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ON LOCATION AND SCALE FUNCTIONS OF A CLASS OF LIMITING PROCESSES WITH APPLICATION TO EXTREME VALUE THEORY

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Let Y_n $(n = 0, 1, \cdots)$ be a random variable and suppose that for suitably chosen constants a_n and b_n $(b_n > 0)$ and each $t \in (0, \infty)$, the random variable $b_n Y_{[nt]} + a_n$ has a limit distribution function G_t. If G_1 is nondegenerate there are only two ways in which G_t is related to G_1: there exist real constants c and θ such that either $G_t(x) = G_1(c + t^\theta(x - c))$ for all $t > 0$ or else $G_t(x) = G_1(x + c \log t)$ for all $t > 0$. This result provides a very short derivation of the three types of extreme value limit distributions.

1. Introduction. We consider here stochastic processes $\{Z_t : t > 0\}$ which are constructed as the limit (in the sense of weak convergence on some metric space) of a sequence $y_n(t) = b_n Y_{[nt]} + a_n$ where $\{Y_n\}$ is a sequence of random variables and a_n and b_n $(b_n > 0)$ are norming constants.

If for each $t \in (0, \infty)$ we have $y_n(t) \rightarrow_D Z_t$ where Z_t is a nondegenerate random variable, then by the *convergence of types theorem* ([2] page 246) all the marginal distribution functions G_t of Z_t belong to the same type, namely, $G_t(x) = G(\beta_t x + \alpha_t)$ for some scale and location functions β_t $(\beta_t > 0)$ and α_t.

Although we are dealing here with a completely general situation, the forms of all the possible α_t and β_t are derived explicitly. This result is then used to give a short derivation of the extreme value limit distributions.

2. Main result.

THEOREM 1. *Suppose*

$$b_n Y_n + a_n \rightarrow_D Z \qquad (n \rightarrow \infty) \tag{1}$$

where Z is a nondegenerate random variable. If for each $t \in (0, \infty)$ there exists a random variable Z_t such that

$$b_n Y_{[nt]} + a_n \rightarrow_D Z_t \qquad (n \rightarrow \infty) \tag{2}$$

then there exist real functions β_t and α_t such that

$$Z_t =_D \beta_t Z + \alpha_t \qquad (\beta_t > 0)\,. \tag{3}$$

Moreover, there exist real θ and c such that

$$\beta_t = t^\theta \tag{4}$$

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and

$$\alpha_t = c \log t \qquad \text{if} \quad \theta = 0 \,, \tag{5}$$

$$\alpha_t = c(1 - t^\theta) \qquad \text{if} \quad \theta \neq 0 \,. \tag{6}$$

PROOF. Let H be the group of all *positive* affine transformations $\gamma = (\beta, \alpha)$ on R (i.e. $\gamma \in H$ iff $\gamma x = \beta x + \alpha$ for all $x \in R$ with $\beta > 0$) and let $h_n = (b_n, a_n)$. We shall first prove that for each $t \in (0, \infty)$ there exists $\gamma(t) = (\beta_t, \alpha_t) \in H$ such that

$$\lim_{n\to\infty} h_n h_{[nt]}^{-1} = \gamma(t) \,. \tag{7}$$

Fix $t > 0$ and put $\gamma_n(t) = h_n h_{[nt]}^{-1}$. The following facts are needed in the proof.

FACT 1. $\gamma_n(t)$ converges to a limit $\gamma(t)$ (possibly not in H). Indeed we have by (1) and (2)

$$W_n = h_{[nt]} Y_{[nt]} \to_D Z$$

and

$$\gamma_n(t) W_n = h_n Y_{[nt]} \to_D Z_t \,.$$

Hence either $Z_t = c$ (a constant) and $\gamma_n(t) \to (0, c)$ or Z_t is nondegenerate, in which case by the *convergence of types theorem* $\gamma_n(t) \to \gamma(t) \in H$ and

$$Z_t =_D \gamma(t) Z \,. \tag{8}$$

Let T be the set of all $t > 0$ for which $\gamma_n(t)$ converges to an element in H and let e be the identity in H. We consider the product $\delta_n(t) = h_n h_{[[nt]/t]}^{-1} = \gamma_n(t)\gamma_{[nt]}(t^{-1})$ for a fixed $t > 0$.

FACT 2. If e is a limit point of $\delta_n(t)$ then $t, t^{-1} \in T$ and $\delta_n(t) \to e$. Indeed by Fact 1 $\gamma_n(t) \to \gamma(t)$ and $\gamma_{[nt]}(t^{-1}) \to \gamma(t^{-1})$. By the assumption of Fact 2 both $\gamma(t)$ and $\gamma(t^{-1})$ belong to H. Hence $\gamma_n(t^{-1}) \to \gamma(t^{-1})$ and both t and t^{-1} lie in T.

FACT 3. If t is rational then $t \in T$. Indeed for rational t, $\delta_n(t) = e$ whenever nt is integral. Thus Fact 3 follows from Fact 2.

FACT 4. $h_n h_{n-1}^{-1} \to e$. For n odd $\delta_n(\frac{1}{2}) = h_n h_{n-1}^{-1}$ but $\delta_n(\frac{1}{2}) \to e$ by Fact 3. (By trivial induction $h_n h_{n+g}^{-1} \to e$ for all integers g.)

FACT 5. If $t > 1$ is irrational then $t \in T$. Indeed in this case $\delta_n(t) = h_n h_{n-1}^{-1}$ for each n and Fact 5 follows from Fact 4.

From Facts 2–5 follows that $T = (0, \infty)$ hence (7) holds and (3) follows from (8).

For given $s, t \in (0, \infty)$ consider the identity

$$\gamma_n(ts) = \gamma_n(t)\gamma_{[nt]}(s) h_{[[nt]s]} h_{[nts]}^{-1} \,.$$

Since $0 \leqq [nts] - [[nt]s] \leqq s + 1$ for all n, Fact 4 implies that $h_{[[nt]s]} h_{[nts]}^{-1} \to e$ and thus $\gamma(ts) = \gamma(t)\gamma(s)$, or equivalently

$$\beta_{ts} = \beta_t \beta_s \,; \qquad \alpha_{ts} = \alpha_t \beta_s + \alpha_s = \alpha_s \beta_t + \alpha_t \,.$$

Since $\gamma(t)$ is the limit of measurable functions it is measurable and the only measurable solution for β is given in (4). If $\theta = 0$ then $\alpha_{ts} = \alpha_t + \alpha_s$ and (5) follows. Otherwise $\alpha_t = (1 - t^\theta)\alpha_s(1 - s^\theta)^{-1}$ ($s \neq 1$). Thus for all $s \neq 1$, $\alpha_s(1 - s^\theta)^{-1}$ is constant and (6) follows. □

Notice that

$$\beta_t = \lim_{n\to\infty} \frac{b_n}{b_{[nt]}}\,; \qquad \alpha_t = \lim_{n\to\infty} \frac{a_n b_{[nt]} - a_{[nt]} b_n}{b_{[nt]}}\,.$$

Thus, if $\theta \neq 0$, by replacing a_n by $a_n - c$ (and Z by $Z - c$) one gets a new $\alpha_t \equiv 0$. Thus, in terms of the marginal distribution functions G_t of Z_t Theorem 1 can be restated as follows.

THEOREM 1*. *Let $\{H_n\}$ be a sequence of* df's *and let $b_n > 0$ and a_n be norming constants such that $H_{[nt]}((x - a_n)/b_n) \to G_t(x)$ (weakly) for all $t > 0$. If G_1 is nondegenerate then one of the following holbs*

(i) *There exists a real constant c such that*

$$G_t(x) = G_1(x + c \log t) \qquad \text{for all} \quad t > 0\,.$$

(ii) *There exist real constants $\theta \neq 0$ and c such that*

$$H_{[nt]}((x - \tilde{a}_n)/b_n) \to \tilde{G}(t^\theta x) \qquad \text{for all} \quad t > 0\,,$$

where $\tilde{a}_n = a_n - c$ and $\tilde{G}(x) = G_1(c + x)$.

REMARK. The function $h(u) = h_{[u]}$ is *regularly varying* since it is measurable and for each $t \in (0, \infty)$

$$\lim_{u\to\infty} h(u)h^{-1}(ut) = \gamma(t)\,.$$

Hence (see [1] page 95) the convergence of $\gamma_n(t)$ is uniform on compact subsets of $(0, \infty)$.

3. The extreme value limit distributions. With Theorem 1* at hand the derivation of the extreme value limit distributions is immediate.

THEOREM 2. (Gnedenko [3]). *The class of nondegenerate limit laws for $F^n((x - a_n)/b_n)$ where $b_n > 0$ and a_n are suitably chosen constants contains only laws of the types*

$$\begin{aligned} \Phi_\alpha(x) &= \exp(-x^{-\alpha}) && (x > 0, \alpha > 0) \\ \Psi_\alpha(x) &= \exp(-(-x)^\alpha) && (x < 0, \alpha > 0) \\ \Lambda(x) &= \exp(-e^{-x}) && (-\infty < x < \infty)\,. \end{aligned}$$

PROOF. If $F^n((x - a_n)/b_n) \to G(x)$ (weakly) then $F^{[nt]}((x - a_n)/b_n) \to G^t(x)$ (weakly) for all $t > 0$. Thus by Theorem 1* we conclude that either

$$G^t(x) = G(t^\theta x) \tag{9}$$

or else

$$G^t(x) = G(x + c \log t)\,. \tag{10}$$

Suppose we have (in (9)) $\theta < 0$. Since $G^t(x)$ is decreasing in t we must have

$G(0-) = 0$ and hence $G(t^\theta) = G^t(1) \equiv \exp(-bt)$ $(b > 0)$ or equivalently $G(x) = \exp(-bx^{1/\theta})$ $(x > 0)$, which is of the same type as $\Phi_\alpha(x)$ with $\alpha = -1/\theta$. Similarly, when $\theta > 0$, G is of Ψ_α-type with $\alpha = 1/\theta$ and when $\theta = 0$ we use (10) to conclude that G is of Λ-type.

Another application of Theorem 1* is made in [4].

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