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CROSS-SPECTRAL ANALYSIS OF PROCESSES WITH STATIONARY INCREMENTS INCLUDING THE STATIONARY $G/G/\infty$ QUEUE[1]

BY DAVID R. BRILLINGER

The University of California, Berkeley

We consider a linear time invariant model relating one process with stationary increments to another such process. The model contains the stationary $G/G/\infty$ queue and a bivariate cluster process as particular cases. The parameters of the model are shown to be identifiable through cross-spectral analysis and estimates are shown to be asymptotically normal under regularity conditions. In the case of the $G/G/\infty$ queue, the parameters considered are the characteristic function and the distribution function of the service time. The estimates are based on a stretch of entry and exit times for the system.

1. Introduction. We consider bivariate processes $X(t) = \{X_1(t), X_2(t)\}$, $-\infty < t < \infty$, having stationary increments. The general theory of such processes is developed in Kolmogorov (1940), Yaglom (1958) and Bochner (1960). A particular case of these processes is the stationary point process considered in Bartlett (1963), Daley and Vere-Jones (1972). Certain additional aspects of processes with stationary increments were presented in Brillinger (1972). In the manner of that paper, suppose that we may write the joint cumulant

$$\text{cum}\,\{dX_{j_1}(t+u_1), \cdots, dX_{j_{k-1}}(t+u_{k-1}), dX_{j_k}(t)\} = C_{j_1\cdots j_k}(du_1, \cdots, du_{k-1})\,dt \tag{1.1}$$

with $C_{j_1\cdots j_k}(du_1, \cdots, du_{k-1})$ a finite measure for $j_1, \cdots, j_k = 1, 2$ and $k = 1, 2, 3, \cdots$. The cumulant spectra of the process may now be defined by

$$f_{j_1\cdots j_k}(\lambda_1, \cdots, \lambda_{k-1}) = (2\pi)^{-k+1} \int \cdots \int \exp\{-i \textstyle\sum_{j=1}^{k-1} \lambda_j u_j\} C_{j_1\cdots j_k}(du_1, \cdots, du_{k-1}) \tag{1.2}$$

for $-\infty < \lambda_1, \cdots, \lambda_{k-1} < \infty$. In particular we suppose that

$$E\,dX_j(t) = C_j\,dt \tag{1.3}$$

$$\text{Cov}\,\{dX_j(t+u), dX_k(t)\} = C_{jk}(du)\,dt \tag{1.4}$$

for a finite measure $C_{jk}(du)$ and that the second-order spectra are given by

$$f_{jk}(\lambda) = (2\pi)^{-1} \int \exp\{-i\lambda u\} C_{jk}(du) \tag{1.5}$$

for $j, k = 1, 2$. In the case of a stationary point process, $f_{jk}(\lambda)$ is the point process spectrum introduced in Bartlett (1963).

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The class of models to be considered in this paper consists of those processes for which

$$E\{dX_2(t) \mid X_1\} = [\mu + \int_{-\infty}^{\infty} a(t-u)\, dX_1(u)]\, dt \tag{1.6}$$

for $-\infty < t < \infty$. The model (1.6) is reminiscent of one for mutually exciting point processes introduced in Hawkes (1972); however it differs in the important respect of not completely describing the probability structure of the process. The model (4.19) of Brillinger (1972) is a particular case of the model (1.6). In the case of stationary point processes, the model is one proposed by John Rice in his doctoral thesis at Berkeley.

By analogy with the terminology of the cross-spectral analysis of ordinary stationary processes, we call $a(u)$ an *impulse response function* and we call its Fourier transform

$$A(\lambda) = \int a(u) \exp\{-i\lambda u\}\, du \tag{1.7}$$

$-\infty < \lambda < \infty$, a *transfer function*. We will see below that this parameter is also of interest.

As an example of a process for which the model (1.6) is satisfied consider the stationary $G/G/\infty$ queue with $X_1(t)$ referring to the times $\cdots, \tau_{-1}, \tau_0, \tau_1, \cdots$ of arrival of customers at a service facility possessing an infinite number of servers and with $X_2(t)$ referring to the times of departure $\cdots, \tau_{-1} + \gamma_{-1}, \tau_0 + \gamma_0, \tau_1 + \gamma_1, \cdots$ of customers, γ_j being the service time of the jth customer. We may write symbolically

$$dX_2(t) = \sum_j \delta(t - \tau_j - \gamma_j)\, dt\,, \tag{1.8}$$

$\delta(t)$ being the Dirac delta function. In the case that the service times are independent of the process $X_1(t)$ and have the same marginal density function $g(u)$ we may conclude from (1.8) that

$$\begin{aligned} E\{dX_2(t) \mid X_1\} &= \sum_j \int \delta(t - \tau_j - u) g(u)\, du\, dt \\ &= \sum_j g(t - \tau_j)\, dt \\ &= [\int g(t-u)\, dX_1(u)]\, dt\,. \end{aligned} \tag{1.9}$$

This has the form of (1.6) with $\mu = 0$ and $a(u) = g(u)$ and indicates the interest of estimating the parameters of that model. If $\Phi(\lambda) = E \exp\{i\lambda\gamma\}$ is the characteristic function of the service time distribution, then the transfer function $A(\lambda) = \Phi(-\lambda)$ here.

As a second example, consider a process in which $X_1(t)$ refers to a primary process of cluster centers while $X_2(t)$ refers to the cluster process itself. In the case that the cluster members are distributed independently of the primary process and in the same stochastic manner from cluster center to cluster center, we can see that the model (1.6) is satisfied with

$$A(\lambda) = E \sum_k \exp\{-i\lambda\sigma_k\} \tag{1.10}$$

the σ_k being the displacements of the cluster members from their center.

We now turn to the problem of identifying the parameters of the model (1.6) from the parameters of the process $X(t)$. By taking repeated expected values in (1.6) we may conclude that the means of the processes are related by

$$C_2 = \mu + A(0)C_1 \tag{1.11}$$

and that the cross-spectrum of the two processes is given by

$$f_{21}(\lambda) = A(\lambda)f_{11}(\lambda) . \tag{1.12}$$

This last shows that we may identify $A(\lambda)$ provided $f_{11}(\lambda) \neq 0$. $a(u)$ may be determined by inverse Fourier transform. μ may be determined from (1.11).

It is useful to set down two further parameters. Expression (1.6) suggests that we define the following process with stationary increments

$$d\varepsilon(t) = dX_2(t) - [\mu + \textstyle\int a(t-u)\, dX_1(u)]\, dt . \tag{1.13}$$

Clearly $E\, d\varepsilon(t) = 0$ and

$$\operatorname{Cov}\{dX_1(t+u), d\varepsilon(t)\} = 0 \tag{1.14}$$

for $-\infty < t, u < \infty$. If the second-order spectrum of the process $\varepsilon(t)$ is $f_{\varepsilon\varepsilon}(\lambda)$ then from (1.14)

$$f_{22}(\lambda) = |A(\lambda)|^2 f_{11}(\lambda) + f_{\varepsilon\varepsilon}(\lambda) \tag{1.15}$$

and so the coherence, $|R_{12}(\lambda)|^2 = |f_{12}(\lambda)|^2/[f_{11}(\lambda)f_{22}(\lambda)]$, of the two processes $X_1(t)$, $X_2(t)$ is given by

$$|R_{12}(\lambda)|^2 = |A(\lambda)|^2 f_{11}(\lambda)/[|A(\lambda)|^2 f_{11}(\lambda) + f_{\varepsilon\varepsilon}(\lambda)] . \tag{1.16}$$

Throughout the paper our concern is with real-valued $X_1(t)$, $X_2(t)$. No serious difficulties appear in extending the results to the vector-valued case considered in Brillinger (1972). The proofs of the theorems set down in the next two sections may be found in Section 4. In connection with the theorems we mention that our concern has not been with finding best possible results, rather it has been with indicating that cross-spectral analysis provides an interesting direct approach to certain point process problems of interest. I would like to thank Dr. Daryl Daley for helpful comments on this paper.

2. Estimation of the parameters. Expression (1.12) suggests that we consider the following estimate of $A(\lambda)$,

$$A^{(T)}(\lambda) = f_{21}^{(T)}(\lambda)/f_{11}^{(T)}(\lambda) \tag{2.1}$$

where $f_{jk}^{(T)}(\lambda)$ is an estimate of $f_{jk}(\lambda)$, $j, k = 1, 2$. We begin this section by determining the asymptotic distribution of this $A^{(T)}(\lambda)$ when the estimate $f_{jk}^{(T)}(\lambda)$ is constructed in the manner of Brillinger (1972).

That construction proceeds as follows. Suppose that the stretch of data $[X_1(t), X_2(t)]$, $0 < t \leqq T$ is available for some $T > 0$. Set

$$d_j^{(T)}(\lambda) = \textstyle\int_0^T \exp\{-i\lambda t\}\, dX_j(t) \tag{2.2}$$

$$I_{jk}^{(T)}(\lambda) = (2\pi T)^{-1} d_j^{(T)}(\lambda) d_k^{(T)}(\lambda)^* \tag{2.3}$$

$$f_{jk}^{(T)}(\lambda) = 2\pi B_T^{-1} T^{-1} \textstyle\sum_{s \neq 0} W(B_T^{-1}[\lambda - 2\pi s/T]) I_{jk}^{(T)}(2\pi s/T) \tag{2.4}$$

for some weight function $W(\alpha)$ and sequence of nonnegative band-width parameters B_T, $T = 1, 2, \cdots$. In order to determine the asymptotic distribution we must set down,

ASSUMPTION 2.1. The function $W(\alpha)$, $-\infty < \alpha < \infty$, is real-valued, even, 0 for $|\alpha| > \frac{1}{2}$, of bounded variation and $\int W(\alpha)\,d\alpha = 1$,
and

ASSUMPTION 2.2. The bivariate process $X(t) = \{X_1(t), X_2(t)\}$, $-\infty < t < \infty$, has stationary increments and the measure $C_{j_1\cdots j_k}(du_1, \cdots, du_{k-1})$ of (1.1) satisfies

$$\int \cdots \int [|u_1| + \cdots + |u_{k-1}|]|C_{j_1\cdots j_k}(du_1, \cdots, du_{k-1})| < \infty \tag{2.5}$$

for $j_1, \cdots, j_k = 1, 2$; $k = 2, 3, \cdots$.
This latter assumption is a form of mixing or asymptotic independence condition on the increments of the process. It will allow us to develop central limit theorems for various statistics based on the process. We begin with

THEOREM 2.1. *Let the process* $\{X_1(t), X_2(t)\}$, $-\infty < t < \infty$, *satisfy Assumption* 2.2. *Suppose* $f_{11}(\lambda) \neq 0$ *and* $|f_{jk}''(\lambda)| < \infty$, $j, k = 1, 2$. *Let* $W(\alpha)$ *satisfy Assumption* 2.1. *Set* $f_{\varepsilon\varepsilon}(\lambda) = f_{22}(\lambda) - |f_{12}(\lambda)|^2/f_{11}(\lambda)$. *Let* $B_T \to 0$, $B_T T \to \infty$, $B_T^5 T \to 0$ *as* $T \to \infty$. *Then* $A^{(T)}(\lambda)$, $\lambda \neq 0$, *is asymptotically complex normal with mean* $A(\lambda)$ *and variance*

$$B_T^{-1}T^{-1}2\pi \int W(\alpha)^2\,d\alpha\, f_{\varepsilon\varepsilon}(\lambda) f_{11}(\lambda)^{-1}\,. \tag{2.6}$$

Also $A^{(T)}(\lambda_1), \cdots, A^{(T)}(\lambda_J)$ *are asymptotically independent normal for distinct* $\lambda_1, \cdots, \lambda_J$.

The proof of this theorem is given in Section 4. The limiting distribution is seen to have the same form as that of the transfer function estimate of ordinary cross-spectral analysis.

We next turn to the problem of estimating the impulse response function $a(u)$. The estimate that we consider is

$$a^{(T)}(u) = [\textstyle\sum_{q=-Q_T}^{Q_T} A^{(T)}(C_T q) \exp\{iuC_T q\}][(1 - \cos C_T u)/(\pi C_T u^2)] \tag{2.7}$$

for some small C_T and large Q_T. For large Q_T one would make use of a Fast Fourier Transform in the computation of this estimate. The final multiplier in (2.7) is introduced in the manner of Bohman (1960). In practice we also find that inserting convergence factors into (2.7) often reduces the bias substantially.

THEOREM 2.2. *Let the conditions of Theorem* 2.1 *be satisfied. Suppose* $f_{11}(\lambda) \geqq \delta > 0$, $j = 1, 2$, $\int |u|^2|a(u)|\,du < \infty$, $\int |\lambda|^r|A(\lambda)|\,d\lambda < \infty$ *for some* $r > 0$. *Suppose* $C_T \geqq B_T$, $Q_T B_T \to 0$, $Q_T^3 B_T^{-1} T^{-1} \to 0$ *as* $T \to \infty$. *Then* $[a^{(T)}(u_1), \cdots, a^{(T)}(u_J)]$ *is asymptotically normal with mean*

$$\begin{aligned}[[\textstyle\sum_{q=-Q_T}^{Q_T} A(C_T q) \exp\{iu_j C_T q\}][(1 - \cos C_T u_j)/(\pi C_T u_j^2)] : j = 1, \cdots, J] \\ = [a(u_j)] + O(C_T^3 Q_T) + O(C_T^{-r} Q_T^{-r})\end{aligned} \tag{2.8}$$

and covariance matrix having the following entry in row j, column k

$$(2.9)\qquad C_T B_T^{-1} T^{-1} \int W(\alpha)^2\, d\alpha \int_{-C_T Q_T}^{C_T Q_T} \exp\{i(u_j - u_k)\alpha\} f_{\varepsilon\varepsilon}(\alpha) f_{11}(\alpha)^{-1}\, d\alpha (2\pi)^{-1}$$

for $j, k = 1, \cdots, J$.

From (2.9) we see that the asymptotic variance of $a^{(T)}(u)$ does not depend on u and is of order $O(C_T^2 Q_T B_T^{-1} T^{-1})$ under the stated conditions. The expression also suggests the following estimate of the covariance (2.9)

$$(2.10)\qquad \begin{aligned} &C_T^2 (2\pi)^{-2} B_T^{-1} T^{-1} \int W(\alpha)^2\, d\alpha \sum_{q=-Q_T}^{Q_T} \exp\{i(u_j - u_k) C_T q\} \\ &\quad \times f_{\varepsilon\varepsilon}^{(T)}(C_T q) f_{11}^{(T)}(C_T q)^{-1} \end{aligned}$$

based on the spectral density estimates. From (2.8) we have,

COROLLARY. *If, in addition to the conditions of the theorem,* $C_T^4 Q_T B_T T \to 0$, $C_T^{-2r-2} Q_T^{-2r-1} B_T T \to 0$ *as* $T \to \infty$, *then* $[a^{(T)}(u_1), \cdots, a^{(T)}(u_J)]$ *is asymptotically normal with mean* $[a(u_1), \cdots, a(u_J)]$ *and covariance structure given by* (2.9).

In the next section we specialize the results of this section to the case of a $G/G/\infty$ queue.

3. The $G/G/\infty$ queue. In Section 1 we saw that a stationary queueing system with an infinite number of servers satisfied the model (1.6) with $\mu = 0$ and $a(u)$ being the density of the service times. In the previous section we developed statistical properties of estimates of the parameters of the model for processes satisfying Assumption 2.2. We begin this section by indicating a set of conditions under which a $G/G/\infty$ queue satisfies Assumption 2.2.

THEOREM 3.1. *Let* $N_1(t)$, $-\infty < t < \infty$, *be a stationary point process whose cumulant measures exist and satisfy* (2.5) *with* $j_1 = \cdots = j_k = 1$ *for* $k = 2, 3, \cdots$. *Let* $N_2(t)$ *be a process constructed from* $N_1(t)$ *by displacing the events of the latter by independent amounts having cumulative distribution function* $G(u)$, $-\infty < u < \infty$. *Suppose* $\int |u|\, dG(u) < \infty$. *Then the process* $\{N_1(t), N_2(t)\}$ $-\infty < t < \infty$ *satisfies Assumption* 2.2.

In other words, the theorems of the previous section apply to the stationary $G/G/\infty$ queue having independent service times with finite mean value. We now proceed to indicate the form that the results of the paper take for this particular case. Let $\Phi(\lambda) = \int \exp\{i\lambda u\}\, dG(u)$ denote the characteristic function of the service time distribution, then from (1.7), (1.9) the transfer function $A(\lambda)$ equals $\Phi(-\lambda)$ here. Cox (1963) showed that

$$(3.1)\qquad f_{22}(\lambda) = |\Phi(\lambda)|^2 f_{11}(\lambda) + (2\pi)^{-1} C_1 [1 - |\Phi(\lambda)|^2]$$

here, (this also follows directly from Theorem 4.4), and so from (1.15) the error spectrum is given by

$$(3.2)\qquad f_{\varepsilon\varepsilon}(\lambda) = (2\pi)^{-1} C_1 [1 - |\Phi(\lambda)|^2]\,.$$

This, in turn, shows that the coherence (1.16) between the input and output series will be nearer to 1 the nearer the service time distribution is to being constant.

Turning to the problem of estimating characteristics of the service time distribution, the characteristic function $\Phi(\lambda)$ would be estimated by $A^{(T)}(-\lambda)$. Expressions (2.6), (3.2) show that the asymptotic variance of this estimate has the form

$$(3.3)\qquad B_T^{-1}T^{-1} \int W(\alpha)^2\, d\alpha\, C_1[1 - |\Phi(\lambda)|^2] f_{11}(\lambda)^{-1}\,.$$

The service time density function $g(u)$, if it exists and satisfies the conditions of Theorem 2.2, would be estimated by $a^{(T)}(u)$. Expressions (2.9), (3.2) show that the asymptotic variance of this estimate is

$$(3.4)\qquad C_T B_T^{-1}T^{-1} \int W(\alpha)^2\, d\alpha \int_{-C_TQ_T}^{C_TQ_T} (2\pi)^{-1} C_1[1 - |\Phi(\alpha)|^2] f_{11}(\alpha)^{-1}\, d\alpha\,.$$

In practice one would undoubtedly be more interested in estimating probabilities involving the service time variate rather than $\Phi(\lambda)$ or $g(u)$. Consider for example the problem of estimating $G(u + h) - G(u - h)$, the probability that the service time falls between $u - h$ and $u + h$. For $u - h$, $u + h$ points of continuity of G,

$$(3.5)\qquad G(u + h) - G(u - h) = \lim_{\Lambda\to\infty} (2\pi)^{-1} \int_{-\Lambda}^{\Lambda} \exp\{-iu\alpha\} \frac{\sin h\alpha}{h\alpha}\, \Phi(\alpha)\, d\alpha\,.$$

This suggests that we estimate the probability (3.5) by

$$(3.6)\qquad G^{(T)}(u + h) - G^{(T)}(u - h) = \left[\sum_{q=-Q_T}^{Q_T} A^{(T)}(C_T q) \exp\{iuC_T q\} \frac{\sin hC_T q}{hC_T q}\right][(1 - \cos C_T u)/(\pi C_T u^2)]\,.$$

In practice this estimate would be computed, for large Q_T, by a Fast Fourier Transform Algorithm. In connection with the estimate we have

THEOREM 3.2. *Let the conditions of Theorem* 3.1 *be satisfied. Suppose also*

(i) $\int |u|^3\, dG(u) < \infty$,
(ii) $\int |\alpha|^{r-1}|\Phi(\alpha)|\, d\alpha < \infty$ *for some* $r > 0$.

Then $G^{(T)}(u_1 + h_1) - G^{(T)}(u_1 - h_1), \cdots, G^{(T)}(u_J + h_J) - G^{(T)}(u_J - h_J)$ *is asymptotically normal with mean*

$$(3.7)\qquad \left[\left[\sum_{q=-Q_T}^{Q_T} A^{(T)}(C_T q) \exp\{-iu_j C_T q\} \frac{\sin h_j C_T q}{h_j C_T q}\right] \times [(1 - \cos C_T u_j)/(\pi C_T u_j^2)] : j = 1, \cdots, J\right] = [G(u_j + h_j) - G(u_j - h_j) : j = 1, \cdots, J] + O(C_T^3 Q_T) + O(C_T^{-r} Q_T^{-r})$$

and covariance matrix having the following entry in row j and column k

$$(3.8)\qquad C_T B_T^{-1} T^{-1} \int W(\alpha)^2\, d\alpha \int_{-C_T Q_T}^{C_T Q_T} \exp\{i(u_j - u_k)\alpha\} \times \frac{\sin h_j \alpha}{h_j \alpha} \frac{\sin h_k \alpha}{h_k \alpha} f_{\varepsilon\varepsilon}(\alpha) f_{11}(\alpha)^{-1}\, d\alpha$$

for $j, k = 1, \cdots, J$.

The asymptotic variance of $G^{(T)}(u + h) - G^{(T)}(u - h)$ is seen to be

$$(3.9)\qquad C_T B_T^{-1} T^{-1} \int W(\alpha)^2\, d\alpha \int_{-C_T Q_T}^{C_T Q_T} \left(\frac{\sin h\alpha}{h\alpha}\right)^2 f_{\varepsilon\varepsilon}(\alpha) f_{11}(\alpha)^{-1}\, d\alpha$$

which is of order $O(C_T B_T^{-1} T^{-1})$. From (3.7) we have,

COROLLARY. *If in addition to the conditions of the theorem,* $C_T^5 Q_T^2 B_T T \to 0$, $C_T^{-2r-1} Q_T^{-2r} B_T T \to 0$ *as* $T \to \infty$, *then* $G^{(T)}(u_1 + h_1) - G^{(T)}(u_1 - h_1), \cdots, G^{(T)}(u_J + h_J) - G^{(T)}(u_J - h_J)$ *is asymptotically normal with mean* $G(u_1 + h_1) - G(u_1 - h_1), \cdots, G(u_J + h_J) - G(u_J - h_J)$ *and covariance structure given above.*

Probabilities involving long intervals may be estimated by accumulating probabilities of the form considered above. For example, Prob $[0 < \text{service time} \leqq 2Lh] = G(2Lh) - G(0)$ may be estimated by

$$(3.10)\qquad \sum_{l=1}^{L} \{G^{(T)}([2l - 1]h + h) - G^{(T)}([2l - 1]h - h)\}\,.$$

The previous theorem indicates that this estimate is asymptotically normal with mean $G(2Lh) - G(0) + O(C_T^3 Q_T) + O(C_T^{-r} Q_T^{-r})$ and variance

$$(3.11)\qquad C_T B_T^{-1} T^{-1} \int W(\alpha)^2\, d\alpha) \int_{-C_T Q_T}^{C_T Q_T} \left(\frac{\sin L\alpha/2}{\sin \alpha/2}\right)^2 \left(\frac{\sin h\alpha}{h\alpha}\right)^2 f_{\varepsilon\varepsilon}(\alpha) f_{11}(\alpha)^{-1}\, d\alpha\,.$$

In the case that the input series is Poisson the expressions of this section simplify because it makes $f_{11}(\lambda) = C_1/2\pi$. The problem is then one concerning an $M/G/\infty$ queueing system. Milne (1970) showed that the service time distribution is identifiable in this case and Brown (1970) considered an estimate of the service time distribution. The discussions of these papers make specific use of properties of the Poisson process and so are not applicable to the $G/G/\infty$ problem considered here.

4. Proofs. In this section we prove the theorems of the paper and develop several theorems of independent interest, namely Theorems 4.2, 4.3, 4.4. We begin with,

THEOREM 4.1. *Let the process* $\{X_1(t), X_2(t)\}$, $-\infty < t < \infty$, *satisfy Assumption* 2.1. *Suppose* $|f_{jk}''(\lambda)| < \infty$, $j, k = 1, 2$. *Let* $W(\alpha)$ *satisfy Assumption* 2.1, $W^{(T)}(\alpha) = B_T^{-1} W(B_T^{-1}\alpha)$. *Then as* $T \to \infty$, *if* $B_T \to 0$ *and* $B_T T \to \infty$

$$(4.1)\qquad Ef_{jk}^{(T)}(\lambda) = f_{jk}(\lambda) + O(B_T^2) + O(B_T^{-1} T^{-1})$$

$$
\begin{aligned}
&\operatorname{Cov}\{f_{j_1k_1}^{(T)}(\lambda_1), f_{j_2k_2}^{(T)}(\lambda_2)\} \\
&\quad = 2\pi T^{-1}\{\textstyle\int W^{(T)}(\lambda_1-\alpha)W^{(T)}(\lambda_2-\alpha)f_{j_1j_2}(\alpha)f_{k_1k_2}(-\alpha)\,d\alpha \\
&\qquad + \textstyle\int W^{(T)}(\lambda_1-\alpha)W^{(T)}(\lambda_2+\alpha)f_{j_1k_2}(\alpha)f_{k_1j_2}(-\alpha)\,d\alpha \\
&\qquad + \textstyle\iint W^{(T)}(\lambda_1-\alpha_1)W^{(T)}(\lambda_2-\alpha_2) \\
&\qquad \times f_{j_1k_1j_2k_2}(\alpha_1, -\alpha_1, -\alpha_2)\,d\alpha_1\,d\alpha_2\} + O(B_T^{-2}T^{-2}) \\
&\quad = [2\pi B_T^{-1}T^{-1}\textstyle\int W(\alpha)^2\,d\alpha][\delta\{\lambda_1-\lambda_2\}f_{j_1j_2}(\lambda_1)f_{k_1k_2}(-\lambda_1) \\
&\qquad + \delta\{\lambda_1+\lambda_2\}f_{j_1k_2}(\lambda_1)f_{k_1j_2}(-\lambda_1)] \\
&\qquad + 2\pi T^{-1}f_{j_1k_1j_2k_2}(\lambda_1, -\lambda_1, -\lambda_2) + O(B_T^{-2}T^{-2}) + O(B_TT^{-1})
\end{aligned} \tag{4.2}
$$

for $|\lambda_1 \pm \lambda_2| \geqq B_T$ *if* $\lambda_1 \pm \lambda_2 \neq 0$

$$
\operatorname{cum}\{f_{j_1k_1}^{(T)}(\lambda_1), \cdots, f_{j_Jk_J}^{(T)}(\lambda_J)\} = O(B_T^{-J+1}T^{-J+1}) \tag{4.3}
$$

for $J = 1, 2, \cdots$. *The error terms are uniform in* $\lambda_1, \cdots, \lambda_J$. ($\delta\{\lambda\} = 1$ *for* $\lambda = 0$ *and* $= 0$ *otherwise*.)

PROOF. This follows in the manner of Theorems 2, 3, 4 in Brillinger and Rosenblatt (1967) and of Theorem 4.3 in Brillinger (1972).

COROLLARY. *Under the conditions of the theorem and if* $B_T^5T \to 0$ *as* $T \to \infty$, $[f_{j_1k_1}^{(T)}(\lambda), \cdots, f_{j_Jk_J}^{(T)}(\lambda_J)]$ *is asymptotically normal with mean* $[f_{j_1k_1}(\lambda_1), \cdots, f_{j_Jk_J}(\lambda_J)]$ *and*

$$
\begin{aligned}
&\lim_{T\to\infty} B_T T \operatorname{cov}\{f_{j_1k_1}^{(T)}(\lambda_1), f_{j_2k_2}^{(T)}(\lambda_2)\} \\
&\quad = 2\pi \textstyle\int W(\alpha)^2\,d\alpha\,[\delta\{\lambda_1-\lambda_2\}f_{j_1k_2}(\lambda_1)f_{k_1k_2}(-\lambda_1) \\
&\qquad + \delta\{\lambda_1+\lambda_2\}f_{j_1k_2}(\lambda_1)f_{k_1j_2}(-\lambda_1)]\,.
\end{aligned} \tag{4.4}
$$

PROOF. The standardized joint cumulants of order greater than 2 tend to 0.

This last theorem and corollary are seen to take the same exact form as corresponding results for ordinary stationary series (see Brillinger and Rosenblatt (1967).)

PROOF OF THEOREM 2.1. From Theorem 4.1 and Corollary 3 of Mann and Wald (1943)

$$
\begin{aligned}
A^{(T)}(\lambda) = A(\lambda) &+ \{[f_{21}^{(T)}(\lambda) - f_{21}(\lambda)] - A(\lambda)[f_{11}^{(T)}(\lambda) - f_{11}(\lambda)]\}/f_{11}(\lambda) \\
&+ O_p(B_T^{-1}T^{-1})\,.
\end{aligned}
$$

This last together with the Corollary above gives the indicated asymptotic normality. The specific expression (2.6) indicated for the asymptotic variance follows from (4.4) by simple algebra.

The next theorem has some independent interest. It will be needed in the proof of Lemma 4.1.

THEOREM 4.2. *Let the conditions of Theorem* 2.1 *be satisfied. Let* Q_T, $T = 1, 2, \cdots$ *be an increasing sequence of positive integers. Then for any* $\varepsilon > 0$

$$
\sup_{q=0}^{Q_T} |f_{jk}^{(T)}(C_Tq) - f_{jk}(C_Tq)| = o_p(Q_T^{\varepsilon}B_T^{-\frac{1}{2}}T^{-\frac{1}{2}}) \tag{4.5}
$$

as $T \to \infty$.

PROOF OF THEOREM 4.2. We have

$$\sup_{q=0}^{Q_T} |f_{jk}^{(T)}(C_T q) - Ef_{jk}^{(T)}(C_T q)|^{2m} \leqq \sum_{q=0}^{Q_T} |f_{jk}^{(T)}(C_T q) - Ef_{jk}^{(T)}(C_T q)|^{2m}$$

for m a positive integer. In Theorem 4.1 we saw that

$$\mathrm{cum}_J \{f_{jk}^{(T)}(\lambda)\} = O(B_T^{-J+1}T^{-J+1})$$

uniformly in λ. Using the classic formula expressing moments in terms of cumulants it follows that

$$E|f_{jk}^{(T)}(\lambda) - Ef_{jk}^{(T)}(\lambda)|^{2m} = O(B_T^{-m}T^{-m}) .$$

Therefore

$$E \sup_q |f_{jk}^{(T)}(C_T q) - Ef_{jk}^{(T)}(C_T q)|^{2m} = O(Q_T B_T^{-m}T^{-m})$$

and so

$$E \sup_q |f_{jk}^{(T)}(C_T q) - Ef_{jk}^{(T)}(C_T q)| = O(Q_T^{1/2m}B_T^{-\frac{1}{2}}T^{-\frac{1}{2}}) .$$

Taking m sufficiently large gives

$$\sup_q |f_{jk}^{(T)}(C_T q) - Ef_{jk}^{(T)}(C_T q)| = o_p(Q_T^{\varepsilon}B_T^{-\frac{1}{2}}T^{-\frac{1}{2}})$$

for any $\varepsilon > 0$. The proof is completed by noting that

$$\sup_q |Ef_{jk}^{(T)}(C_T q) - f_{jk}(C_T q)| = o(Q_T^{\varepsilon}B_T^{-\frac{1}{2}}T^{-\frac{1}{2}})$$

from Theorem 4.1.

We next state,

LEMMA 4.1. *Suppose the conditions of Theorem* 2.1 *are satisfied. Suppose* $f_{11}(\lambda) \geqq \delta > 0$ *for some* δ. *Let* Q_T, $T = 1, 2, \cdots$ *be an increasing sequence of positive integers, then*

$$\text{(4.6)} \quad \sup_{q=0}^{Q_T} |A^{(T)}(C_T q) - A(C_T q) - \{[f_{21}^{(T)}(C_T q) - f_{21}(C_T q)] - A(C_T q)[f_{11}^{(T)}(C_T q) - f_{11}(C_T q)]/f_{11}(C_T q)\}| = o_p(Q_T^{2\varepsilon}B_T^{-1}T^{-1})$$

for any $\varepsilon > 0$.

PROOF. We have the identity

$$\frac{a}{b} - \frac{\alpha}{\beta} - \frac{(a-\alpha)}{\beta} + \frac{\alpha}{\beta^2}(b-\beta) = \frac{(b-\beta)}{\beta}\left[\frac{\alpha(b-\beta)}{\beta} - (a-\alpha)\right]\frac{1}{b} .$$

Using this identity with $a = f_{21}^{(T)}$, $b = f_{11}^{(T)}$, $\alpha = f_{21}$, $\beta = f_{11}$, we see that the left-hand side of expression (4.6) is bounded by

$$\delta^{-1} \sup |f_{21}^{(T)} - f_{21}|[\delta^{-1} \sup |f_{21}| \sup |f_{11}^{(T)} - f_{11}| + \sup |f_{21}^{(T)} - f_{21}|]/\inf f_{11}^{(T)} .$$

Now $\inf f_{11}^{(T)} \geqq \inf f_{11} - \sup |f_{11}^{(T)} - f_{11}| = \inf f_{11} + o_p(Q_T^{\varepsilon}B_T^{-\frac{1}{2}}T^{-\frac{1}{2}})$ and so $1/\inf f_{11}^{(T)} = O_p(1)$. The Lemma now follows from Theorem 4.2.

Before proving Theorem 2.2, we develop the limiting distribution of a class of spectral averages. Consider

$$\mathscr{F}_{jk}^{(T)}(H) = C_T \sum_{q=-Q_T}^{Q_T} H(C_T q)[f_{jk}^{(T)}(C_T q) - Ef_{jk}^{(T)}(C_T q)]$$

for a function $H(\lambda)$, $-\infty < \lambda < \infty$, of interest.

THEOREM 4.3. *Let the process* $\{X_1(t), X_2(t)\}$ *satisfy Assumption* 2.2. *Suppose* $|f''_{jk}(\lambda)| < \infty$, $j, k = 1, 2$. *Let* $W(\alpha)$ *satisfy Assumption* 2.1. *Suppose* $|H_j(\lambda)|$ *is bounded. Let* $B_T \to 0$, $B_T T \to \infty$, $C_T \geqq B_T$ *as* $T \to \infty$. *Then* $E\mathscr{F}_{jk}^{(T)}(\lambda) = 0$ *and*

$$\begin{aligned}
&\mathrm{Cov}\,\{\mathscr{F}_{j_1k_1}^{(T)}(H_1), \mathscr{F}_{j_2k_2}^{(T)}(H_2)\} \\
&\quad = C_T^2 B_T^{-1} T^{-1} 2\pi \textstyle\int W(\alpha)^2\, d\alpha\, \{\sum_q H_1(C_T q) H_2(C_T q)^* f_{j_1j_2}(C_T q) \\
&\qquad \times f_{k_1k_2}(-C_T q) + \textstyle\sum_q H_1(C_T q) H_2(-C_T q)^* f_{j_1j_2}(C_T q) f_{k_1j_2}(-C_T q)\} \\
&\qquad + C_T^2 2\pi T^{-1} \textstyle\sum_{q_1}\sum_{q_2} H_1(C_T q_1) H_2(C_T q_2)^* \\
&\qquad \times f_{j_1k_1j_2k_2}(C_T q_1, -C_T q_1, -C_T q_2) + O(Q_T^2 C_T^2 B_T^{-2} T^{-2}) \\
&\qquad + O(Q_T^2 C_T^2 B_T T^{-1})\,.
\end{aligned} \tag{4.7}$$

If in addition, $\mathrm{Var}\,\mathscr{F}_{jk}^{(T)}(H) \geqq C_T^2 B_T^{-1} T^{-1} \sum |H(C_T q)|^2 \varepsilon$ *for some* $\varepsilon > 0$, $Q_T B_T = O(1)$, $Q_T^3 B_T^{-1} T^{-1} \to 0$, *then* $[\mathscr{F}_{j_1k_1}^{(T)}(H_1), \cdots, \mathscr{F}_{j_Jk_J}^{(T)}(H_J)]$ *is asymptotically normal.*

PROOF. Clearly $E\mathscr{F}_{jk}^{(T)}(H) = 0$. Using (4.2) the covariance (4.7) equals

$$\begin{aligned}
&C_T^2 \textstyle\sum_{q_1}\sum_{q_2} H_1(C_T q_1) H_2(C_T q_2)^* \{[B_T^{-1} T^{-1} 2\pi \int W(\alpha)^2\, d\alpha]\} \\
&\quad \times [\delta\{q_1 - q_2\} f_{j_1j_2}(C_T q_1) f_{k_1k_2}(-C_T q_1) + \delta\{q_1 + q_2\} f_{j_1k_2}(C_T q_1) f_{k_1j_2}(-C_T q_1)] \\
&\qquad + 2\pi T^{-1} f_{j_1k_1j_2k_2}(C_T q_1, -C_T q_1, -C_T q_2) + O(B_T^{-2} T^{-2}) + O(B_T T^{-1})
\end{aligned}$$

giving (4.7).
Next,

$$\begin{aligned}
&|\mathrm{cum}\,\{\mathscr{F}_{j_1k_1}^{(T)}(H_1), \cdots, \mathscr{F}_{j_Jk_J}^{(T)}(H_J)\}| \\
&\quad = |C_T^J \textstyle\sum_{q_1} \cdots \sum_{q_J} H_1(C_T q_1) \cdots H_J(C_T q_J) \\
&\qquad \times \mathrm{cum}\,\{f_{j_1k_1}^{(T)}(C_T q_1), \cdots, f_{j_Jk_J}^{(T)}(C_T q_J)\}| \\
&\quad \leqq C_T^J B_T^{-J+1} T^{-J+1} [\textstyle\sum_q |H_1(C_T q)| \cdots \sum_q |H_J(C_T q)|]\,.
\end{aligned}$$

Because

$$\textstyle\sum_q |H(C_T q)| \leqq [2Q + 1]^{\frac{1}{2}} [\sum_q |H(C_T q)|^2]^{\frac{1}{2}}$$

it follows that the standardized joint cumulant is $O(B_T^{1-J/2} T^{1-J/2} Q_T^{J/2})$. This tends to 0 for $J = 3, 4, \cdots$ provided $Q_T^3 B_T^{-1} T^{-1} \to 0$ as $T \to \infty$. This completes the proof of Theorem 4.3.

We next record a lemma that is essentially contained in the work of Bohman (1960).

LEMMA 4.2. *Let* $f(u)$, $-\infty < u < \infty$, *be such that*

(i) $\int |f(u)|\, du < \infty$ *and*
(ii) $\int |u|^2 |f(u)|\, du < \infty$.

Let

$$\phi(\lambda) = \textstyle\int_{-\infty}^{\infty} \exp\{i\lambda u\} f(u)\, du\,.$$

Then for $T, C > 0$

$$\left|(2\pi)^{-1} \int_{-T}^{T} \exp\{-i\lambda u\}\phi(\lambda)\, d\lambda - \frac{1 - \cos Cu}{\pi C u^2} \sum_{q=-T/C}^{T/C} \phi(Cq) \exp\{-iuCq\}\right| \leqq (12\pi)^{-1} T C^2 \int u^2 |f(u)|\, du\,.$$

PROOF. See pages 118–120 in Bohman (1960).

PROOF OF THEOREM 2.2. Using Lemma 4.1 and (4.1) we may write

$$a^{(T)}(u) = \frac{1 - \cos C_T u}{\pi C_T u^2} \sum_q \exp\{iuC_T q\} A(C_T q) + \zeta_T(u) + o_p(C_T Q_T^{1+2\varepsilon} B_T^{-1} T^{-1})$$

for any $\varepsilon > 0$, where

$$\zeta_T(u) = \frac{1 - \cos C_T u}{\pi C_T u^2} \sum_q \exp\{-iuC_T q\}\{[f_{21}^{(T)} - Ef_{21}^{(T)}] - A[f_{11}^{(T)} - Ef_{11}^{(T)}]/f_{11}\} .$$

Using Theorem 4.3, and some algebra, $[\zeta_T(u_1), \cdots, \zeta_T(u_J)]$ is seen to be asymptotically normal with mean $[0, \cdots, 0]$ and covariance structure

$$B_T^{-1}T^{-1}2\pi \int W(\alpha)^2\, d\alpha \frac{[1 - \cos C_T u_j][1 - \cos C_T u_k]}{\pi^2 C_T^2 u_j^2 u_k^2} \times \sum_q \exp\{i(u_j - u_k)C_T q\} f_{\varepsilon\varepsilon}(C_T q) f_{11}(C_T q)^{-1} .$$

The terms involving order 4 spectra drop out as we assume $B_T Q_T \to 0$. From Lemma 4.2

$$\frac{1 - \cos C_T(u_j - u_k)}{\pi C_T(u_j - u_k)^2} \sum_q \exp\{i(u_j - u_k)C_T q\} f_{\varepsilon\varepsilon}(C_T q) f_{11}(C_T q)^{-1} = \int_{-C_T Q_T}^{C_T Q_T} \exp\{i(u_j - u_k)\lambda\} f_{\varepsilon\varepsilon}(\lambda) f_{11}(\lambda)^{-1}\, d\lambda + O(C_T^3 Q_T) .$$

For T sufficiently large $1 - \cos C_T(u_j - u_k)$ does not vanish and the last expression is equivalent to (2.9).

The mean function (2.8) follows from Lemma 4.1 and the fact that $\int |\lambda|^r |A(\lambda)|\, d\lambda < \infty$.

We next set down a theorem of independent interest that will be used in the proof of Theorem 3.1.

THEOREM 4.4. *Under the conditions of Theorem* 3.1, *the probability generating functional of the process* $[N_1(t), N_2(t)]$ *is given by*

$$\text{(4.8)} \qquad \prod\nolimits_{12} [\xi_1, \xi_2] = E[\exp\{\int \log \xi_1(t)\, dN_1(t) + \int \log \xi_2(t)\, dN_2(t)\}] = \prod\nolimits_1 [\xi_1(\xi_2 * G^-)]$$

with $\prod_1 [\cdot]$ *denoting the probability generating functional of the process* $N_1(t)$, $*$ *denoting convolution and* $G^-(u) = 1 - G(-u)$.

PROOF. We have

$$\prod\nolimits_{12} [\xi_1, \xi_2] = E[\prod\nolimits_j \{\xi_1(\tau_j)\xi_2(\tau_j + \gamma_j)\}] = E[\prod\nolimits_j \{\xi_1(\tau_j) \int \xi_2(\tau_j + \gamma)\, dG(\gamma)\}]$$

giving the indicated expression as

$$\int \xi_2(\tau_j + \gamma)\, dG(\gamma) = \int \xi_2(\tau_j - \gamma)\, dG^-(\gamma) .$$

Probability generating functionals are discussed in Vere–Jones (1968). In that paper he evaluated $\prod_2 [\xi_2] = \prod_1 [\xi_2 * G^-]$.

PROOF OF THEOREM 3.1. The cumulant measures of the process appear in the expansion of the cumulant generating functional

$$\log \prod_{12} [\exp \eta_1, \exp \eta_2] = \sum_{j,k} \frac{1}{j!\, k!} \int \cdots \int \eta_1(t_1) \cdots \eta_1(t_j)\eta_2(u_1) \cdots \eta_2(u_k) \times \operatorname{cum}\{dN_1(t_1), \cdots, dN_1(t_j), dN_2(u_1), \cdots, dN_2(u_k)\}.$$

Using (4.8) this is given by

$$\prod_1 [\eta_1 + \log (e^{\eta_2} * G^-)] = \sum_j \frac{1}{j!} \int \cdots \int [\eta_1(t_1) + \log (e^{\eta_2} * G^-)(t_1)] \cdots [\eta_1(t_j) + \log (e^{\eta_2} * G^-)(t_j)] \operatorname{cum}\{dN_1(t_1), \cdots, dN_1(t_j)\}. \tag{4.9}$$

Making the expansions

$$e^{\eta} * G^-(t) = 1 + \eta * G^-(t) + \frac{1}{2!}\eta^2 * G^-(t) + \cdots$$

$$\log (e^{\eta} * G^-)(t) = \left\{\eta * G^-(t) + \frac{1}{2!}\eta^2 * G^-(t) + \cdots\right\} - \frac{1}{2}\left\{\eta * G^-(t) + \frac{1}{2!}\eta^2 * G^-(t) + \cdots\right\}^2 + \cdots$$

we see that the term of (4.9) of order m in η_1 and n in η_2 is the sum of terms that are simple multiples of

$$\int \cdots \int \eta_1(t_1) \cdots \eta_1(t_m)[\eta_2(w_{11})^{\alpha_1} \cdots \eta_2(w_{1\beta_1})^{\alpha_1}] \cdots [\eta_2(w_{k1})^{\alpha_k} \cdots \eta_2(w_{k\beta_k})^{\alpha_k}][dG(w_{11} - u_1) \cdots dG(w_{1\beta_1} - u_1)] \cdots [dG(w_{k1} - u_k) \cdots dG(w_{k\beta_k} - u_k)] \times \operatorname{cum}\{dN_1(t_1), \cdots, dN_1(t_m), dN_1(u_1), \cdots, dN_1(u_k)\}$$

where $\alpha_1\beta_1 + \cdots + \alpha_k\beta_k = n$. As $\int |u|\, dG(u) < \infty$, and (2.5) holds for the process $N_1(t)$, Fubini's Theorem applies and yields

$$\int \cdots \int [|t_1| + \cdots + |t_m| + |w_{11}| + \cdots + |w_{k\beta_k - 1}|] \times [dG(w_{11} - u_1) \cdots dG(w_{1\beta_1} - u_1)] \cdots [dG(w_{k1} - u_k) \cdots dG(w_{k\beta_k - 1} - u_k)\, dG(-u_k)] \times |\operatorname{cum}\{dN_1(t_1), \cdots, dN_1(t_m), dN_1(u_1), \cdots, dN_k(u_k)\}/du_k| < \infty$$

which is what is needed for (2.5).

PROOF OF THEOREM 3.2. Proved in the same manner as Theorem 2.2. Integration by parts shows that condition (i) of the Theorem implies condition (ii) of Lemma 4.2.

REFERENCES

BARTLETT, M. S. (1963). The spectral analysis of point processes. *J. Roy. Statist. Soc. Ser. B*, **25** 264–281.

BOCHNER, S. (1960). *Harmonic Analysis and the Theory of Probability.* Univ. of California Press, Berkeley.

BOHMAN, H. (1960). Approximate Fourier analysis of distribution functions. *Ark. Mat.* **4** 99–157.

BRILLINGER, D. R. (1972). The spectral analysis of stationary interval functions. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press, **1** 483–513.

BRILLINGER, D. R. and ROSENBLATT, M. (1967). Asymptotic theory of estimates of kth order spectra, in *Spectral Analysis of Time Series*, ed. B. Harris. Wiley, New York, 153–188.

BROWN, M. (1970). An $M/G/\infty$ estimation problem. *Ann. Math. Statist.* **41** 651–654.

COX, D. R. (1963). Some models for series of events. *Bull. Inst. Internat. Statist.* **40** 737–746.

DALEY, D. J. and VERE-JONES, D. (1972). A summary of the theory of point processes, in *Stochastic Point Processes*, ed. P. A. W. Lewis. Wiley, New York, 299–383.

HAWKES, A. G. (1972). Spectra of some mutually exciting point processes with associated variables, in *Stochastic Point Processes*, ed. P. A. W. Lewis. Wiley, New York, 261–271.

KOLMOGOROV, A. N. (1940). Curves in Hilber space invariant with regard to a one parameter group of motions. *Dokl. Akad. Nauk SSSR* **26** 6–9.

MANN, H. B. and WALD, A. (1943). On stochastic limit and order relationships. *Ann. Math. Statist.* **14** 217–226.

MILNE, R. K. (1970). Identifiability for random translations of Poisson processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **15** 195–201.

VERE-JONES, D. (1968). Some applications of probability generating functionals to the study of input-output streams. *J. Roy. Statist. Soc. Ser. B* **30** 321–333.

YAGLOM, A. M. (1958). Correlation theory of processes with stationary nth order increments. *Amer. Math. Soc. Trans.* **8.** Providence, 87–142.

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720