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A NOTE ON THE "LACK OF MEMORY" PROPERTY OF THE EXPONENTIAL DISTRIBUTION

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The exponential distribution is often characterized as the only distribution with lack of memory. This note points out a stronger result: the exponential is the only distribution that is occasionally forgetful.

Let X be a nonnegative random variable with tail distribution $R(x) = P[X \geqq x]$. Suppose that we say the distribution of X is *forgetful at* t_0 if $P[X \geqq t_0] > 0$ and

$$P[X \geqq t_0 + x \mid X \geqq t_0] = R(x) \qquad x \geqq 0\,;$$

that is, given that $X \geqq t_0$, the amount that X exceeds t_0 (the residual lifetime) has the same distribution as the unconditioned X.

Put in the form of a functional equation, since

$$P[X \geqq t_0 + x \mid X \geqq t_0] = R(t_0 + x)/R(t_0)\,,$$

we see that X (or its distribution) is forgetful at t_0 if $R(t_0) \neq 0$ and

$$R(t_0 + x) = R(t_0)R(x)\,, \qquad x \geqq 0\,.$$

Now if X is forgetful for every $t \geqq 0$ then $R(t + x) = R(t)R(x)$ for all $t, x \geqq 0$ and the standard method of developing R on the rationals and invoking right-continuity shows that R must be exponential.

The question is: for what set of t's can a distribution be forgetful? Our principal result is as follows:

THEOREM 1. *If the distribution of X is forgetful at two incommensurable values $t_1 < t_2$ (that is, t_1/t_2 is irrational) then X is exponential*:

$$P[X \geqq x] = R(x) = e^{-\beta x}\,, \qquad x \geqq 0\,.$$

We prove this theorem by establishing a general result on the set of values t for which the functional equation $R(t + x) = R(t)R(x)$, $x \geqq 0$ is valid.

THEOREM 2. *Let $f(x)$ be an arbitrary real-valued function defined for $x \geqq 0$ with $f(0) = 1$. Let T be the set of all nonnegative t's for which*:

$$f(t + x) = f(t)f(x) \qquad \textit{for all } x \geqq 0\,. \tag{1}$$

Then $t = 0$ is in T. If T contains at least one other value then one of these three conditions must hold:

(A) *T is dense in the interval* $[0, \infty)$.

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(B) *T consists of the point* $t = 0$ *and for some* $\gamma > 0$ *either the open interval* (γ, ∞) *or the closed interval* $[\gamma, \infty)$.

(C) *T is a lattice of the form* $0, \delta, 2\delta, 3\delta, \cdots$.

In particular, if T contains two incommensurable values $t_1 < t_2$ *where f does not vanish then T is dense in the interval* $[0, \infty)$.

In case A, *the additional condition that f be right-continuous ensures that the only nonconstant solutions to* (1) *are exponential*: $f(x) = e^{-\beta x}$, $x \geqq 0$.

In case B, *f is arbitrary for* $0 < x < \gamma$ *and f vanishes for* $x > \gamma$.

In case C, *all solutions to* (1) *can be constructed by arbitrarily assigning* $f(x)$ *in the first lattice interval* $0 < x \leqq \delta$ *and then f is completely determined by the requirements*

$$f(n\delta + x) = f(\delta)^n f(x) \qquad \text{for } 0 < x \leqq \delta \text{ and } n = 1, 2, 3, \cdots .$$

The proof depends on the fact that T is closed under addition and (ordered) subtraction—that is, if s, t are both in T then

(i) $t + s$ is in T; in particular $s, 2s, 3s, \cdots$ are all in T.

(ii) $t - s$ is in T provided $s < t$ and $f(s) \neq 0$.

Proof of (i) is trivial, while (ii) follows from writing, for all $x \geqq 0$,

$$f(s)f(t-s)f(x) = f(t)f(x) = f(t+x) = f(s + (t - s + x)) = f(s)f(t - s + x),$$

then dividing by $f(s)$.

Proof of the theorem now separates according to whether f vanishes in T:

If f vanishes in T, let $\gamma = \inf\{v : v \in T, f(v) = 0\}$. Then $\gamma = 0$ implies case A—that T is dense in $[0, \infty)$, since T contains arbitrarily small numbers and, by (i), all positive integral multiples of them. If $\gamma > 0$ then case B holds, since $f(x) = 0$ for $x > \gamma$ and if there were a $t \in T$ with $t < \gamma$ then $f(t), f(2t), f(3t), \cdots$ could not vanish.

If f does not vanish on T, let $\delta = \inf\{s, s \in T\}$. Then $\delta = 0$ implies that T is dense, for it contains arbitrary small numbers and all their multiples, while $\delta > 0$ implies that T is the lattice $0, \delta, 2\delta, 3\delta, \cdots$ since T is closed under ordered differences and any points in T between the lattice points would lead to a smaller δ.

Since if t_1/t_2 is not rational, the lattices

$$t_1, 2t_1, 3t_1, 4t_1, \cdots \qquad \text{and} \qquad t_2, 2t_2, 3t_2, 4t_2, \cdots$$

come arbitrarily close to one another, it follows that incommensurable points t_1 and t_2 in T (where f does not vanish) lead to $\delta = 0$.

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