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A GAUSSIAN PARADOX: DETERMINISM AND DISCONTINUITY OF SAMPLE FUNCTIONS[1]

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A real stochastic process $\{X(t): 0 \leqq t \leqq 1\}$, is called window-deterministic if the points $(t, X(t))$ on its graph belonging to a "window" $\{(t, x): 0 \leqq t \leqq 1, a < x < b\}$ stochastically determine all other points on the graph. Here it is shown that a large class of Gaussian processes with discontinuous sample functions has this property.

1. Introduction. Let $X(t)$, $0 \leqq t \leqq 1$, be a real separable measurable Gaussian stochastic process with a continuous covariance function. We assume that $EX(t) = 0$ for all t. According to a theorem of the author [2], if

$$\int_0^1 \int_0^1 \exp\{b/E(X(t) - X(s))^2\}\, ds\, dt < \infty \tag{1.1}$$

for some $b > 0$, then the sample functions have the "Caratheodory property": the inverse image of a set of positive measure has an intersection of positive measure with every open subinterval of $[0, 1]$. Such functions have graphs which wildly oscillate in the neighborhood of every point. The condition (1.1) is satisfied, in particular, if $E(X(t) - X(s))^2$ tends to 0, as $t - s \to 0$, no faster than $|\log |t - s||^{-1}$. The assumption of stationarity is not necessary.

Determinism in a Gaussian process or any second order process signifies that some random variables of the process are perfectly predictable on the basis of the values of others. Determinism in the context of prediction theory is discussed in [7], Chapter 12. The deterministic character of Brownian motion over Hilbert space was discovered by Lévy [9] page 355. His work was extended to more general linear spaces by Bretagnolle and Dacunha-Castelle [6]. The concept of local nondeterminism, which is a local antithesis of determinism, was recently introduced by the author in [4] and [5].

Now we define another version of the concept of determinism. For an arbitrary nonempty open interval A on the real line, consider the portion of the graph of the sample function that appears in the "window" $[0, 1] \times A$. We call the process *window-deterministic* if this portion of the graph almost surely determines $X(t)$ for every t and A.

Our main result is that a process which satisfies (1.1) is window-deterministic. This is surprising: The sample function, as noted above, is badly discontinuous

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and unbounded everywhere, and we hardly expect to be able to find the complete graph of such a function on the basis of an arbitrary window view. We offer this intuitive explanation: Under the Caratheodory property the inverse image of any nonempty open interval A in the range is dense in the domain. Since the process is stochastically continuous (though not sample function continuous) the values on the dense set "determine" those on the entire set. The gap here is that the dense set varies with the sample function. In the following section we precisely define window-determinism and give a proof of our result based on local times.

2. Window fields and local times. Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space for the process X defined above. For an arbitrary nonempty open interval A on the line let $\mathscr{F}_A$ be the sub-σ-field of $\mathscr{F}$ generated by the $\mathscr{F}$-measurable subsets of Ω of the form $\{\omega: X(t, \omega) \in B \cap A\}$, where $X(t, \omega)$ is the sample function $X(t)$ corresponding to the point $\omega \in \Omega$, and where t ranges over $[0, 1]$ and B over all linear Borel sets. $\mathscr{F}_A$ is called the window field corresponding to A.

DEFINITON. X is called window-deterministic if for every t and A there is an $\mathscr{F}_A$-measurable random variable $\tilde{X}(t)$ such that $X(t) = \tilde{X}(t)$ except for an $\mathscr{F}$-set of probability 0.

This means that the process is completely determined by the information that can be obtained by observing the part of the graph in the window $[0, 1] \times A$ together with the information about all null $\mathscr{F}$-sets. (The latter are not necessarily $\mathscr{F}_A$-sets.)

For an arbitrary subinterval I of $[0, 1]$ and an arbitrary linear Borel set B, put $\nu(B, I) =$ measure of $X^{-1}(B) \cap I$. X is said to have a local time if, for each I, $\nu(\cdot, I)$ is absolutely continuous as a set function of B almost surely. The local time is defined to be the Radon–Nikodym derivative, and is denoted $\phi(x, I)$. Since this depends on the sample function, it may, for fixed I, be considered as a stochastic process with the time parameter x, $-\infty < x < \infty$. It follows from the definition of the window field that $\nu(B \cap A, I)$ is $\mathscr{F}_A$-measurable for every Borel set B and open interval A; therefore, the random variables $\phi(x, I)$, $x \in A$, are $\mathscr{F}_A$-measurable.

3. The main result.

THEOREM. *If* (1.1) *holds, then the process is window-deterministic.*

PROOF. Under (1.1), and according to the result stated in [2] and refined in [3], the extension of $\phi(x, I)$ to the complex plane is analytic in a strip containing the real line, for all I with rational endpoints, almost surely. If $\phi(x, I)$ is given for $x \in A$, for any nonempty open interval A, then, as an analytic function, it is determined for all x. Therefore, $\phi(x, I)$ is $\mathscr{F}_A$-measurable for all x, I and A.

The continuity of the covariance function implies that the integral $\int_I X(s)\, ds$ exists and is finite almost surely for all I. It may be represented as the mean of the local time density,

$$\int_I X(s)\,ds = \int_{-\infty}^{\infty} x\phi(x, I)\,dx\,,$$

except for a null $\mathscr{F}$-set (see [2]). The right-hand integral inherits $\mathscr{F}_A$-measurability from ϕ; therefore, the left-hand integral is equal to an $\mathscr{F}_A$-measurable function except on a null $\mathscr{F}$-set.

For any t in $(0, 1)$, $X(t)$ is the quadratic mean limit of the average

$$(2h)^{-1} \int_{-h}^{h} X(t + s)\,ds$$

for $h \to 0$ because the covariance is continuous; hence, $X(t)$ is the almost sure limit of some subsequence; therefore, like the above average, $X(t)$ is almost surely equal to an $\mathscr{F}_A$-measurable random variable. The proof for $t = 0, 1$ is similar.

We remark that the integrand in (1.1) may be considered as a "potential" kernel, and the double integral as the corresponding "energy." The condition (1.1) states that the time set $[0, 1]$ has a positive capacity. This signifies that the set is large enough so that the sample functions have sufficient "room" for their wild dance. It appears that (1.1) cannot be substantially weakened. Gaussian processes usually have either continuous sample functions or everywhere discontinuous ones. This was first proved by Belyaev [1] for the stationary case, and has been extended in recent years to more general Gaussian processes (see, for example [8], Section 5). To illustrate the sharpness of (1.1) (though not the necessity of the condition) consider a stationary Gaussian process with covariance function r such that $1 - r(t) \sim |\log |t||^{-1-p}$, for $t \to 0$. If $p > 0$, then the sample functions are continuous [1]; however, if $p = 0$, then (1.1) is satisfied. Evidently a process with continuous sample functions cannot be window-deterministic because most windows are empty.

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