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AN ESTIMATE OF THE MODULUS OF THE CHARACTERISTIC FUNCTION OF A LATTICE DISTRIBUTION WITH APPLICATION TO REMAINDER TERM ESTIMATES IN LOCAL LIMIT THEOREMS

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We prove an inequality for characteristic functions of lattice distributions, which can be used to give "explicit" remainder term estimates in local central limit theorems.

1. Introduction. Let X be an integer valued random variable with distribution

$$P(X = \nu) = p_\nu , \qquad \nu = 0, \pm 1, \pm 2, \cdots \tag{1.1}$$

and characteristic function

$$\varphi(t) = \sum_{\nu=-\infty}^{\infty} p_\nu e^{i\nu t} , \qquad -\infty < t < \infty . \tag{1.2}$$

Consider the following condition:

(1.3) The support of the distribution $\{p_\nu\}_{\nu=-\infty}^{\infty}$ is not contained in a coset of a proper subgroup of $\mathbb{Z}$.

It is well known, see e.g. Feller [1] page 475, that (1.3) can be characterized in terms of φ as follows: (1.3) holds iff φ attains the value 1 on $[-\pi, \pi]$ only for $t = 0$. We shall recast this criterion somewhat and we introduce

$$\omega(\delta) = \max_{\delta \leqq |t| \leqq \pi} |\varphi(t)| , \qquad 0 \leqq \delta \leqq \pi . \tag{1.4}$$

An equivalent way of expressing the criterion is: (1.3) holds iff

$$\omega(\delta) < 1 , \qquad 0 < \delta \leqq \pi . \tag{1.5}$$

Condition (1.3) is relevant in many contexts, one being in connection with so-called local limit theorems, i.e., results concerning the asymptotic (as $n \to \infty$) local behavior of the distribution of $S_n = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \cdots$ are independent random variables all having the distribution (1.1). When (1.3) holds $P(S_n = x)$ will be positive for all integers x sufficiently near ES_n (the expectation of S_n) and for n large enough, which is not true if (1.3) is not fulfilled.

When proving local limit theorems with the aid of characteristic functions, see e.g., Gnedenko–Kolmogorov [2] page 234, one meets the integral

$$\int_{\delta \leqq |t| \leqq \pi} |\varphi(t)|^n \, dt , \tag{1.6}$$

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which is dominated (see (1.4)) by

$$2\pi\omega(\delta)^n . \tag{1.7}$$

If (1.3) is fulfilled we can conclude that the term in (1.7) tends to zero as $n \to \infty$ for every $\delta > 0$, thereby obtaining one of the necessary ingredients in the proof.

From a "practical" point of view the bound (1.7) is unpleasant because $\omega(\delta)$ is a rather implicit function of the distribution $\{p_\nu\}_{\nu=-\infty}^{\infty}$. Our aim is to give an estimate of $\omega(\delta)$ which is more explicit in terms of the given distribution. The main result is Theorem 1, which is stated and proved in the next section. From the previous argument it should be clear that this estimate can be useful for instance in deriving "explicit" bounds for the remainder term in local central limit theorems.

We note that the assumption that X is integer valued is only a "standardization" of the more general situation that X is lattice distributed, i.e., takes its values in a set of the type $\{hk + r \mid k = 0, \pm 1, \pm 2, \cdots\}$. Our results can easily be transformed to this more general situation.

2. Bounds for $\omega(\delta)$. The support of the distribution of X is $S = \{k \in \mathbb{Z} \mid p_k > 0\}$. Let Δ denote the set

$$\Delta = \{|\nu - \mu| \mid \nu, \mu \in S, \nu \neq \mu\} \tag{2.1}$$

which we will call the difference set.

Now consider the following two statements about the support S of the distribution $\{p_\nu\}_{\nu=-\infty}^{\infty}$.

(2.2) (i) S is not contained in a coset of a proper subgroup of $\mathbb{Z}$.

(ii) There exists a finite subset $T \subset \Delta$ and integers q_j such that

$$\textstyle\sum_{j \in T} jq_j = 1 . \tag{2.3}$$

It is easily verified that (i) and (ii) are equivalent.

We can now give the main result on the estimation of $\omega(\delta)$.

THEOREM 1. *Put*

$$\eta_j = \textstyle\sum_{\nu=-\infty}^{\infty} p_{\nu+j}\, p_\nu \tag{2.4}$$

and let T and q_j be as in (2.3). *Then we have the estimate*:

$$\omega(\delta) = \max_{\delta \leqq |t| \leqq \pi} |\varphi(t)| \leqq 1 - \frac{2}{\pi^2} \frac{\delta^2}{\sum_{j \in T} |q_j|^2/\eta_j} . \tag{2.5}$$

PROOF. In terms of η_j, the set Δ can be represented as:

$$\Delta = \{j \in \mathbb{Z} \mid j \geqq 1, \eta_j > 0\} , \tag{2.6}$$

so there is no problem with denominators being 0 in (2.5).

We define $H(x)$, $x \in \mathbb{R}$, by

$$H(x) = \min_{n \in \mathbb{Z}} |x - 2\pi n| . \tag{2.7}$$

This function H has the following properties, which are all easily verified.

(i) It is subadditive, i.e.,

$$H(x + y) \leqq H(x) + H(y) . \tag{2.8}$$

(ii) For $n \in \mathbb{Z}$,

$$H(nx) \leqq |n| H(x) . \tag{2.9}$$

(iii)

$$1 - \cos x \geqq \frac{2}{\pi^2} H^2(x) , \qquad x \in \mathbb{R} . \tag{2.10}$$

We now multiply (2.3) by t and use (2.8) and (2.9) to conclude that

$$H(t) = H(\textstyle\sum_{j \in T} j q_j t) \leqq \sum_{j \in T} |q_j| H(jt) . \tag{2.11}$$

By (1.2) we get

$$\begin{aligned} |\varphi(t)|^2 &= |\textstyle\sum_{-\infty}^{\infty} p_\nu e^{i\nu t}|^2 = \sum_{\nu,\mu} p_\nu p_\mu e^{i(\nu-\mu)t} \\ &= \textstyle\sum_{\nu=-\infty}^{\infty} p_\nu{}^2 + 2 \sum_{\nu<\mu} p_\nu p_\mu \cos(\nu - \mu)t \\ &= (\textstyle\sum_{-\infty}^{\infty} p_\nu)^2 - 2 \sum_{\nu<\mu} p_\nu p_\mu (1 - \cos(\nu - \mu)t) \\ &= 1 - 2 \textstyle\sum_{j=1}^{\infty} \eta_j (1 - \cos jt) . \end{aligned} \tag{2.12}$$

From (2.10), Cauchy's inequality and (2.11) we get

$$\begin{aligned} |\varphi(t)|^2 &\leqq 1 - \frac{4}{\pi^2} \textstyle\sum_{j \in T} \eta_j H^2(jt) \\ &\leqq 1 - \frac{4}{\pi^2} \frac{1}{\sum_{j \in T} |q_j|^2/\eta_j} (\textstyle\sum_{j \in T} |q_j| H(jt))^2 \\ &\leqq 1 - \frac{4}{\pi^2} \frac{H^2(t)}{\sum_{j \in T} |q_j|^2/\eta_j} . \end{aligned} \tag{2.13}$$

From the observation that for $\delta \leqq |t| \leqq \pi$, $H(t) \geqq \delta$ and the elementary inequality $(1 - x)^{\frac{1}{2}} \leqq 1 - \frac{1}{2}x$, we finally conclude that (2.5) holds.

In fact Theorem 1 can be improved by considering several subsets T_d of Δ for which (2.3) is satisfied.

THEOREM 1′. *Assume that there exist integers q_{dj} such that*

$$\textstyle\sum_{j \in T_d} j q_{dj} = 1 \tag{2.14}$$

and let η_{dj} be positive and real such that

$$\textstyle\sum_d \eta_{dj} = \eta_j . \tag{2.15}$$

Then we have the estimate

$$\omega(\delta) \leqq 1 - \frac{2\delta^2}{\pi^2} \sum_d \frac{1}{\sum_{j \in T_d} |q_{dj}|^2/\eta_{dj}} . \tag{2.16}$$

This theorem can be proved exactly as the preceding one.

In the following example, we shall see how good the estimate (2.16) becomes in a (rather arbitrarily chosen) concrete situation.

EXAMPLE. Consider the distribution

$$p_\nu = \frac{1}{2N+1}, \qquad \nu = -N, -N+1, \cdots, N, \tag{2.17}$$
$$= 0 \qquad \text{otherwise}.$$

In this case the characteristic function is

$$\varphi(t) = \frac{1}{2N+1} \frac{\sin (N+\frac{1}{2})t}{\sin t/2}. \tag{2.18}$$

From $\varphi(\pi) = (-1)^N/(2N+1)$ it is easily seen that the constant C in the majorization

$$\omega(\delta) \leqq 1 - C\delta^2 \tag{2.19}$$

at least for even N must satisfy $C \leqq \pi^{-2}$. We now compare with the estimate obtained from (2.16). We get

$$\eta_j = \frac{2N-j+1}{(2N+1)^2}, \qquad j = 1, 2, \cdots, 2N. \tag{2.20}$$

Choosing the sets $T_1 = \{1\}$, $T_2 = \{2, 3\}$, $\cdots$, $T_{N-1} = \{2N-2, 2N-1\}$ yields

$$\begin{aligned} &\sum_d \frac{1}{\sum_{j \in T_d} |q_{dj}|^2/\eta_{dj}} \\ &\quad \geqq \frac{2N}{(2N+1)^2} + \frac{2N-2}{2(2N+1)^2} + \cdots + \frac{2}{2(2N+1)^2} \\ &\quad = \frac{N(N+3)}{2(2N+1)^2} \geqq \frac{1}{8}. \end{aligned} \tag{2.21}$$

From (2.16) and (2.21) we get

$$\omega(\delta) \leqq 1 - \frac{\delta^2}{4\pi^2}. \tag{2.22}$$

Hence we see that C in (2.19) can be taken to be $1/4\pi^2$ to be compared with the upper bound $C \leqq 1/\pi^2$ given above.

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