A NOTE ON THE "LACK OF MEMORY" PROPERTY OF THE EXPONENTIAL DISTRIBUTION

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The exponential distribution is often characterized as the only distribution with lack of memory. This note points out a stronger result: the exponential is the only distribution that is occasionally forgetful.

Let $X$ be a nonnegative random variable with tail distribution $R(x) = P[X \geq x]$. Suppose that we say the distribution of $X$ is forgetful at $t_0$ if $P[X \geq t_0] > 0$ and

$$P[X \geq t_0 + x | X \geq t_0] = R(x) \quad x \geq 0;$$

that is, given that $X \geq t_0$, the amount that $X$ exceeds $t_0$ (the residual lifetime) has the same distribution as the unconditioned $X$.

Put in the form of a functional equation, since

$$P[X \geq t_0 + x | X \geq t_0] = R(t_0 + x)/R(t_0),$$

we see that $X$ (or its distribution) is forgetful at $t_0$ if $R(t_0) \neq 0$ and

$$R(t_0 + x) = R(t_0)R(x), \quad x \geq 0.$$

Now if $X$ is forgetful for every $t \geq 0$ then $R(t + x) = R(t)R(x)$ for all $t, x \geq 0$ and the standard method of developing $R$ on the rationals and invoking right-continuity shows that $R$ must be exponential.

The question is: for what set of $t$'s can a distribution be forgetful? Our principal result is as follows:

**Theorem 1.** If the distribution of $X$ is forgetful at two incommensurable values $t_1 < t_2$ (that is, $t_1/t_2$ is irrational) then $X$ is exponential:

$$P[X \geq x] = R(x) = e^{-x}, \quad x \geq 0.$$

We prove this theorem by establishing a general result on the set of values $t$ for which the functional equation $R(t + x) = R(t)R(x), x \geq 0$ is valid.

**Theorem 2.** Let $f(x)$ be an arbitrary real-valued function defined for $x \geq 0$ with $f(0) = 1$. Let $T$ be the set of all nonnegative $t$'s for which:

$$(1) \quad f(t + x) = f(t)f(x) \quad \text{for all} \quad x \geq 0.$$

Then $t = 0$ is in $T$. If $T$ contains at least one other value then one of these three conditions must hold:

(A) $T$ is dense in the interval $[0, \infty)$.  

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(B) $T$ consists of the point $t = 0$ and for some $\gamma > 0$ either the open interval $(\gamma, \infty)$ or the closed interval $[\gamma, \infty)$.

(C) $T$ is a lattice of the form $0, \delta, 2\delta, 3\delta, \ldots$.

In particular, if $T$ contains two incommensurable values $t_1 < t_2$ where $f$ does not vanish then $T$ is dense in the interval $[0, \infty)$.

In case A, the additional condition that $f$ be right-continuous ensures that the only nonconstant solutions to (1) are exponential: $f(x) = e^{-\beta x}, x \geq 0$.

In case B, $f$ is arbitrary for $0 < x < \gamma$ and $f$ vanishes for $x > \gamma$.

In case C, all solutions to (1) can be constructed by arbitrarily assigning $f(x)$ in the first lattice interval $0 < x \leq \delta$ and then $f$ is completely determined by the requirements

$$f(n\delta + x) = f(\delta)^n f(x) \quad \text{for} \quad 0 < x \leq \delta \quad \text{and} \quad n = 1, 2, 3, \ldots$$

The proof depends on the fact that $T$ is closed under addition and (ordered) subtraction—that is, if $s$, $t$ are both in $T$ then

(i) $t + s$ is in $T$; in particular $s$, $2s$, $3s$, $\ldots$ are all in $T$.

(ii) $t - s$ is in $T$ provided $s < t$ and $f(s) \neq 0$.

Proof of (i) is trivial, while (ii) follows from writing, for all $x \geq 0$,

$$f(s)f(t - s)f(x) = f(t)f(x) = f(t + x) = f(s + (t - s + x)) = f(s)f(t - s + x),$$

then dividing by $f(s)$.

Proof of the theorem now separates according to whether $f$ vanishes in $T$:

If $f$ vanishes in $T$, let $\gamma = \inf \{v : v \in T, f(v) = 0\}$. Then $\gamma = 0$ implies case A—that $T$ is dense in $[0, \infty)$, since $T$ contains arbitrarily small numbers and, by (i), all positive integral multiples of them. If $\gamma > 0$ then case B holds, since $f(x) = 0$ for $x > \gamma$ and if there were a $t \in T$ with $t < \gamma$ then $f(t), f(2t), f(3t), \ldots$ could not vanish.

If $f$ does not vanish on $T$, let $\delta = \inf \{s : s \in T\}$. Then $\delta = 0$ implies that $T$ is dense, for it contains arbitrary small numbers and all their multiples, while $\delta > 0$ implies that $T$ is the lattice $0, \delta, 2\delta, 3\delta, \ldots$ since $T$ is closed under ordered differences and any points in $T$ between the lattice points would lead to a smaller $\delta$.

Since if $t_1/t_2$ is not rational, the lattices

$$t_1, 2t_1, 3t_1, 4t_1, \ldots \quad \text{and} \quad t_2, 2t_2, 3t_2, 4t_2, \ldots$$

come arbitrarily close to one another, it follows that incommensurable points $t_1$ and $t_2$ in $T$ (where $f$ does not vanish) lead to $\delta = 0$.