\cos\theta) \sin^{n-1}\theta\, d\theta , \tag{3.5}$$

for $m = 0, 1, \cdots$. *Then* $q(t) \in \Omega_n$ [*and* (3.1) *holds*] *if and only if there exists a distribution function* $Q_n(v)$ *on* $v \geqq 0$ *satisfying* $Q_n(+0) = Q_n(0) = 0$ *and*

$$q_{nm} = \int_0^\infty \exp[-m(m + n - 1)v]Q_n(dv) \tag{3.6}$$

for $m = 0, 1, \cdots$.

Note that the exponents in (3.6) satisfy $\sum 1/m(m + n - 1) < \infty$, so that the Hausdorff criterion [6] for the existence of Q_n does not apply while, in general, Hallenbach's [4] conditions are too complicated to be useful.

PROOF. Assume the existence of a distribution function Q_n on $v \geqq 0$, satisfying $Q_n(+0) = Q_n(0) = 0$ and (3.6). We shall verify that $q(t) \in \Omega_n$ and (3.1) holds for $0 < \theta \leqq \pi$. Let $\varepsilon > 0$ and put

$$q_{nm}(\varepsilon) = \int_\varepsilon^\infty \exp[-m(m + n - 1)v]Q_n(dv) , \qquad m = 0, 1, \cdots , \tag{3.7}$$

so that $0 \leqq q_{nm}(\varepsilon) \leqq \exp[-m(m + n - 1)\varepsilon]$. Hence the series in

$$q_\varepsilon(\theta) = \omega_n^{-1} \sum_{m=0}^\infty N_{nm} q_{nm}(\varepsilon)P_{nm}(\cos\theta) \tag{3.8}$$

is uniformly and absolutely convergent for $0 \leqq \theta \leqq \pi$. Hence one may put (3.7) in (3.8) and interchange the order of summation and integration to obtain

$$q_\varepsilon(\theta) = \int_\varepsilon^\infty f_n(\theta, v)Q_n(dv) . \tag{3.9}$$

Since $f_n \geqq 0$, $q_\varepsilon(\theta)$ is non-increasing with respect to ε, and

$$\int_{S^n} q_\varepsilon(\theta)\, d\omega_n = \int_\varepsilon^\infty Q_n(dv) \leqq 1 \qquad \text{for } \varepsilon > 0 , \tag{3.10}$$

it follows from a theorem of Lebesgue (or B. Levi) that

$$\lim_{\varepsilon\to0} q_\varepsilon(\theta) = \int_0^\infty f_n(\theta, v)Q_n(dv) \tag{3.11}$$

exists on $0 < \theta \leqq \pi$ and is an $L^1(S^n)$ function. But (3.10) and $q_{nm}(\varepsilon) \to q_{nm}$ as $\varepsilon \to 0$ for $m = 0, 1, \cdots$ imply that $q_\varepsilon(\theta)$ tends weakly in $L^1(S^n)$ to $C_n[q]q(-\cos\theta)$ as $\varepsilon \to 0$. Consequently, (3.11) is the function $C_n[q]q(-\cos\theta)$, and $q(t) \in \Omega_n$.

Conversely, if $q(t) \in \Omega_n$ and (3.1) holds, define $q_\varepsilon(\theta)$ by (3.9); hence (3.7) and (3.8) hold. Thus $0 \leqq q_\varepsilon(\theta) \leqq C_n[q]q(-\cos\theta)$ implies that $q_\varepsilon(\theta)$ tends to $C_n[q]q(-\cos\theta)$ strongly, and so weakly, in $L^1(S^n)$ as $\varepsilon \to 0$. Since $q_{nm}(\varepsilon)$ is the

mth Fourier coefficient of $q_\varepsilon(\theta)$, we have that $q_{nm}(\varepsilon) \to q_{nm}$ as $\varepsilon \to 0$ for $m = 0, 1, \cdots$, that is, (3.6) holds. This completes the proof.

Whereas Theorem 3.2 gives a necessary condition for $q(t) \in \Omega_n$, a sufficient condition, independent of n, is contained in the following.

THEOREM 3.5. *Let $x > 0$ and suppose that $q(t)$ has the properties*

(i) $0 \not\equiv q(t) \in C^0[-x, \infty)$,

(ii) $q(t) \to 0$ *as* $t \to \infty$,

(iii) $q(t)$ *completely monotone for* $t \geqq -x$, (*that is, suppose* $q(t)$ *has a representation as a Laplace–Stieltjes transform*

$$q(t) = \int_0^\infty \exp(-st)\alpha(ds) \qquad \textit{for } t \geqq -x, \tag{3.12}$$

$\alpha(+0) = \alpha(0) = 0$, *with* $\alpha(ds) \geqq 0$), *then* $q(xt) \in \Omega_n$ *and*

$$Q_n(v) = Q_n(v, x) = C_n[q(x\cdot)] \int_0^\infty [W_n(v, sx)/c_n(sx)]\alpha(ds), \tag{3.13}$$

where $c_n(x)$ *is given by* (1.3) *and* $W_n(v, x)$ *occurs in* (1.6). *The relation* (3.13) *is equivalent to*

$$\partial Q_n(v, x)/\partial v = C_n[q(x\cdot)] \int_0^\infty [(\partial W_n(v, sx)/\partial v)/c_n(sx)]\alpha(ds). \tag{3.14}$$

In the proof of the first part of this theorem, we shall use Theorem 1.1 (to be proved in the next section). Had Theorem 3.5 been proved *ab initio*, Theorem 1.1 would have followed as the special case $q(t) = \exp(-t)$. The equivalence of (3.13) and (3.14) is an immediate consequence of properties of W_n which follow from (1.6) and Theorem 4.1.

PROOF. By (3.12), we have

$$q(-x\cos\theta) = \int_0^\infty \exp(sx\cos\theta)\alpha(ds) \qquad \text{for} \quad 0 \leqq \theta \leqq \pi.$$

Hence Theorem 1.1, (1.5) and $\alpha(+0) = \alpha(0)$ give

$$q(-x\cos\theta) = \int_0^\infty [c_n(sx)]^{-1}[\int_0^\infty f_n(\theta, v)W_n(dv, sx)]\alpha(ds).$$

Since the functions and measures in this iterated integral are nonnegative, Fubini's theorem implies that the integral in (3.13) is convergent and that we can change the order of integration in the last integral to obtain

$$C_n[q(x\cdot)]q(-x\cos\theta) = \int_0^\infty f_n(\theta, v)Q_n(dv, x).$$

This argument is more transparent and gives (3.14) if we use that $W_n(dv, sx) = (\partial W_n(v, sx)/\partial v)\, dv$; cf. Theorem 4.1 and (1.6).

4. The von Mises–Fisher distribution and Brownian motion. Theorem 1.1 asserts that, for fixed $x > 0$, the von Mises–Fisher density $g_n(\theta, x)$ is of class Ω_n, the mixing distribution being denoted by $W_n(v) = W_n(v, x)$. We shall deduce this from Theorem 3.4 and Theorem 1.3.

PROOF OF THEOREM 1.1. The case $n = 1$ of (1.5) follows with $W_1 = W$ from Theorem 3.4, in view of (1.2.1) and (1.8) in Theorem 1.3. In order to obtain the cases $n > 1$, we note that for $\nu \geqq 0$,

$$I_{\nu+(n-1)/2}(x)/I_{(n-1)/2}(x) = [I_0(x)/I_{(n-1)/2}(x)] \int_0^\infty \exp[-(\nu + (n-1)/2)^2 t] W(dt, x) ,$$

by (1.8). But this can be written as

$$I_{\nu+(n-1)/2}(x)/I_{(n-1)/2}(x) = \int_0^\infty \exp[-\nu(\nu + n - 1)t] W_n(dt, x) ,$$

where W_n is defined by (1.6). Hence, Theorem 1.1 for $n > 1$ follows from Theorem 3.4 in view of (1.2).

PROOF OF PROPOSITION 1.2. Suppose, if possible, that there exists a distribution function U satisfying (1.7) for almost all θ. Formula (1.3) and the asymptotic behavior of $I_\nu(x)$ imply that $c_n(x) \sim 1/2\pi^{(n+1)/2}$ and $c_n(x) \sim \exp(-x)(x/2\pi)^{n/2}$ as $x \to 0$ and $x \to \infty$, respectively. Thus $g_n(\theta, x) = c_n(x) \exp(x \cos \theta)$ is uniformly bounded for $0 \leqq \theta \leqq \pi$ and $x > 0$, while $g_n(\theta, x) \sim (\nu/2\pi)^n \exp[x(1 - \cos \theta)]$ as $x \to \infty$, uniformly in θ. Hence (1.7) is uniformly convergent on every interval $0 < \theta_0 \leqq \theta \leqq \pi$.

Let $p = U(+0) - U(0) \geqq 0$ and write (1.7) as

$$f_n(\theta, v) = p/\omega_n + \int_{+0}^\infty g_n(\theta, x) U(dx) ,$$

since $g_n(\theta, x) \to 1/\omega_n$ as $x \to +0$ uniformly in θ. By (1.5),

$$f_n(\theta, v) = p/\omega_n + \int_{+0}^\infty [\int_0^\infty f_n(\theta, u) W_n(du, x)] U(dx) ,$$

where W_n is given by (1.6) and $W_1 = W$ of Theorem 1.3. By Fubini's theorem, the last formula can be written as

$$f_n(\theta, v) = p/\omega_n + \int_0^\infty f_n(\theta, u) V(du) , \tag{4.1}$$

where $V(+0) = V(0) = 0$, $V(du) \geqq 0$, and

$$V(u) = \int_{+0}^\infty W_n(u, x) U(dx) .$$

The function V is continuous for $u \geqq 0$. Thus Proposition 1.2 will be proved if we verify the following "uniqueness" theorem.

PROPOSITION 4.2. *Let $v > 0$ and $n = 1, 2, \cdots$ be fixed. Let $p \geqq 0$ and $V(u)$ a non-decreasing function such that $V(+0) = V(0) = 0$ and that (4.1) holds uniformly on every interval $0 < \theta_0 \leqq \theta \leqq \pi$. Then*

$$p = 0 \quad \textit{and} \quad V(u) = \varepsilon(u - v) , \tag{4.2}$$

where $\varepsilon(u - v)$ is the Dirac distribution with jump 1 at $u = v$.

PROOF. Arguing as in the proof of Theorem 3.4, we obtain the following relationship between the Fourier coefficients of $f_n(\theta, v)$ and $f_n(\theta, u)$,

$$e^{-m(m+n-1)v} = p\delta_{m0} + \int_0^\infty e^{-m(m+n-1)u} V(du) \qquad \text{for } m = 0, 1, \cdots , \tag{4.3}$$

where $\delta_{00} = 1$ and $\delta_{m0} = 0$ if $m > 0$. The case $m = 0$ shows that V is bounded and

$$1 = p + V(\infty) . \tag{4.4}$$

Let $0 < t < v$ and $m > 0$. Then

$$e^{-m(m+n-1)v} \geqq \int_0^t e^{-m(m+n-1)u} V(du) \geqq e^{-m(m+n-1)t} V(t) .$$

On letting $m \to \infty$, it follows that $V(t) = 0$ for $0 < t < v$. Thus (4.3) becomes

$$e^{-m(m+n-1)v} = \int_u^\infty e^{-m(m+n-1)u} V(du) \qquad \text{for } m = 1, 2, \cdots . \tag{4.5}$$

Consequently, $V(u)$ has a jump of 1 at $u = v$ and is constant for $u > v$. This proves the proposition, in view of (4.4).

5. Proof of Theorem 1.3. It is possible to give a direct proof based on the modified Bessel differential equation and the asymptotic expansions for $I_\nu(x)$ and its partial derivatives $\partial^n I_\nu(x)/\partial \nu^n$ as $x \to \infty$. Instead we give a short proof based on the theory of differential operators and an observation of McKean [8] page 524, since this proof may have wider applications.

The functions $I_\nu(x)$ and $K_\nu(x)$ are solutions of the modified Bessel equation of order ν,

$$L[y] \equiv x^2 y'' + xy' - x^2 y = \mu y , \tag{5.1}$$

where $\mu = \nu^2$, and their Wronskian determinant satisfies

$$x(I_\nu K_\nu' - I_\nu' K_\nu) = -1 ; \tag{5.2}$$

cf. [14] pages 77 and 80. In order to obtain a formally self-adjoint operator, make the change of independent variables $x \to s$,

$$ds = dx/x , \qquad \text{i.e.,} \quad x = e^s . \tag{5.3}$$

Thus $0 < x < \infty$ becomes $-\infty < s < \infty$ and (5.1) becomes

$$L[y] \equiv \frac{d^2 y}{ds^2} - e^{2s} y = \mu y , \tag{5.4}$$

which has the solutions $I_\nu(e^s)$, $K_\nu(e^s)$. The differential operator L in (5.4) can be considered a self-adjoint operator on $L^2(-\infty, \infty)$. No boundary conditions are needed since $s = \pm\infty$ are in the limit point case. Also, the half-line $\mu > 0$ is in the resolvent set. In fact, by (5.2), $(\mu - L)^{-1}$ is a bounded integral operator and the corresponding Green kernel is

$$\begin{aligned} G(s, t, \mu) &= K_\nu(e^s) I_\nu(e^t) \qquad \text{if} \quad -\infty < t \leqq s < \infty , \\ G(s, t, \mu) &= K_\nu(e^t) I_\nu(e^s) \qquad \text{if} \quad -\infty < s \leqq t < \infty , \end{aligned} \tag{5.3}$$

where $\mu = \nu^2$. The standard expansion of $(\mu - L)^{-1}$ about a point $\mu = \mu_0$ in the resolvent set,

$$(\mu - L)^{-1} = \sum_{m=0}^\infty (-1)^m (\mu - \mu_0)^m (\mu_0 - L)^{-m-1} ,$$

in the uniform operator topology shows that $(d/d\mu)^m (\mu - L)^{-1} = (-1)^m m! \, (\mu - L)^{-m-1}$ or that

$$(-1)^m \left(\frac{\partial}{\partial \mu}\right)^m G(s, t, \mu) = m! \, G^{(m+1)}(s, t, \mu) \qquad \text{for } m = 0, 1, \cdots$$

and $\mu > 0$, where $G^{(1)} = G$ and

$$G^{(m+1)}(s, t, \mu) = \int_{-\infty}^{\infty} G(s, r, \mu)G^{(m)}(r, t, \mu)\, dr > 0 .$$

Thus, for fixed (s, t), $G(s, t, \mu)$ is a completely monotone function of $\mu > 0$.

In particular, for $0 < x \leqq y$, $K_\nu(y)I_\nu(x)$ is a completely monotone function of ν^2. Since $e^y(2y/\pi)^{\frac{1}{2}}K_\nu(y)I_\nu(x) \rightarrow I_\nu(x)$ as $y \rightarrow \infty$ holds uniformly on ν-compacts (for $x > 0$ fixed), Theorem 1.3 follows.

REMARK. A special case of McKean's main result [8] is that

$$G(x, y, \mu) = \int_0^\infty p(x, y, t)e^{-\mu t}\, dt ,$$

where $p(x, y, t)$ is a fundamental solution for the parabolic equation $L\sigma = \sigma_t$. The relation between $p(x, y, t)$ and $W(t, x)$ in Theorem 1.3 is

$$I_0(x)\partial W(t, x)/\partial t = \lim_{y\rightarrow\infty} e^y(2y/\pi)^{\frac{1}{2}}p(x, y, t) .$$

6. The function $W(t, x)$ in Theorem 1.3. The formula (1.8) in Theorem 1.3 permits us to deduce a number of properties of $W_1 = W$.

THEOREM 6.1. *Let $W(t, x)$ for $t \geqq 0$, $x > 0$ be given by Theorem* 1.3 *and extend its definition to $-\infty < t < \infty$, $x \geqq 0$ by*

(6.1) $$W(t, x) = 0 \quad \text{if} \quad t < 0 \quad \text{or} \quad x = 0 .$$

Then $W(t, x)$ has the following properties: $W(t, x) \in C^\infty((-\infty, \infty) \times (0, \infty)) \cap C^0((-\infty, \infty) \times [0, \infty))$;

(6.2) $$I_\nu(x)/I_0(x) = \int_0^\infty [\partial W(t, x)/\partial t] \exp(-\nu^2 t)\, dt \qquad \text{for} \quad \nu \geqq 0$$

and for $x > 0$; $\sigma(t, x) = W(t, x)I_0(x)$ satisfies the parabolic equation

(6.3) $$(x\partial/\partial x)^2\sigma = x^2\sigma + \partial\sigma/\partial t ;$$

$W(t, x)$ is a distribution function on $0 \leqq t < \infty$ for fixed $x > 0$, and on $0 \leqq x < \infty$ for fixed $t > 0$, in fact,

(6.4) $$W(t, x) \quad \text{is increasing in} \quad t(\geqq 0) \quad \text{and in} \quad x(\geqq 0) ,$$

(6.5) $$W(0, x) = 0 \quad \text{and} \quad W(\infty, x) = 1 \qquad \text{for} \quad x > 0 ,$$

(6.6) $$W(t, 0) = 0 \quad \text{and} \quad W(t, \infty) = 1 \qquad \text{for} \quad t > 0 ;$$

$\partial W/\partial t$ satisfies

(6.7) $$\int_0^\infty x^{-2}[\partial W(t, x)/\partial t)]\, dx < \infty \qquad \text{for fixed} \quad t ;$$

there exists a constant C_1 (independent of x) such that, for fixed $x > 0$,

(6.8) $$I_0(x)W(t, x) \leqq C_1(ext^{\frac{1}{2}}/2)^{1/t^{\frac{1}{2}}}t^{\frac{1}{4}} \qquad \text{for small} \quad t > 0 ,$$

(6.9) $$1 - W(t, x) \leqq C_1 K_0(x)/I_0(x)t^{\frac{1}{2}} \qquad \text{for large} \quad t > 0 ;$$

finally,

(6.10) $$t^{-\frac{1}{2}} \int_0^t sW(ds, x) \rightarrow K_0(x)/\pi^{\frac{1}{2}}I_0(x) > 0 \qquad \text{as} \quad t \rightarrow \infty .$$

In particular, (6.10) shows that the first moment of $W(\cdot, x)$ is ∞; in fact, that

$$\int_0^\infty t^{\frac{1}{2}+\varepsilon} W(dt, x) = \infty \,, \qquad \text{but} \qquad \int_0^\infty t^{\frac{1}{2}-\varepsilon} W(dt, x) < \infty$$

for $\varepsilon > 0$ and $x > 0$.

PROOF. Multiplying (1.8) by $I_0(x)$, introducing

$$\sigma(t) = \sigma(t, x) = I_0(x) W(t, x) \,, \tag{6.11}$$

and writing $I_\nu(x) \equiv I(\nu, x)$, we have

$$I(\nu, x) = \int_0^\infty \exp(-\nu^2 t)\sigma(dt) \,. \tag{6.12}$$

Let $c > 0$. By the inversion theorem for Laplace–Stieltjes transforms, we have at continuity points of $\sigma(t)$,

$$\sigma(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I((c + iu)^{\frac{1}{2}}, x)(c + iu)^{-1} e^{(c+iu)t}\, du \,. \tag{6.13}$$

We verify that this integral can be differentiated arbitrarily often with respect to both t and x; in fact

$$\int_{-\infty}^{\infty} |\partial^n I((c + iu)^{\frac{1}{2}}, x)/\partial x^n| |c + iu|^k \, du < \infty \tag{6.14}$$

is uniformly convergent on compacts of $x > 0$ for arbitrarily fixed k and $n = 0, 1, \cdots$. To this end we use the integral representation (cf. [14] page 79),

$$I(\nu, x) = [(x/2)^\nu / \pi^{\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2})] \int_{-1}^{1} (1 - s^2)^{\nu - \frac{1}{2}} e^{xs}\, ds \,. \tag{6.15}$$

Define $\theta = \theta(u)$ by

$$\nu = (c + iu)^{\frac{1}{2}} = (c^2 + u^2)^{\frac{1}{4}} e^{i\theta(u)} \,, \qquad |\theta(u)| < \pi/2 \,. \tag{6.16}$$

Then $\theta(u) \to \pm\pi/4$ as $u \to \pm\infty$; in fact $e^{i\theta(u)} = e^{\pm i\pi/4} + O((c^2 + u^2)^{-\frac{1}{2}})$ as $u \to \pm\infty$. Thus

$$\operatorname{Re} \nu \sim |u/2|^{\frac{1}{2}} \,, \qquad \operatorname{Im} \nu \sim \pm |u/2|^{\frac{1}{2}} \qquad \text{as} \quad u \to \pm\infty \,. \tag{6.17}$$

It follows from (6.17) that there exist constants $\beta = \beta(\varepsilon, n)$, $C = C(\varepsilon, n)$ such that

$$|\partial^n I(\nu, x)/\partial x^n| \leqq C(1 + |\nu|^{n-1}) e^{\beta \operatorname{Re} \nu} / \Gamma(\nu + \tfrac{1}{2}) \tag{6.18}$$

for $0 < \varepsilon < x < 1/\varepsilon$, $\nu = (c + iu)^{\frac{1}{2}}$, and large $|u|$. By Stirling's formula (cf. [7] pages 3 and 4) and (6.16), (6.17) and (6.18),

$$\Gamma(\nu + \tfrac{1}{2}) = (2\pi)^{\frac{1}{2}} \exp(\nu \log(\nu + \tfrac{1}{2}) - \nu)[1 + O(|\nu|^{-1})]$$

as $|u| \to \infty$. Hence, if $0 < a < 2^{-\frac{3}{2}}$,

$$1/\Gamma(\nu + \tfrac{1}{2}) = O(\exp(-a|u|^{\frac{1}{2}} \log |u|)) \qquad \text{as} \quad |u| \to \infty \,.$$

Thus, by (6.18) for fixed k and n,

$$|\nu|^{2k} |\partial^n I(\nu, x)/\partial x^n| = O(\exp(-a|u|^{\frac{1}{2}} \log |u|)) \qquad \text{as} \quad |u| \to \infty \,,$$

uniformly for $0 < \varepsilon < x < 1/\varepsilon$. This implies the assertions concerning (6.14) and hence the fact that $\sigma(t, x) \in C^\infty$ and $W(t, x) \in C^\infty$ for $-\infty < t < \infty$ and $x > 0$.

In particular (1.8) can be written as (6.2). By an integration by parts applied to (6.12),

(6.19) $$I_\nu(x) = \nu^2 \int_0^\infty \sigma(t, x) \exp(-\nu^2 t)\, dt\,, \qquad \nu \geqq 0,\ x > 0\,.$$

Since $I_\nu(x)$ is a solution of

(6.20) $$L[y] \equiv x^2 y'' + xy' - x^2 y = \nu^2 y\,,$$

an application of L to (6.19) gives

(6.21) $$\nu^2 I_\nu(x) = \nu^2 \int_0^\infty L[\sigma(t, x)] \exp(-\nu^2 t)\, dt\,.$$

But (6.2) and the uniqueness theorem for Laplace transforms, with (6.21), implies that $L[\sigma(t, \cdot)] = \partial\sigma/\partial t$ which gives (6.3).

Since the right side of (6.3) is nonnegative,

$$\partial^2\sigma(t, e^u)/\partial u^2 \equiv (x\partial/\partial x)^2 \sigma(t, x) \geqq 0\,,$$

where $x = \exp u$. Hence $\sigma(t, e^u) \geqq 0$ is a convex function of u and so

$$0 \leqq \lim_{u\to-\infty} \sigma(t, e^u) = \lim_{x\to+0} \sigma(t, x) \leqq \infty$$

exists and is a non-decreasing function of t. Fatou's lemma and (6.19) imply that

$$0 \geqq \nu^2 \int_0^\infty [\lim_{x\to+0} \sigma(t, x)] \exp(-\nu^2 t)\, dt \qquad \text{for } \nu > 0\,.$$

This gives $\sigma(t, +0) = 0$, hence $W(t, +0) = 0$ and $W(t, x)$ is continuous for $-\infty < t < \infty$ and $x \geqq 0$. It is clear that (6.4) holds.

Since $W(t, x) = \sigma(t, x)/I_0(x)$, $L(I_0) = 0$ and (6.3) show that

$$W'' + [2I_0'/I_0 + 1/x]W' = W_t/x^2$$

where $W_t = \partial W/\partial t \geqq 0$. Hence

(6.22) $$(W' I_0^2 x)' = I_0^2 W_t/x \geqq 0 \qquad \text{for } t \geqq 0,\ x > 0\,.$$

Since $W \geqq 0$ and $W(t, +0) = 0$, a convexity argument shows that $W(t, x)$ is non-decreasing in x. In particular, by (6.5), $0 \leqq \lim W(t, x) \leqq 1$ exists, as $x \to \infty$, and is non-decreasing function of t. From (6.19),

$$1 - I_\nu(x)/I_0(x) = \nu^2 \int_0^\infty [1 - W(t, x)] \exp(-\nu^2 t)\, dt \qquad \text{for } \nu > 0\,.$$

Since $1 - I_\nu(x)/I_0(x) \to 0$ as $x \to \infty$, it follows from Fatou's lemma that $W(t, +\infty) = 1$ for $t > 0$. Thus (6.4), (6.5), (6.6) hold.

Two quadratures of (6.20) give

$$\int^\infty (I_0^2(r) r)^{-1} (\int^r I_0^2 W_t(t, x) x^{-1}\, dx)\, dr < \infty$$

by (6.6). On interchanging the order of integration and using $I_0(x) \sim e^x (2\pi x)^{-\frac{1}{2}}$ as $x \to \infty$, we obtain (6.7).

From the relation

$$I_\nu(x)/I_0(x) \geqq \int_0^{1/\nu^2} \exp(-\nu^2 t) W(dt, x) \geqq e^{-1} W(1/\nu^2, x)\,,$$

we obtain the inequality $W(t, x) \leqq e I_\nu(x)/I_0(x)$ for $\nu = t^{-\frac{1}{2}} > 0$. Thus (6.8) follows from a standard asymptotic formula for $I_\nu(x)$ as $\nu \to \infty$ (with $x > 0$ fixed), easily obtained from (6.15) and Stirling's formula,

$$I_\nu(x) \sim (ex/2\nu)^\nu/(2\pi)^{\frac{1}{2}}\nu^{\frac{1}{2}} ;$$

Horn (cf. [14] page 225).

In order to prove (6.10), we use

$$(6.23) \qquad -[\partial I_\nu(x)/\partial\nu]/I_0(x) = 2\nu \int_0^\infty t \exp(-\nu^2 t) W(dt, x)$$

and the formula (cf. [14] page 78),

$$(6.24) \qquad -\partial I_\nu(x)/\partial v \to K_0(x) \qquad \text{as } \nu \to 0 .$$

Thus, if we let $\nu^2 = \mu$ in (6.23), we obtain

$$\int_0^\infty te^{-\mu t} W(dt, x) \sim K_0(x)/2I_0(x)\mu^{\frac{1}{2}} \qquad \text{as } \mu \to \infty .$$

Hence, (6.10) follows from a standard Tauberian theorem; cf. [15] page 197.

We now obtain (6.9) as a consequence of (6.10) and the inequalities

$$\int_0^t sW(ds, x) \geqq \int_{t/2}^t sW(ds, x) \geqq (t/2)[W(t, x) - W(t/2, x)] .$$

By (6.10), $W(t, x) - W(t/2, x) \leqq 3K_0(x)/\pi^{\frac{1}{2}} I_0(x) t^{\frac{1}{2}}$ for large t. Hence, for large t,

$$\begin{aligned} 1 - W(t, x) &= \textstyle\sum_{n=0}^\infty [W(2^{n+1}t, x) - W(2^n t, x)] \\ &\leqq [3K_0(x)/\pi^{\frac{1}{2}} I_0(x) t^{\frac{1}{2}}] \textstyle\sum_{n=0}^\infty 2^{-(n+1)/2} . \end{aligned}$$

This gives (6.9) and completes the proof of Theorem 6.1.

7. On distributions related to $I_\nu(x)$ and $K_\nu(x)$. A version of the argument showing that $W(t, x)$ is a distribution function on $x \geqq 0$ (for fixed t) will be used to obtain the following.

PROPOSITION 7.1. *Let $0 \leqq \mu < \nu$ be fixed. Then $I_\nu(x)/I_\mu(x)$ and $K_\mu(x)/K_\nu(x)$ are (continuous) distribution function on $x \geqq 0$.*

PROOF. Since $I_\nu(x) \sim (x/2)^\nu/\Gamma(1+\nu)$, $K_\nu(x) \sim 2(x/2)^{-\nu}/\Gamma(\nu)$ as $x \to 0$ and $I_\nu(x) \sim e^x/(2\pi x)^{\frac{1}{2}}$, $K_\nu(x) \sim e^{-x}(\pi/2x)^{\frac{1}{2}}$ as $x \to \infty$, it follows that $I_\nu(x)/I_\mu(x)$ and $K_\mu(x)/K_\nu(x)$ tend to 0 or 1 as $x \to +0$ or ∞; cf. [14] pages 202–203. Thus it only remains to verify that these functions are increasing for $x \geqq 0$.

Since $y = I_\nu(x)$ is a positive solution of (6.20) for $x > 0$ the function $z = I_\nu(x)/I_\mu(x)$ is a solution of

$$(xI_\mu^2 z')' = (\nu^2 - \mu^2) I_\mu^2 z/x > 0$$

for $x > 0$, satisfying $z > 0$ and $z(+0) = 0$, $z(\infty) = 1$. Rewrite this equation as

$$d^2z/ds^2 = (\nu^2 - \mu^2) I_\mu^4 z > 0 , \qquad \text{where } ds = dx/xI_\mu^2 .$$

Thus z is a convex function of s and, in particular, cannot take a maximum. Since $z > 0$ and varies from $z = 0$ to $z = 1$ as s increases, z must be monotone. In fact, $dz/ds > 0$, hence $z' > 0$ for $x > 0$.

Similarly, $z = K_\nu(x)/K_\mu(x) > 0$ satisfies $z(+0) = \infty$, $z(\infty) = 1$ and, as a function of s, is a solution of

$$d^2z/ds^2 = (\nu^2 - \mu^2)K_\mu^4 z > 0 , \qquad \text{where } ds = dx/xK_\mu^2 .$$

Note that $s \to \infty$ as $x \to \infty$ since $1/xK_\mu^2 \sim 2e^{2x}/\pi$ as $x \to \infty$. Since z is a convex function of s which is bounded as $s \to \infty$, it follows that z is monotone. In fact, $dz/ds < 0$, hence $z' < 0$ for $x > 0$.

PROPOSITION 7.2. *Let $\nu \geqq \frac{1}{2}$ and $0 \leqq \mu \leqq \nu$. Then $2xI_\nu(x)K_\mu(x)$ is a (continuous) distribution function on $x \geqq 0$.*

This is false for $\mu > \nu \geqq 0$ or $0 \leqq \nu < \frac{1}{2}$. In fact, the asymptotic developments for I_ν, K_ν show that, as $x \to \infty$,

$$2xI_\nu(x)K_\mu(x) \sim 1 + (\mu^2 - \nu^2)/2x + \cdots ,$$
$$2xI_\nu(x)K_\nu(x) \sim 1 - (4\nu^2 - 1)/8x^2 + \cdots ;$$

cf. [14] pages 202–203. Thus if $\mu > \nu \geqq 0$ or $0 \leqq \nu < \frac{1}{2}$, we have $2xI_\nu K_\mu > 1$ or $2xI_\nu K_\nu > 1$ for large x.

PROOF. In view of Proposition 7.1, it suffices to consider the case $\mu = \nu \geqq \frac{1}{2}$. If $\nu = \frac{1}{2}$, then $2xI_\nu(x)K_\nu(x) = 1 - e^{-2x}$, and the result is trivial. We shall therefore deal only with the case $\mu = \nu > \frac{1}{2}$. The asymptotic behavior of I_ν, K_ν shows that $2xI_\nu(x)K_\nu(x) \to 0$ or 1 according as $x \to 0$ or ∞. Thus it suffices to show that $(xI_\nu K_\nu)' = [(x^{\frac{1}{2}}I_\nu)(x^{\frac{1}{2}}K_\nu)]' > 0$ for $x > 0$, equivalently, that if $r_1(x)$, $r_2(x)$ are the logarithmic derivatives of $x^{\frac{1}{2}}K_\nu$, $x^{\frac{1}{2}}I_\nu$, then $r_1(x) + r_2(x) > 0$ for $x > 0$.

In the differential equation (6.20) for $y = I_\nu$, K_ν introduce the new dependent variable $z = x^{\frac{1}{2}}y$. Thus $z = x^{\frac{1}{2}}I_\nu$, $x^{\frac{1}{2}}K_\nu$ are solutions of

$$z'' - q(x)z = 0 , \qquad \text{where } q = 1 + (\nu^2 - \tfrac{1}{4})/x^2 . \tag{7.1}$$

The corresponding Riccati equation for $r = z'/z$ is

$$r' = q(x) - r^2 . \tag{7.2}$$

Hence r_1, r_2 are solutions of (7.2) satisfying

$$r_1(x) \to -1 \quad \text{and} \quad r_2(x) \to 1 \qquad \text{as } x \to \infty . \tag{7.3}$$

From the behavior of I_ν and its derivative at $x = 0$,

$$r_2(x) \sim (\nu + \tfrac{1}{2})/x \qquad \text{as } x \to 0 . \tag{7.4}$$

The zero-level curves for the right side of (7.2) are $r = \pm q^{\frac{1}{2}}(x)$, where $q^{\frac{1}{2}}(x) > 1$ satisfies $q^{\frac{1}{2}\prime}(x) < 0$ for $x > 0$. Since $q(x) - r^2 < 0$ when $r < -q^{\frac{1}{2}}(x)$, it follows that if $r = r(x)$ is a solution of (7.2) satisfying $r(x_0) \leqq -q^{\frac{1}{2}}(x_0) < -1$ for some $x_0 > 0$, then $r(x)$ is decreasing (in particular $r(x) < r(x_0) < -1$) for $x > x_0$ on its interval of existence. Hence the first part of (7.3) implies that $r_1(x) > -q^{\frac{1}{2}}(x)$ for $x > 0$.

Since $q^{\frac{1}{2}\prime}(x) < 0$ while $q(x) - r^2 = 0$ if $r = q^{\frac{1}{2}}(x)$, it follows that if $r = r(x)$ is a solution of (7.2) satisfying $r(x_0) > q^{\frac{1}{2}}(x_0)$ for some $x_0 > 0$, then $r(x) > q^{\frac{1}{2}}(x)$ for

$x > x_0$. Note that $q^{\frac{1}{2}}(x) \sim (\nu^2 - \frac{1}{4})^{\frac{1}{2}}/x$ as $x \to 0$. Since $(\nu^2 - \frac{1}{4})^{\frac{1}{2}} < \nu + \frac{1}{2}$, (7.4) implies that $r_2(x) > q^{\frac{1}{2}}(x)$ for small $x > 0$. Hence $r_2(x) > q^{\frac{1}{2}}(x)$ for all $x > 0$. Consequently, $r_1(x) + r_2(x) > 0$ for $x > 0$, and the proof is complete.

REFERENCES

[1] BREITENBERGER, E. (1963). Analogues of the normal distribution on the circle and the sphere. *Biometrika* **50** 81–88.

[2] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications* **2**. Wiley, New York.

[3] FISHER, R. A. (1953). Dispersion on a sphere. *Proc. Roy. Soc. Ser. A* **217** 295–300.

[4] HALLENBACH, F. (1933). Zur theorie der limitierungsverfahren von Doppelfolgen. Ph. D. dissertation, Bonn Univ.

[5] HARTMAN, P. and WINTNER, A. (1940). On the spherical approach to the normal distribution law. *Amer. J. Math.* **62** 759–779.

[6] HAUSDORFF, F. (1921). Summationsmethoden und momentfolgen II. *Math. Z.* **9** 280–299.

[7] MAGNUS, W., OBERHETTINGER, F. and SONI, R. T. (1966). *Formulas and Theorems for Special Functions of Mathematical Physics*, 3rd. ed. Springer-Verlag, Berlin.

[8] MCKEAN, H. P., Jr. (1950). Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.* **82** 519–548.

[9] MÜLLER, C. (1966). *Spherical Harmonics, Lecture Notes in Mathematics*, No. 17. Springer-Verlag, Berlin.

[10] ROBERTS, P. H. and URSELL, H. D. (1960). Random walk on a sphere and on a Riemannian manifold. *Philos. Trans. Roy. Soc. London Ser. A* **252** 317–356.

[11] SCHOENBERG, I. J. (1938). Metric spaces and completely monotone functions. *Ann. Math.* **39** 811–841.

[12] STEPHENS, M. A. (1963). Random walk on a circle. *Biometrika* **50** 385–390.

[13] VON MISES, R. (1918). Ueber die ganzzahligkeit der Atomgewichte und verwandte fragen. *Phys. Z.* **19** 490–500.

[14] WATSON, G. N. (1958). *A Treatise on the Theory of Bessel Functions.* 2nd Ed., Cambridge.

[15] WIDDER, D. (1941). *The Laplace Transform.* Princeton.

[16] YOSIDA, K. (1949). Brownian motion on the surface of a 3-sphere. *Ann. Math. Statist.* **20** 292–296.

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